NUCLEAR SCIENCE AND ENGINEERING: **84**, 285-290 (1983)

Particle Transport Theory in a Finite Sphere Containing a Spherical-Shell Source

C. E. Siewert* and J. R. Thomas, Jr.

Università di Bologna
Laboratorio di Ingegneria Nucleare
Bologna, Italy

and

Virginia Polytechnic Institute and State University
Nuclear Engineering Group
Blacksburg, Virginia 24061

Received November 22, 1982
Accepted January 20, 1983

The method of elementary solutions is used to compute the radiation field due to a spherical-shell source located in a finite sphere. In addition, exact analysis and the F_N method are used to establish an approximate solution that can be evaluated with modest computational effort to yield results accurate to five significant figures.

I. INTRODUCTION

In a paper published in 1968, Erdmann and Siewert[1] showed that the solution to the integral form of the particle transport equation

$$r\rho(r_0;r) = \frac{c}{2}\int_0^R r'\rho(r_0;r')K(r'\to r)\,dr' + \frac{S}{2r_0}K(r_0\to r) \ , \quad 0\leqslant r,r_0\leqslant R \ , \quad (1)$$

with

$$K(x\to y) = E_1(|x-y|) - E_1(x+y) \ , \quad (2)$$

where

$$E_1(x) = \int_0^1 \exp(-x/\nu)\,\frac{d\nu}{\nu} \ , \quad (3)$$

relevant to a spherical-shell source located at $r = r_0$ in a sphere of radius R, could be expressed as

$$\rho(r_0;r) = \frac{S}{2rr_0}\,[\phi(r_0;r) - \phi(r_0;-r)] \ , \quad 0\leqslant r,r_0\leqslant R \ , \quad (4)$$

where $\phi(r_0;r)$ is the equivalent result for a plane-parallel medium; i.e., $\phi(r_0;r)$ is defined by

$$\phi(r_0;r) = \frac{c}{2}\int_{-R}^R \phi(r_0;r')E_1(|r'-r|)\,dr' + E_1(|r_0-r|) \ , \quad -R\leqslant r,r_0\leqslant R \ . \quad (5)$$

In that same work[1] the method of elementary solutions[2] was used to establish $\phi(r_0;r)$ and hence the desired solution $\rho(r_0;r)$; however, no numerical results were given. In a recent paper by Spiga et al.,[3] an equation mathematically identical to Eq. (1) was obtained, as a special case, in the context of the diffusion of charged test particles in a finite slab.

*Permanent address: North Carolina State University, Mathematics Department, Raleigh, North Carolina 27650.

[1]R. C. ERDMANN and C. E. SIEWERT, *J. Math. Phys.*, **9**, 81 (1968).

[2]K. M. CASE, *Ann. Phys.*, **9**, 1 (1960).

[3]G. SPIGA, V. C. BOFFI, and A. MAGNAVACCA, *Z. Angew. Math. Phys.*, **33**, 783 (1982).

Having established the equivalent to Eq. (1), Spiga et al.[3] went on to use a projection technique to obtain numerical results for some cases of interest. Inspired by this new use of Eq. (1), in this paper we will follow two previous works[4,5] concerning the case $r_0 = 0$ and numerically evaluate the exact (at least in principle) solution reported by Erdmann and Siewert[1] and will use the F_N method[6] to provide an alternative approximate solution.

II. GENERAL ANALYSIS

We start with the integrodifferential form of the transport equation and boundary conditions appropriate to the Green's function for a finite slab, i.e.,

$$\mu \frac{\partial}{\partial r}\Psi(r_0;r,\mu) + \Psi(r_0;r,\mu) = \frac{c}{2}\int_{-1}^{1}\Psi(r_0;r,\mu')\,d\mu' + \delta(r - r_0) \tag{6}$$

and

$$\Psi(r_0;\mp R,\pm\mu) = 0\ , \quad \mu > 0\ . \tag{7}$$

We let $\Psi_\infty(r_0;r,\mu)$ denote the infinite-medium result and use the elementary solutions[2,7] to write

