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Radiation transport in plane-parallel media with non-uniform surface illumination

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I. Introduction

In a recent paper [1] we considered the inverse problem in radiative transfer for infinite plane-parallel media and for a class of boundary conditions that allows the incident radiation to vary over the two free surfaces. Here we investigate the direct radiation transport problem for isotropic scattering, and thus we seek a solution of

$$\mu \frac{\partial}{\partial z} I(z, \varrho, \boldsymbol{\Omega}) + \boldsymbol{\omega} \cdot \frac{\partial}{\partial \boldsymbol{\varrho}} I(z, \varrho, \boldsymbol{\Omega}) + I(z, \varrho, \boldsymbol{\Omega}) = \frac{c}{4\pi} \iint I(z, \varrho, \boldsymbol{\Omega}')\,\mathrm{d}\boldsymbol{\Omega}' \tag{1}$$

subject to the boundary conditions

$$I(0, \varrho, \boldsymbol{\Omega}) = I_1(\varrho, \boldsymbol{\Omega}), \quad \mu > 0, \quad \varphi \in [0, 2\pi], \tag{2a}$$

and

$$I(a, \varrho, \boldsymbol{\Omega}) = I_2(\varrho, \boldsymbol{\Omega}), \quad \mu < 0, \quad \varphi \in [0, 2\pi], \tag{2b}$$

where $I_1(\varrho, \boldsymbol{\Omega})$ and $I_2(\varrho, \boldsymbol{\Omega})$ are assumed to be given and to have two-dimensional Fourier transforms. We use a notational scheme similar to that used by Rybicki [2] in his study of the classical searchlight problem. Thus z and ϱ, which lies in the $x - y$ plane, locate in optical units the position in the homogeneous medium and $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mu, \varphi)$, with $\mu = \cos(\theta)$, is a unit vector that defines the direction of propagation (see Fig. 1). In addition $\boldsymbol{\omega}$ is the projection of $\boldsymbol{\Omega}$ in the $x - y$ plane and $c < 1$ is the mean number of secondary particles per collision.

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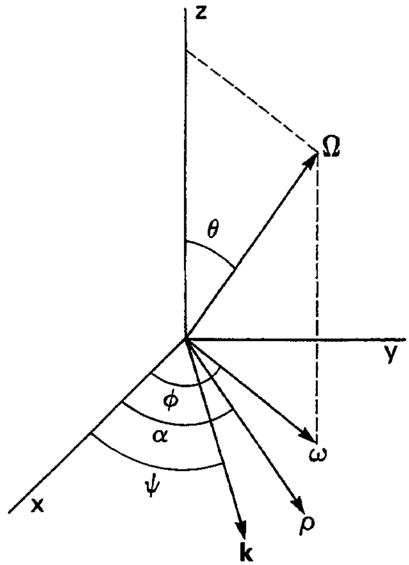

Figure 1
The geometry for $\boldsymbol{\Omega}$, $\boldsymbol{\omega}$, $\boldsymbol{\varrho}$, and $\boldsymbol{k}$.

We can multiply Eqs. (1) and (2) by $\exp(i\,\boldsymbol{k}\cdot\boldsymbol{\varrho})$ and integrate, for fixed z, over the $x-y$ plane to find

$$\mu\frac{\partial}{\partial z}\Psi(z,\mu,\varphi)+[1-if(\mu,\varphi)]\,\Psi(z,\mu,\varphi)=\frac{c}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\Psi(z,\mu',\varphi')\,\mathrm{d}\varphi'\,\mathrm{d}\mu' \quad (3)$$

and, for $\mu>0$ and $\varphi\in[0,2\pi]$,

$$\Psi(0,\mu,\varphi)=\Psi_1(\mu,\varphi) \quad (4\text{a})$$

and

$$\Psi(a,-\mu,\varphi)=\Psi_2(\mu,\varphi). \quad (4\text{b})$$

Here we suppress the dependence on the vector $\boldsymbol{k}$, which is in the $x-y$ plane as shown in Fig. 1 and write

