4. The F_N Method for Neutron Transport in Nonplanar Geometries, *J. R. Thomas, Jr. (VPI&SU), C. E. Siewert (North Carolina State Univ)*

The F_N approximation for neutron transport theory[1] has been shown to be an elegant approximate method for transport calculations in plane geometry, often yielding results of benchmark accuracy with remarkably low-order approximations. The work summarized here demonstrates the extension of this method to spherical and cylindrical geometry.

For both spherical and cylindrical geometries, the F_N method can be developed from the transform technique of Mitsis.[2] We illustrate the development here for the particular problems of a spherical-shell source in a finite sphere and the critical problem for an infinite cylinder.

TABLE I

The Flux $\rho(r_0;r)$ for $c = 0.9$, $r_0 = 0.45$, and $R = 1$

r	$\rho(r_0,r)$			
	F_4	F_5	F_6	Exact
0.0	6.66634	6.66634	6.66633	6.66632
0.1	6.74908	6.74908	6.74906	6.74906
0.2	7.03212	7.03212	7.03210	7.03210
0.3	7.67856	7.67856	7.67854	7.67854
0.4	9.57776	9.57776	9.57774	9.57774
0.5	7.96178	7.96178	7.96176	7.96176
0.6	4.59191	4.59191	4.59189	4.59189
0.7	3.09519	3.09519	3.09517	3.09517
0.8	2.19478	2.19479	2.19477	2.19476
0.9	1.57673	1.57673	1.57671	1.57671
1.0	1.08384	1.08385	1.08382	1.08381

For the problem of a sphere of radius R containing a spherical-shell source located at r_0, Erdmann and Siewert[3] showed that the neutron flux $\rho(r_0;r)$ could be expressed in the form

$$\rho(r_0;r) = \frac{S}{2rr_0}\left[\phi(r_0;r) - \phi(r_0;-r)\right] , \quad 0 \leq r, r_0 \leq R , \tag{1}$$

where $\phi(r_0;r)$ is the flux in the equivalent planar problem. The flux $\phi(r_0;r)$ is related to the angular density $\Psi(r_0;r,\mu)$ through the usual relation

$$\phi(r_0;r) = \int_{-1}^{1} \Psi(r_0;r,\mu)\, d\mu . \tag{2}$$

We find[4] that the solution $\Psi(r_0;r,\mu)$ can be expressed in the form

$$\Psi(r_0;r,\mu) = \Psi_\infty(r_0;r,\mu) - \Psi_C(r_0;r,\mu) , \tag{3}$$

where Ψ_∞ is the solution of the corresponding infinite-medium problem and Ψ_C is the remainder, both of which may be expressed in terms of the elementary solutions of Case and Zweifel.[5] The infinite-medium solution can be written out explicitly,[4] whereas Ψ_C can be written as an expansion in elementary solutions. A half-range orthogonality theorem[5] can be used to derive Fredholm integral equations for the expansion coefficients,[3,4] which are amenable to numerical solution by iteration.[4]

In the F_N method, on the other hand, we use full-range orthogonality[5] to derive a system of singular-integral equations[6] for Ψ_C on the surfaces:

$$\int_{-1}^{1} \mu\phi(\xi,\mu)\Psi_C(r_0;-R,-\mu)\, d\mu + \exp(-2R/\xi) \times \int_{-1}^{1} \mu\phi(-\xi,\mu)\Psi_C(r_0;R,\mu)\, d\mu = 0 \tag{4}$$

and

$$\int_{-1}^{1} \mu\phi(\xi,\mu)\Psi_C(r_0;R,\mu)\, d\mu + \exp(-2R/\xi) \times \int_{-1}^{1} \mu\phi(-\xi,\mu)\Psi_C(r_0;-R,-\mu)\, d\mu = 0 . \tag{5}$$

TABLE II

Critical Radius in Mean-Free-Paths

C	F-5	F-6	F-7	F-8	F-9	Exact	
1.01	13.1255	13.1255	13.1255	13.1255	13.1255	13.1255	(S)
1.05	5.41128	5.41128	5.41129	5.41129	5.41129	5.41129	(W)
1.1	3.57738	3.57739	3.57739	3.57739	3.57739	3.57739	(S,W)
1.2	2.28719	2.28721	2.28721	2.28721	2.28721	2.28721	(W)
1.3	1.72498	1.72500	1.72500	1.72500	1.72500	1.72500	(S)
1.4	1.39696	1.39697	1.39698	1.39698	1.39698	1.39698	(W)
1.5	1.17835	1.17833	1.17834	1.17834	1.17834	1.17834	(S)
1.6	1.02087	1.02082	1.02084	1.02084	1.02084	1.02084	(W)
1.8	0.807484	0.807419	0.807418	0.807426	0.807427	0.807427	(W)
2.0	0.668655	0.668626	0.668601	0.668610	0.668613	0.668613	(S,W)

(W) - Westfall.[6]
(S) - Sanchez.[8]