$$\Psi_\infty(r_0;r,\mu) = \frac{1}{N(\nu_0)}\,\phi(\pm\nu_0,\mu)\exp(-|r - r_0|/\nu_0) + \int_0^1 \frac{1}{N(\nu)}\,\phi(\pm\nu,\mu)\exp(-|r - r_0|/\nu)\,d\nu\ , \quad r \gtrless r_0\ , \tag{8}$$

where

$$N(\nu_0) = \frac{c}{2}\,\nu_0^3\left(\frac{c}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right) \tag{9a}$$

and

$$N(\nu) = \nu\left[(1 - c\nu\tanh^{-1}\nu)^2 + \frac{1}{4}\,\pi^2c^2\nu^2\right]\ . \tag{9b}$$

We then let

$$\Psi(r_0;r,\mu) = \Psi_\infty(r_0;r,\mu) - \Psi_c(r_0;r,\mu)\ , \tag{10}$$

where $\Psi_c(r_0;r,\mu)$ is the solution of

$$\mu\frac{\partial}{\partial r}\Psi_c(r_0;r,\mu) + \Psi_c(r_0;r,\mu) = \frac{c}{2}\int_{-1}^{1}\Psi_c(r_0;r,\mu')\,d\mu'\ , \tag{11}$$

subject to the boundary conditions

$$\Psi_c(r_0;\mp R,\pm\mu) = \Psi_\infty(r_0;\mp R,\pm\mu)\ , \quad \mu > 0\ . \tag{12}$$

Using the fact that

$$\phi(r_0;r) = \int_{-1}^{1}\Psi(r_0;r,\mu)\,d\mu\ , \tag{13}$$

we substitute Eq. (10) into Eq. (4) to obtain

$$\rho(r_0;r) = \rho_\infty(r_0;r) - \rho_c(r_0;r) \tag{14}$$

where

$$\rho_\infty(r_0;r) = \frac{S}{2rr_0}\left(\frac{1}{N(\nu_0)}\{\exp(-|r - r_0|/\nu_0) - \exp[-(r + r_0)/\nu_0]\} + \int_0^1 \frac{1}{N(\nu)}\{\exp(-|r - r_0|/\nu) - \exp[-(r + r_0)/\nu]\}\,d\nu\right) \tag{15}$$

is the infinite-medium result found by Erdmann and Siewert.[1] Here we have

$$\rho_c(r_0;r) = \frac{S}{2rr_0}\,[\phi_c(r_0;r) - \phi_c(r_0;-r)] \tag{16}$$

where

$$\phi_c(r_0;r) = \int_{-1}^{1}\Psi_c(r_0;r,\mu)\,d\mu\ . \tag{17}$$

To complete the solution we must now solve the problem defined by Eqs. (11) and (12).

III. THE EXACT SOLUTION

Since our (formally exact) solution to Eqs. (11) and (12) is essentially that of Case and Zweifel,[7] we simply sketch the development in order to define the notation we use. We write

$$\Psi_c(r_0;r,\mu) = A(\nu_0)\phi(\nu_0,\mu)\exp[-(R + r)/\nu_0] + A(-\nu_0)\phi(-\nu_0,\mu)\exp[-(R - r)/\nu_0] + \int_0^1 \{A(\nu)\phi(\nu,\mu)\exp[-(R + r)/\nu] + A(-\nu)\phi(-\nu,\mu)\exp[-(R - r)/\nu]\}\,d\nu\ , \tag{18}$$

where the expansion coefficients $A(\pm\xi)$ are to be found so that

$$F_1(\mu) = A(\nu_0)\phi(\nu_0,\mu) + A(-\nu_0)\phi(-\nu_0,\mu)\exp(-2R/\nu_0) + \int_0^1 [A(\nu)\phi(\nu,\mu) + A(-\nu)\phi(-\nu,\mu)\exp(-2R/\nu)]\,d\nu \tag{19a}$$

and

---

[4] C. E. SIEWERT and P. GRANDJEAN, *Nucl. Sci. Eng.*, **70**, 96 (1979).