$$\Psi(z,\mu,\varphi)=\int\!\!\int I(z,\boldsymbol{\varrho},\boldsymbol{\Omega})\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\varrho}}\,\mathrm{d}\boldsymbol{\varrho}, \quad (5)$$

$$\Psi_1(\mu,\varphi)=\int\!\!\int I_1[\boldsymbol{\varrho},\boldsymbol{\Omega}(\mu,\varphi)]\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\varrho}}\,\mathrm{d}\boldsymbol{\varrho} \quad (6\text{a})$$

and

$$\Psi_2(\mu,\varphi)=\int\!\!\int I_2[\boldsymbol{\varrho},\boldsymbol{\Omega}(-\mu,\varphi)]\,\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{\varrho}}\,\mathrm{d}\boldsymbol{\varrho}. \quad (6\text{b})$$

In addition

$$f(\mu,\varphi)=\boldsymbol{k}\cdot\boldsymbol{\omega}=k(1-\mu^2)^{1/2}\cos(\varphi-\psi), \quad (7)$$

with $k=|\boldsymbol{k}|$. It is apparent from Eqs. (3) and (4) that we can write, for $\mu>0$ and

$\varphi \in [0, 2\pi]$,

$$\Psi(z, \mu, \varphi) = \Psi_1(\mu, \varphi)\,\mathrm{e}^{-u(\mu,\varphi)z/\mu} + \frac{c}{4\pi\mu}\int_0^z \Psi(z')\,\mathrm{e}^{-u(\mu,\varphi)(z-z')/\mu}\,\mathrm{d}z' \tag{8a}$$

and

$$\Psi(z, -\mu, \varphi) = \Psi_2(\mu, \varphi)\,\mathrm{e}^{-u(\mu,\varphi)(a-z)/\mu} + \frac{c}{4\pi\mu}\int_z^a \Psi(z')\,\mathrm{e}^{-u(\mu,\varphi)(z'-z)/\mu}\,\mathrm{d}z' \tag{8b}$$

where

$$u(\mu, \varphi) = 1 - if(\mu, \varphi) \tag{9}$$

and

$$\Psi(z) = \int_{-1}^{1}\int_0^{2\pi} \Psi(z, \mu', \varphi')\,\mathrm{d}\varphi'\,\mathrm{d}\mu'. \tag{10}$$

We thus can integrate Eqs. (8) and add the two resulting equations to deduce the integral equation [2, 3]

$$\Psi(z) = F(z) + \frac{c}{2}\int_0^a \Psi(z')K(|z - z'|)\,\mathrm{d}z' \tag{11}$$

where the known term is

$$F(z) = \int_0^1\int_0^{2\pi} [\Psi_1(\mu, \varphi)\,\mathrm{e}^{-u(\mu,\varphi)z/\mu} + \Psi_2(\mu, \varphi)\,\mathrm{e}^{-u(\mu,\varphi)(a-z)/\mu}]\,\mathrm{d}\varphi\,\mathrm{d}\mu \tag{12}$$

and the kernel is

$$K(\xi) = \int_0^1 (1 + k^2\mu^2)^{-1/2}\,\mathrm{e}^{-\xi(1+k^2\mu^2)^{1/2}/\mu}\,\frac{\mathrm{d}\mu}{\mu}. \tag{13}$$

At this point we can make use of an idea recently reported by Williams [3] and consider, for $z \in [0, a]$ and $\mu \in [-1, 1]$, a pseudo problem defined by

$$\mu(1 + k^2\mu^2)^{1/2}\frac{\partial}{\partial z}\Phi(z, \mu) + (1 + k^2\mu^2)\Phi(z, \mu) = \frac{c}{2}\int_{-1}^{1}\Phi(z, \mu')\,\mathrm{d}\mu' + \frac{1}{2}F(z) \tag{14}$$