We then substitute the approximations

$$\Psi_c(r_0;-R,-\mu) = F_2(\mu)\exp(-2R/\mu) + \frac{c}{2}\sum_{\alpha=0}^{N} a_\alpha \mu^\alpha \quad (6)$$

and

$$\Psi_c(r_0;R,\mu) = F_1(\mu)\exp(-2R/\mu) + \frac{c}{2}\sum_{\alpha=0}^{N} b_\alpha \mu^\alpha \ , \quad (7)$$

and subsequently evaluate Eqs. (4) and (5) at N + 1 points $\xi_\beta \in \nu_0 U[0,1]$ to yield 2N + 2 linear algebraic equations for the 2N + 2 coefficients a_α, b_α. Here ν_0 represents the usual eigenvalue of one-speed transport theory,[5] and $F_1(\mu), F_2(\mu)$ are combinations of elementary solutions given elsewhere.[4] The system of linear equations can be solved straightforwardly for $\{a_\alpha\}$ and $\{b_\alpha\}$ so that the solution $\rho(r_0;r)$ can be readily reconstructed.[4]

For the critical problem in an infinite cylinder, Mitsis[2] has shown that the flux $\phi(r)$ can be expressed in terms of a transform function $\Phi(r,\mu)$ according to

$$\phi(r) = \int_0^1 \Phi(r,\mu)\frac{d\mu}{\mu^2} \ , \quad (8)$$

where $\Phi(r,\mu)$ is the solution of the integro-differential equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{\mu^2}\right)\Phi(r,\mu) = -c\int_0^1 \Phi(r,\mu')\frac{d\mu'}{\mu'^2} \quad (9)$$

subject to the conditions $\Phi(0,\mu) < \infty$ and

$$K_1(R/\mu)\Phi(R,\mu) + \mu K_0(R/\mu)\frac{\partial}{\partial r}\Phi(r,\mu)\Big|_{r=R} = 0, \mu \in [0,1] \ . \quad (10)$$

Thus, the solution can be expressed in terms of elementary solutions[2]:

$$\Phi(r,\mu) = \mu^2\left\{A(\nu_0)[\phi(\nu_0,\mu) + \phi(-\nu_0,\mu)]I_0(r/\nu_0) + \int_0^1 A(\nu)[\phi(\nu,\mu) + \phi(-\nu,\mu)]I_0(r/\nu)\,d\nu\right\} \ . \quad (11)$$

Westfall[6] used half-range orthogonality to deduce singular integral equations for $A(\nu_0)$ and $A(\nu)$ in Eq. (11), which he solved numerically to yield accurate values for the critical radius R and the flux $\phi(r)$. For the F_N method we use full-range orthogonality to derive the singular-integral equation

$$\int_0^1 [\phi(\xi,\mu) - \phi(-\xi,\mu)]\left[\mu + \xi\frac{I_0(R/\xi)}{I_1(R/\xi)}\frac{K_1(R/\mu)}{K_0(R/\mu)}\right] \times \Phi(R,\mu)\frac{d\mu}{\mu^2} = 0 \ , \quad (12)$$

where the $I_n(x)$ and $K_n(x)$ are modified Bessel functions of the first and second kind, respectively. Substituting the approximation

$$\Phi(R,\mu) = \mu^2\sum_{\alpha=0}^{N} a_\alpha \mu^\alpha \quad (13)$$

into Eq. (12) and evaluating at N + 1 points $\xi_\beta \in \nu_0 U[0,1]$, we find N + 1 equations for the coefficients a_α, $\alpha = 1, \ldots,$ N and R (we choose $a_0 = -1$ for convenience since the equations are homogeneous in the $\{a_\alpha\}$).

In Table I, we show values for the neutron flux as a function of r in the spherical-shell source problem with c = 0.9, r_0 = 0.45, and R = 1, for several values of N in comparison with the exact solution.[4] Note that in most cases, six digit accuracy is obtained with an F-6 approximation.

Similarly, in Table II we give values for the critical radius of an infinite cylinder for several values of c, in comparison with the accurate values of Westfall[6] and Sanchez.[8] It can be seen that for $1.01 \leqslant c \leqslant 1.6$, an F-7 approximation yields six-digit accuracy, whereas for $c > 1.8$, an F-9 approximation is needed.

We have shown that the F_N method can be conveniently applied to transport problems in cylindrical and spherical geometry, as well as plane geometry, to yield results of benchmark accuracy with relatively low-order approximations. Work is under way to develop multigroup and possibly multiregion F_N approximations in nonplanar geometries.

1. C. E. SIEWERT and P. BENOIST, *Nucl. Sci. Eng.*, **69**, 156 (1979).

2. G. J. MITSIS, "Transport Solutions to the Monoenergetic Critical Problems," ANL-6787, Argonne National Lab. (1963).

3. R. C. ERDMANN and C. E. SIEWERT, *J. Math. Phys.*, **9**, 81 (1968).

4. C. E. SIEWERT and J. R. THOMAS, Jr., *Nucl. Sci. Eng.*, **84**, 285 (1983).

5. K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley, Reading, Massachusetts (1967).

6. R. M. WESTFALL, PhD Thesis, University of Virginia (1974).

7. J. R. THOMAS, Jr., J. D. SOUTHERS, and C. E. SIEWERT, *Nucl. Sci. Eng.*, **84**, 79 (1983).

8. R. SANCHEZ, "Generalisation of Asaoka's Method to Linearly Anisotropic Scattering: Benchmark Data in Cylindrical Geometry," CEA-N-1831, Centre d'Etudes Nucléaires de Saclay, Gif-sur-Yvette, France (1975).