[5] C. E. SIEWERT and J. R. MAIORINO, *J. Quant. Spectrosc. Radiat. Transfer*, **22**, 435 (1979).

[6] C. E. SIEWERT and P. BENOIST, *Nucl. Sci. Eng.*, **69**, 156 (1979).

[7] K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley, Reading, Massachusetts (1967).

$$F_2(\mu) = A(-\nu_0)\phi(\nu_0,\mu) + A(\nu_0)\phi(-\nu_0,\mu)\exp(-2R/\nu_0) + \int_0^1 [A(-\nu)\phi(\nu,\mu) + A(\nu)\phi(-\nu,\mu)\exp(-2R/\nu)]\,d\nu \quad (19b)$$

for $\mu \in [0,1]$. Here

$$F_1(\mu) = \frac{1}{N(\nu_0)}\phi(-\nu_0,\mu)\exp[-(R+r_0)/\nu_0] + \int_0^1 \frac{1}{N(\nu)}\phi(-\nu,\mu)\exp[-(R+r_0)/\nu]\,d\nu \quad (20a)$$

and

$$F_2(\mu) = \frac{1}{N(\nu_0)}\phi(-\nu_0,\mu)\exp[-(R-r_0)/\nu_0] + \int_0^1 \frac{1}{N(\nu)}\phi(-\nu,\mu)\exp[-(R-r_0)/\nu]\,d\nu \ . \quad (20b)$$

We can now substitute Eq. (18) into Eq. (17) and use Eq. (16) to express the remaining component of our desired solution as

$$\rho_c(r_0;r) = \frac{S}{2rr_0}\Big(E(\nu_0)\{\exp[-(R-r)/\nu_0] - \exp[-(R+r)/\nu_0]\} + \int_0^1 E(\nu)\{\exp[-(R-r)/\nu] - \exp[-(R+r)/\nu]\}\,d\nu\Big) \quad (21)$$

where

$$E(\xi) = A(-\xi) - A(\xi) \ . \quad (22)$$

Therefore, since only $E(\xi)$ is required here, we can subtract Eq. (19a) from Eq. (19b) for $\mu \in [0,1]$ to obtain

$$K(\mu) = E(\nu_0)[\phi(\nu_0,\mu) - \phi(-\nu_0,\mu)\exp(-2R/\nu_0)] + \int_0^1 E(\nu)[\phi(\nu,\mu) - \phi(-\nu,\mu)\exp(-2R/\nu)]\,d\nu \ , \quad (23)$$

where

$$K(\mu) = \frac{1}{N(\nu_0)}\phi(-\nu_0,\mu)\{\exp[-(R-r_0)/\nu_0] - \exp[-(R+r_0)/\nu_0]\} + \int_0^1 \frac{1}{N(\nu)}\phi(-\nu,\mu)\{\exp[-(R-r_0)/\nu] - \exp[-(R+r_0)/\nu]\}\,d\nu \ . \quad (24)$$

We can now multiply Eq. (23) by $\mu H(\mu)\phi(\xi,\mu)$, for $\xi \in \nu_0 \cup [0,1]$, integrate over μ from 0 to 1 and use the half-range orthogonality relations of Kuščer et al.[8] to find

$$E(\nu_0)[H(\nu_0)N(\nu_0) - J(\nu_0,\nu_0)\exp(-2R/\nu_0)] = F(\nu_0) + \int_0^1 E(\nu')J(\nu_0,\nu')\exp(-2R/\nu')\,d\nu' \quad (25a)$$

and

$$E(\nu)H(\nu)N(\nu) - \int_0^1 E(\nu')J(\nu,\nu')\exp(-2R/\nu')\,d\nu' = F(\nu) + E(\nu_0)J(\nu,\nu_0)\exp(-2R/\nu_0) \ . \quad (25b)$$