and the boundary conditions

$$\Phi(0, \mu) = 0, \quad \mu > 0, \tag{15a}$$

and

$$\Phi(a, -\mu) = 0, \quad \mu > 0. \tag{15b}$$

It is clear from Eq. (14) that, for $\mu > 0$,

$$\Phi(z, \mu) = \frac{1}{2\mu}(1 + k^2\mu^2)^{-1/2}\int_0^z S(z')\,\mathrm{e}^{-(z-z')(1+k^2\mu^2)^{1/2}/\mu}\,\mathrm{d}z' \tag{16a}$$

and

$$\Phi(z,-\mu)=\frac{1}{2\mu}(1+k^2\mu^2)^{-1/2}\int_z^a S(z')\,e^{-(z'-z)(1+k^2\mu^2)^{1/2}/\mu}\,dz' \tag{16b}$$

where

$$S(z)=c\,\Phi(z)+F(z) \tag{17}$$

with

$$\Phi(z)=\int_{-1}^{1}\Phi(z,\mu)\,d\mu. \tag{18}$$

Thus on integrating Eqs. (16) we find

$$\Phi(z)=\frac{1}{2}\int_0^a S(z')K(|z-z'|)\,dz' \tag{19}$$

or

$$S(z)=F(z)+\frac{c}{2}\int_0^a S(z')K(|z-z'|)\,dz'. \tag{20}$$

It follows that a solution of Eqs. (14) and (15) yields, by way of Eq. (17), a solution to Eq. (20), and so we conclude that

$$\Psi(z)=c\,\Phi(z)+F(z) \tag{21}$$

satisfies Eq. (11). Of course once $\Psi(z)$ is established, the Fourier transform of the angular flux $\Psi(z,\mu,\varphi)$ is available from Eqs. (8).

II. The pseudo problem

Considering now the pseudo problem defined by Eqs. (14) and (15), we multiply Eq. (14), with μ changed to $-\mu$, by $\exp(-z/s)$, integrate over z from 0 to a, multiply by $W(\mu,s)=s[s(1+k^2\mu^2)-\mu(1+k^2\mu^2)^{1/2}]^{-1}$, integrate over μ from -1 to 1 and use Eqs. (15) to obtain

$$\Lambda(s)S^*(s)=F^*(s)+cs\int_0^1\frac{\mu\,\Phi(0,-\mu)}{\mu-s(1+k^2\mu^2)^{1/2}}\,d\mu$$

$$-cs\,e^{-a/s}\int_0^1\frac{\mu\,\Phi(a,\mu)}{\mu+s(1+k^2\mu^2)^{1/2}}\,d\mu \tag{22}$$

where

$$\Lambda(s)=1+\frac{1}{2}cs\int_{-1}^{1}\frac{(1+k^2\mu^2)^{-1/2}}{\mu-s(1+k^2\mu^2)^{1/2}}\,d\mu, \tag{23}$$

$$F^*(s)=\int_0^a F(z)\,e^{-z/s}\,dz \tag{24}$$

and

$$S^*(s) = \int_0^a S(z)\,\mathrm{e}^{-z/s}\,\mathrm{d}z \tag{25}$$

or

$$S^*(s) = \int_0^a \Psi(z)\,\mathrm{e}^{-z/s}\,\mathrm{d}z. \tag{26}$$

Changing the integration variable in Eqs. (22) and (23), we find the convenient forms

$$\Lambda(s)\,S^*(s) = F^*(s) + c\,s\int_0^\gamma \tau\,\varphi^3(\tau)\,\Phi[0, -p(\tau)]\,\frac{\mathrm{d}\tau}{\tau - s}$$

$$- c\,s\,\mathrm{e}^{-a/s}\int_0^\gamma \tau\,\varphi^3(\tau)\,\Phi[a, p(\tau)]\,\frac{\mathrm{d}\tau}{\tau + s} \tag{27}$$

and

$$\Lambda(s) = 1 + \frac{1}{2}c\,s\int_{-\gamma}^{\gamma} \varphi(\tau)\,\frac{\mathrm{d}\tau}{\tau - s} \tag{28}$$