Here $H(\mu)$ is Chandrasekhar's H function,[9]

$$J(\xi,\zeta) = \int_0^1 \mu H(\mu)\phi(\xi,\mu)\phi(-\zeta,\mu)\,d\mu \quad (26)$$

and

$$F(\xi) = \frac{1}{N(\nu_0)}J(\xi,\nu_0)\{\exp[-(R-r_0)/\nu_0] - \exp[-(R+r_0)/\nu_0]\} + \int_0^1 \frac{1}{N(\nu)}J(\xi,\nu) \times \{\exp[-(R-r_0)/\nu] - \exp[-(R+r_0)/\nu]\}\,d\nu \ . \quad (27)$$

As Eqs. (25) can be solved by iteration, for example, the formally exact solution[1] is now in a convenient form to be evaluated numerically.

IV. AN F_N SOLUTION

We now develop an F_N solution[6] of Eqs. (11) and (12). If we substitute, for $\mu > 0$, the approximations

$$\Psi_c(r_0;-R,-\mu) = F_2(\mu)\exp(-2R/\mu) + \frac{c}{2}\sum_{\alpha=0}^{N} a_\alpha \mu^\alpha \quad (28a)$$

and

$$\Psi_c(r_0;R,\mu) = F_1(\mu)\exp(-2R/\mu) + \frac{c}{2}\sum_{\alpha=0}^{N} b_\alpha \mu^\alpha \quad (28b)$$

into the system of singular integral equations and constraints[10]

$$\int_{-1}^1 \mu\phi(\xi,\mu)\Psi_c(r_0;-R,-\mu)\,d\mu + \exp(-2R/\xi)\int_{-1}^1 \mu\phi(-\xi,\mu)\Psi_c(r_0;R,\mu)\,d\mu = 0 \quad (29a)$$

[8] I. KUŠČER, N. J. McCORMICK, and G. C. SUMMERFIELD, *Ann. Phys.*, **30**, 411 (1964).

[9] S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, London (1950).

[10] P. GRANDJEAN and C. E. SIEWERT, *Nucl. Sci. Eng.*, **69**, 161 (1979).

and

$$\int_{-1}^{1} \mu\phi(\xi,\mu)\Psi_c(r_0;R,\mu)\,d\mu + \exp(-2R/\xi)\int_{-1}^{1} \mu\phi(-\xi,\mu)\Psi_c(r_0;-R,-\mu)\,d\mu = 0 \tag{29b}$$

and consider $\xi = \xi_\beta \in \nu_0 \cup [0,1]$, we find

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi_\beta) + \exp(-2R/\xi_\beta) c b_\alpha A_\alpha(\xi_\beta)] = 2\int_0^1 \mu[F_1(\mu)S(\mu,\xi_\beta) + F_2(\mu)C(\mu,\xi_\beta)]\,d\mu \tag{30a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi_\beta) + \exp(-2R/\xi_\beta) c a_\alpha A_\alpha(\xi_\beta)] = 2\int_0^1 \mu[F_2(\mu)S(\mu,\xi_\beta) + F_1(\mu)C(\mu,\xi_\beta)]\,d\mu \ . \tag{30b}$$

We use here the collocation strategy suggested by Garcia and Siewert,[11] i.e., $\xi_0 = \nu_0$ and

$$\xi_\beta = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\beta-1}{2N}\pi\right) \ , \quad \beta = 1, 2, \ldots, N \ . \tag{31}$$

We note that the functions

$$A_\alpha(\xi) = \frac{2}{c\xi}\int_0^1 \mu^{\alpha+1}\phi(-\xi,\mu)\,d\mu \tag{32a}$$

and

$$B_\alpha(\xi) = \frac{2}{\xi}\int_0^1 \mu^{\alpha+1}\phi(\xi,\mu)\,d\mu \tag{32b}$$

are given by

$$A_0(\xi) = 1 - \xi\ln\left(1 + \frac{1}{\xi}\right) \ , \tag{33a}$$

$$A_\alpha(\xi) = -\xi A_{\alpha-1}(\xi) + \frac{1}{\alpha+1} \ , \quad \alpha \geqslant 1 \ , \tag{33b}$$

$$B_0(\xi) = 2 - c - c\xi\ln\left(1 + \frac{1}{\xi}\right) \ , \tag{33c}$$

and

$$B_\alpha(\xi) = \xi B_{\alpha-1}(\xi) - \frac{c}{\alpha+1} \ , \quad \alpha \geqslant 1 \ . \tag{33d}$$