where

$$\varphi(\tau) = (1 - k^2\tau^2)^{-1/2}, \tag{29}$$

$$\gamma = (1 + k^2)^{-1/2} \tag{30}$$

and

$$p(\tau) = \tau(1 - k^2\tau^2)^{-1/2}. \tag{31}$$

We now observe from Eqs. (8) and (26) that the first of our desired results, the Fourier transforms of the angular fluxes exiting the medium, can be expressed, for $\mu > 0$, as

$$\Psi(0, -\mu, \varphi) = \Psi_2(\mu, \varphi)\,\mathrm{e}^{-au(\mu,\varphi)/\mu} + \frac{c}{4\pi\mu}\,S^*[\mu/u(\mu, \varphi)] \tag{32a}$$

and

$$\Psi(a, \mu, \varphi) = \Psi_1(\mu, \varphi)\,\mathrm{e}^{-au(\mu,\varphi)/\mu} + \frac{c}{4\pi\mu}\,\mathrm{e}^{-au(\mu,\varphi)/\mu}\,S^*[-\mu/u(\mu, \varphi)]. \tag{32b}$$

It is apparent that we can readily compute $S^*(s)$ from Eq. (27), for any s for which $\Lambda(s) \neq 0$, provided we first establish the pseudo angular fluxes at the boundary. We therefore now focus our attention on deducing $\Phi(0, -\mu)$ and $\Phi(a, \mu)$, $\mu > 0$.

Since $\Lambda(s)$ and the right-hand side of Eq. (27) are analytic in the s plane cut along the real axis from $-\gamma$ to γ we use the Plemelj formulas [4] to deduce from Eq. (27) that

$$\left[\lambda(\nu) \pm \frac{c}{2}\pi i\,\nu\,\varphi(\nu)\right] S^*(\nu) = F^*(\nu) + c\,\nu\,P\int_0^\gamma \tau\,\varphi^3(\tau)\,\Phi[0, -p(\tau)]\,\frac{\mathrm{d}\tau}{\tau - \nu}$$

$$\pm \pi i\,c\,\nu^2\,\varphi^3(\nu)\,\Phi[0, -p(\nu)] - c\,\nu\,\mathrm{e}^{-a/\nu}\int_0^\gamma \tau\,\varphi^3(\tau)\,\Phi[a, p(\tau)]\,\frac{\mathrm{d}\tau}{\tau + \nu} \tag{33a}$$

and

$$\left[\lambda(\nu) \pm \frac{c}{2}\pi i \nu \varphi(\nu)\right] S^*(-\nu) \mathrm{e}^{-a/\nu} = F^*(-\nu) e^{-a/\nu}$$
$$+ c \nu P \int_0^\gamma \tau \varphi^3(\tau) \Phi[a, p(\tau)] \frac{\mathrm{d}\tau}{\tau - \nu} \pm \pi i c \nu^2 \varphi^3(\nu) \Phi[a, p(\nu)]$$
$$- c \nu \mathrm{e}^{-a/\nu} \int_0^\gamma \tau \varphi^3(\tau) \Phi[0, -p(\tau)] \frac{\mathrm{d}\tau}{\tau + \nu} \tag{33b}$$

for $\nu \in [0, \gamma]$. Now from Eqs. (33) we conclude that

$$\lambda(\nu) S^*(\nu) = F^*(\nu) + c \nu P \int_0^\gamma \tau \varphi^3(\tau) \Phi[0, -p(\tau)] \frac{\mathrm{d}\tau}{\tau - \nu}$$
$$- c \nu \mathrm{e}^{-a/\nu} \int_0^\gamma \tau \varphi^3(\tau) \Phi[a, p(\tau)] \frac{\mathrm{d}\tau}{\tau + \nu}, \tag{34a}$$

$$S^*(\nu) = 2 \nu \varphi^2(\nu) \Phi[0, -p(\nu)], \tag{34b}$$

$$\lambda(\nu) S^*(-\nu) \mathrm{e}^{-a/\nu} = F^*(-\nu) \mathrm{e}^{-a/\nu} + c \nu P \int_0^\gamma \tau \varphi^3(\tau) \Phi[a, p(\tau)] \frac{\mathrm{d}\tau}{\tau - \nu}$$
$$- c \nu \mathrm{e}^{-a/\nu} \int_0^\gamma \tau \varphi^3(\tau) \Phi[0, -p(\tau)] \frac{\mathrm{d}\tau}{\tau + \nu} \tag{35a}$$

and

$$S^*(-\nu) \mathrm{e}^{-a/\nu} = 2 \nu \varphi^2(\nu) \Phi[a, p(\nu)] \tag{35b}$$

for $\nu \in [0, \gamma]$. Here

$$\lambda(\nu) = 1 + \frac{1}{2} c \nu P \int_{-\gamma}^{\gamma} \varphi(\tau) \frac{\mathrm{d}\tau}{\tau - \nu}. \tag{36}$$

Clearly we can eliminate between Eqs. (34) and (35) to find

$$2 \nu \lambda(\nu) \varphi^2(\nu) \Phi[0, -p(\nu)] = F^*(\nu) + c \nu P \int_0^\gamma \tau \varphi^3(\tau) \Phi[0, -p(\tau)] \frac{\mathrm{d}\tau}{\tau - \nu}$$
$$- c \nu \mathrm{e}^{-a/\nu} \int_0^\gamma \tau \varphi^3(\tau) \Phi[a, p(\tau)] \frac{\mathrm{d}\tau}{\tau + \nu} \tag{37a}$$

and

$$2 \nu \lambda(\nu) \varphi^2(\nu) \Phi[a, p(\nu)] = F^*(-\nu) \mathrm{e}^{-a/\nu} + c \nu P \int_0^\gamma \tau \varphi^3(\tau) \Phi[a, p(\tau)] \frac{\mathrm{d}\tau}{\tau - \nu}$$
$$- c \nu \mathrm{e}^{-a/\nu} \int_0^\gamma \tau \varphi^3(\tau) \Phi[0, -p(\tau)] \frac{\mathrm{d}\tau}{\tau + \nu} \tag{37b}$$

for $\nu \in [0, \gamma]$. The limiting values $\Lambda^{\pm}(\nu)$ of $\Lambda(s)$ cannot vanish for $\nu \in [-\gamma, \gamma]$; however, the argument principle [5] can be used to show that $\Lambda(s)$ has exactly two zeros $\pm s_0$, and thus we supplement Eqs. (37) with the constraints

$$F^*(s_0) + c s_0 \int_0^\gamma \tau \varphi^3(\tau) \Phi[0, -p(\tau)] \frac{d\tau}{\tau - s_0}$$
$$- c s_0 e^{-a/s_0} \int_0^\gamma \tau \varphi^3(\tau) \Phi[a, p(\tau)] \frac{d\tau}{\tau + s_0} = 0 \tag{38a}$$

and

$$F^*(-s_0) e^{-a/s_0} + c s_0 \int_0^\gamma \tau \varphi^3(\tau) \Phi[a, p(\tau)] \frac{d\tau}{\tau - s_0}$$
$$- c s_0 e^{-a/s_0} \int_0^\gamma \tau \varphi^3(\tau) \Phi[0, -p(\tau)] \frac{d\tau}{\tau + s_0} = 0. \tag{38b}$$

Equations (37) and (38) represent a system of singular-integral equations and constraints that can be solved, at least in principle, to yield the required pseudo boundary fluxes $\Phi(0, -\mu)$ and $\Phi(a, \mu)$, $\mu > 0$.