In addition,

$$C(\mu,\xi) = \frac{\exp(-2R/\mu) - \exp(-2R/\xi)}{\mu - \xi} \tag{34a}$$

and

$$S(\mu,\xi) = \frac{1 - \exp(-2R/\mu)\exp(-2R/\xi)}{\mu + \xi} \ . \tag{34b}$$

We can now use Case's full-range theory[2,7] to deduce from Eq. (18) that

$$E(\xi)N(\xi) = \Xi(\xi) + \frac{c\xi}{2}\left[\Upsilon(\xi) + \frac{c}{2}\sum_{\alpha=0}^{N} e_\alpha A_\alpha(\xi)\right] \tag{35}$$

for $\xi \in \nu_0 \cup [0,1]$. Here $e_\alpha = a_\alpha - b_\alpha$,

$$\Upsilon(\xi) = \int_0^1 \mu[F_2(\mu) - F_1(\mu)]\exp(-2R/\mu)\frac{d\mu}{\mu+\xi} \ , \tag{36}$$

and

$$\Xi(\xi) = \int_0^1 \mu[F_2(\mu) - F_1(\mu)]\phi(\xi,\mu)\,d\mu \ . \tag{37}$$

As we require only $\{e_\alpha\}$ in Eq. (35), we can subtract Eq. (30b) from Eq. (30a) to find

$$\sum_{\alpha=0}^{N} [B_\alpha(\xi_\beta) - \exp(-2R/\xi_\beta) c A_\alpha(\xi_\beta)] e_\alpha = 2\int_0^1 \mu[F_2(\mu) - F_1(\mu)][C(\mu,\xi_\beta) - S(\mu,\xi_\beta)]\,d\mu \ , \tag{38}$$

which we can readily solve to find the desired constants $\{e_\alpha\}$. In the following section, we compare our F_N results to the exact solution developed in Sec. III.

V. NUMERICAL RESULTS

To begin our numerical calculation, we first compute, for a given value of $c < 1$, the discrete eigenvalue ν_0, i.e., the positive zero of

$$\Lambda(z) = 1 + \frac{c}{2}z\int_{-1}^{1}\frac{d\mu}{\mu - z} \ . \tag{39}$$

We then use Gaussian integration to evaluate numerically the integral term in Eq. (15). In the columns labeled $\rho_\infty(r_0;r)$ in Tables I through VI, we take $S = 1$ and list our infinite-medium results obtained by evaluating Eq. (15) for combinations of $c = 0.3$ and $c = 0.9$ with $r_0 = 0.05$, $r_0 = 0.45$, and $r_0 = 0.95$ for $R = 1$. Having established the infinite-medium result, we then approximate the integrals in Eqs. (25) in a Gaussian manner and subsequently solve the equations iteratively to deduce $E(\nu_0)$ and $E(\nu)$ at a set of Gauss points. With computed values of $E(\nu_0)$ and $E(\nu)$, we evaluate the correction term $\rho_c(r_0;r)$ from Eq. (21) and our finite-medium result $\rho(r_0;r)$ from Eq. (14). In the columns labeled as exact for $\rho(r_0;r)$ in Tables I through VI, we list the flux found in this manner. Results of our F_N calculations of the flux $\rho(r_0;r)$ are also included in Tables I through VI. Of course our F_N calculation yields only the correction term $\rho_c(r_0;r)$, which is used with $\rho_\infty(r_0;r)$ to obtain the flux $\rho(r_0;r)$. It is clear that our F_6 results agree very well with our "exact" results.

---

[11] R. D. M. GARCIA and C. E. SIEWERT, *Nucl. Sci. Eng.*, **81**, 474 (1982).