III. The F_N method

Rather than pursue exact analysis, we wish to use the F_N method [6, 7, 8] to develop approximate, but accurate, solutions to Eqs. (37) and (38). We therefore choose to approximate the exiting boundary fluxes by the representations, for $\mu > 0$,

$$\Phi(0, -\mu) = \sum_{\alpha=0}^{N} a_\alpha S_\alpha(\mu) \tag{39a}$$

and

$$\Phi(a, \mu) = \sum_{\alpha=0}^{N} b_\alpha S_\alpha(\mu) \tag{39b}$$

where the basis functions $S_\alpha(\mu)$ are to be selected. We can now substitute Eqs. (39) into Eqs. (37) and (38) to find

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi) + c e^{-a/\xi} b_\alpha A_\alpha(\xi)] = \Xi(\xi) \tag{40a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi) + c e^{-a/\xi} a_\alpha A_\alpha(\xi)] = T(\xi) \tag{40b}$$

for $\xi \in P = s_0 \, U [0, \gamma]$. Here we have defined

$$A_\alpha(\xi) = \int_0^\gamma \tau \varphi^3(\tau) S_\alpha[p(\tau)] \frac{d\tau}{\xi + \tau}, \quad \xi \notin [-\gamma, 0), \tag{41}$$

$$B_\alpha(s_0) = c \int_0^\gamma \tau \varphi^3(\tau) S_\alpha[p(\tau)] \frac{d\tau}{s_0 - \tau} \tag{42a}$$

and

$$B_\alpha(\nu) = c P \int_0^\gamma \tau \varphi^3(\tau) S_\alpha[p(\tau)] \frac{d\tau}{\nu - \tau} + 2\lambda(\nu)\varphi^2(\nu) S_\alpha[p(\nu)] \tag{42b}$$

for $\nu \in [0, \gamma]$. In addition the known terms in Eqs. (40) are

$$\Xi(\xi) = \frac{1}{\xi} F^*(\xi) \tag{43a}$$

and

$$T(\xi) = \frac{1}{\xi} F^*(-\xi) e^{-a/\xi}. \tag{43b}$$

It is apparent that we can now consider Eqs. (40) at $N + 1$ selected values of ξ, say ξ_β, $\beta = 0, 1, 2, \ldots, N$, to obtain the system of linear algebraic equations

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi_\beta) + c\, e^{-a/\xi_\beta} b_\alpha A_\alpha(\xi_\beta)] = \Xi(\xi_\beta) \tag{44a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi_\beta) + c\, e^{-a/\xi_\beta} a_\alpha A_\alpha(\xi_\beta)] = T(\xi_\beta) \tag{44b}$$

that can be solved to yield the constants $\{a_\alpha\}$ and $\{b_\alpha\}$ required in Eqs. (39). It is also clear that the choice of basis functions $S_\alpha(\mu)$ and the collocation strategy used to define the ξ_β play vital roles in this approximate solution.

In order to continue with our F_N calculation we now must select a set of basis functions $S_\alpha(\mu)$ and establish an efficient way to evaluate the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ defined by Eqs. (41) and (42). We let $P_\alpha(z)$ denote the Legendre polynomial of degree α and elect to use

$$S_\alpha(\mu) = \varphi^{-2}[\mu(1 + k^2\mu^2)^{-1/2}] P_\alpha[2\mu(1 + k^2\mu^2)^{-1/2} - 1] \tag{45}$$

which yields

$$S_\alpha[p(\tau)] = \varphi^{-2}(\tau) P_\alpha(2\tau - 1). \tag{46}$$

We can now substitute Eq. (46) into Eq. (41) to find, for $\xi \notin [-\gamma, 0)$,

$$A_\alpha(\xi) = \int_0^\gamma \tau \varphi(\tau) P_\alpha(2\tau - 1) \frac{d\tau}{\tau + \xi}. \tag{47}$$