TABLE I

The Flux $\rho(r_0;r)$ for $c = 0.3$, $r_0 = 0.05$, and $R = 1$

r	$\rho_\infty(r_0;r)$ Exact	$\rho(r_0;r)$			
		F_4	F_5	F_6	Exact
0.0	3.93585(+2)[a]	3.93562(+2)	3.93562(+2)	3.93562(+2)	3.93562(+2)
0.1	1.06317(+2)	1.06294(+2)	1.06294(+2)	1.06294(+2)	1.06294(+2)
0.2	2.35360(+1)	2.35126(+1)	2.35126(+1)	2.35126(+1)	2.35126(+1)
0.3	9.78396	9.75994	9.75994	9.75994	9.75994
0.4	5.16424	5.13930	5.13930	5.13930	5.13930
0.5	3.09685	3.07065	3.07065	3.07065	3.07065
0.6	2.01080	1.98288	1.98288	1.98288	1.98288
0.7	1.37843	1.34816	1.34816	1.34816	1.34816
0.8	9.82854(−1)	9.49287(−1)	9.49286(−1)	9.49286(−1)	9.49286(−1)
0.9	7.22009(−1)	6.83354(−1)	6.83354(−1)	6.83353(−1)	6.83353(−1)
1.0	5.42922(−1)	4.92354(−1)	4.92353(−1)	4.92353(−1)	4.92353(−1)

[a]Read as 3.93585×10^2.

TABLE II

The Flux $\rho(r_0;r)$ for $c = 0.3$, $r_0 = 0.45$, and $R = 1$

r	$\rho_\infty(r_0;r)$ Exact	$\rho(r_0;r)$			
		F_4	F_5	F_6	Exact
0.0	3.92973	3.90424	3.90424	3.90424	3.90424
0.1	4.01569	3.99007	3.99007	3.99007	3.99007
0.2	4.30877	4.28276	4.28276	4.28276	4.28276
0.3	4.97329	4.94659	4.94659	4.94659	4.94659
0.4	6.90162	6.87390	6.87390	6.87390	6.87390
0.5	5.66570	5.63656	5.63656	5.63656	5.63656
0.6	2.85313	2.82207	2.82207	2.82207	2.82207
0.7	1.77100	1.73731	1.73731	1.73731	1.73731
0.8	1.19591	1.15851	1.15851	1.15851	1.15851
0.9	8.48948(−1)[a]	8.05832(−1)	8.05831(−1)	8.05831(−1)	8.05831(−1)
1.0	6.23628(−1)	5.67012(−1)	5.67011(−1)	5.67011(−1)	5.67011(−1)

[a]Read as 8.48948×10^{-1}.

TABLE III

The Flux $\rho(r_0;r)$ for $c = 0.3$, $r_0 = 0.95$, and $R = 1$

r	$\rho_\infty(r_0;r)$ Exact	$\rho(r_0;r)$			
		F_4	F_5	F_6	Exact
0.0	6.23383(−1)[a]	5.80785(−1)	5.80784(−1)	5.80784(−1)	5.80784(−1)
0.1	6.27753(−1)	5.84927(−1)	5.84927(−1)	5.84926(−1)	5.84926(−1)
0.2	6.41211(−1)	5.97693(−1)	5.97693(−1)	5.97692(−1)	5.97692(−1)
0.3	6.64898(−1)	6.20183(−1)	6.20183(−1)	6.20182(−1)	6.20182(−1)
0.4	7.01060(−1)	6.54570(−1)	6.54569(−1)	6.54569(−1)	6.54569(−1)
0.5	7.53797(−1)	7.04831(−1)	7.04830(−1)	7.04830(−1)	7.04830(−1)
0.6	8.30879(−1)	7.78527(−1)	7.78526(−1)	7.78526(−1)	7.78526(−1)
0.7	9.48821(−1)	8.91808(−1)	8.91807(−1)	8.91807(−1)	8.91807(−1)
0.8	1.15201	1.08833	1.08833	1.08832	1.08832
0.9	1.65511	1.58087	1.58087	1.58087	1.58087
1.0	1.49695	1.39571	1.39570	1.39570	1.39570

[a]Read as 6.23383×10^{-1}.

TABLE IV

The Flux $\rho(r_0;r)$ for $c = 0.9$, $r_0 = 0.05$, and $R = 1$

r	$\rho_\infty(r_0;r)$ Exact	$\rho(r_0;r)$			
		F_4	F_5	F_6	Exact
0.0	4.24415(+2)[a]	4.23809(+2)	4.23809(+2)	4.23809(+2)	4.23809(+2)
0.1	1.21933(+2)	1.21327(+2)	1.21327(+2)	1.21327(+2)	1.21327(+2)
0.2	3.13960(+1)	3.07871(+1)	3.07871(+1)	3.07871(+1)	3.07871(+1)
0.3	1.49747(+1)	1.43618(+1)	1.43618(+1)	1.43618(+1)	1.43618(+1)
0.4	8.98626	8.36750	8.36751	8.36749	8.36749
0.5	6.08085	5.45405	5.45405	5.45403	5.45403
0.6	4.42717	3.78959	3.78959	3.78957	3.78957
0.7	3.38448	2.73237	2.73238	2.73236	2.73236
0.8	2.67856	2.00626	2.00626	2.00624	2.00624
0.9	2.17508	1.47203	1.47203	1.47201	1.47201
1.0	1.80145	1.02648	1.02649	1.02646	1.02646

[a]Read as 4.24415×10^2.

TABLE V

The Flux $\rho(r_0;r)$ for $c = 0.9$, $r_0 = 0.45$, and $R = 1$

r	$\rho_\infty(r_0;r)$ Exact	$\rho(r_0;r)$			
		F_4	F_5	F_6	Exact
0.0	7.28862	6.66634	6.66634	6.66633	6.66632
0.1	7.37214	6.74908	6.74908	6.74906	6.74906
0.2	7.65758	7.03212	7.03212	7.03210	7.03210
0.3	8.30814	7.67856	7.67856	7.67854	7.67854
0.4	1.02134(+1)[a]	9.57776	9.57776	9.57774	9.57774
0.5	8.60572	7.96178	7.96178	7.96176	7.96176
0.6	5.24701	4.59191	4.59191	4.59189	4.59189
0.7	3.76533	3.09519	3.09519	3.09517	3.09517
0.8	2.88587	2.19478	2.19479	2.19477	2.19476
0.9	2.29978	1.57673	1.57673	1.57671	1.57671
1.0	1.88202	1.08384	1.08385	1.08382	1.08381

[a]Read as 1.02134×10.

TABLE VI

The Flux $\rho(r_0;r)$ for $c = 0.9$, $r_0 = 0.95$, and $R = 1$

r	$\rho_\infty(r_0;r)$ Exact	$\rho(r_0;r)$			
		F_4	F_5	F_6	Exact
0.0	1.97409	1.24723	1.24723	1.24720	1.24720
0.1	1.97843	1.25061	1.25060	1.25058	1.25058
0.2	1.99180	1.26102	1.26101	1.26099	1.26099
0.3	2.01527	1.27942	1.27941	1.27939	1.27939
0.4	2.05102	1.30769	1.30769	1.30766	1.30766
0.5	2.10301	1.34936	1.34935	1.34933	1.34933
0.6	2.17881	1.41120	1.41120	1.41117	1.41117
0.7	2.29461	1.50807	1.50806	1.50804	1.50803
0.8	2.49433	1.68106	1.68105	1.68103	1.68102
0.9	2.99168	2.13676	2.13675	2.13672	2.13672
1.0	2.74461	1.78488	1.78488	1.78485	1.78484

ACKNOWLEDGMENTS

C. E. Siewert is grateful to V. C. Boffi and the Università di Bologna for their kind hospitality during recent visits to Italy and to D. H. Roy for the interest shown in this investigation. In addition, the authors wish to express their gratitude to W. L. Dunn for communicating Monte Carlo solutions that increased their confidence in the numerical results reported here.

This work was supported in part by the Babcock and Wilcox Company and the U.S. National Science Foundation.