On multiplying Eq. (47) by $(2\xi + 1)$ and using the recursion formula

$$(2\alpha + 1)(2\tau - 1) P_\alpha(2\tau - 1) = (\alpha + 1) P_{\alpha+1}(2\tau - 1) + \alpha P_{\alpha-1}(2\tau - 1), \tag{48}$$

we find

$$(2\alpha + 1)(2\xi + 1) A_\alpha(\xi) + (\alpha + 1) A_{\alpha+1}(\xi) + \alpha A_{\alpha-1}(\xi) = 2(2\alpha + 1)\Delta_\alpha \tag{49}$$

where

$$\Delta_\alpha = \int_0^\gamma \tau\,\varphi(\tau)\,P_\alpha(2\tau - 1)\,\mathrm{d}\tau. \tag{50}$$

In a similar manner we find from Eqs. (42) that the functions $B_\alpha(\xi)$, with $S_\alpha(\mu)$ given by Eq. (45), satisfy, for $\xi \in P$, the recursion formula

$$(2\alpha + 1)(2\xi - 1)\,B_\alpha(\xi) - (\alpha + 1)\,B_{\alpha+1}(\xi) - \alpha B_{\alpha-1}(\xi) = 2(2\alpha + 1)\,c\,\Delta_\alpha. \tag{51}$$

In order to initiate the use of Eqs. (49) and (51) in the forward direction we clearly require $A_0(\xi)$ and $B_0(\xi)$ for $\xi \in P$. The starting value

$$A_0(\xi) = \int_0^\gamma \tau\,\varphi(\tau)\,\frac{\mathrm{d}\tau}{\tau + \xi} \tag{52}$$

follows from Eq. (47), and from Eqs. (42) and (28) we find we can express the other starting value in the convenient form

$$B_0(\xi) = 2\Lambda(\infty) + c\,A_0(\xi), \tag{53}$$

for $\xi \in P$. Here

$$\Lambda(\infty) = 1 - \frac{c}{k}\,\mathrm{Tan}^{-1}\,k. \tag{54}$$

For $\xi \in [0, 1/k]$ we can carry out the integration in Eq. (52) to obtain

$$A_0(\xi) = \frac{1}{k}\,\mathrm{Tan}^{-1}\,k - \xi\,\varphi(\xi)\ln[1 + 2/f(\xi)] \tag{55}$$

where

$$f(\xi) = [1 + (1 + k^2)^{1/2}]\,\xi\,\varphi(\xi) + \varphi(\xi) - 1. \tag{56}$$

We note that Eq. (23) can be written as

$$\Lambda(s) = 1 + c\,s^2(1 - k^2 s^2)^{-1} \int_0^1 \frac{\mathrm{d}\mu}{\mu^2 - s^2(1 - k^2 s^2)^{-1}}, \tag{57}$$

so that the positive zero s_0 of $\Lambda(s)$ can be expressed as

$$s_0 = \nu_0(1 + k^2 \nu_0^2)^{-1/2} \tag{58}$$

where ν_0 is the positive zero of the usual ($k = 0$) dispersion function

$$\Lambda_0(z) = 1 + \frac{1}{2}\,c\,z \int_{-1}^{1} \frac{\mathrm{d}\mu}{\mu - z}. \tag{59}$$

It is therefore clear that

$$(1 + k^2)^{-1/2} < s_0 < k^{-1} \tag{60}$$

for all k and all $c \in (0, 1)$. For the conservative case $c = 1$ we observe that

$$s_0 = k^{-1}. \tag{61}$$

It therefore follows that Eqs. (53) and (55) provide for $c \in (0, 1]$ the desired starting expressions for all $\xi \in P$.

IV. The desired solution

Here we assume that the F_N method has been used as discussed in the foregoing section to deduce the exit boundary fluxes for the pseudo problem, and we proceed to express the boundary fluxes $I[0, \varrho, \boldsymbol{\Omega}(-\mu, \varphi)]$ and $I[a, \varrho, \boldsymbol{\Omega}(\mu, \varphi)]$ for $\mu > 0$ and all φ in terms of the F_N results. First of all we can use Eqs. (27) and (39) in Eqs. (32) to obtain, for $\mu > 0$ and all φ,

$$\begin{aligned}\Psi(0, -\mu, \varphi) = \Psi_2(\mu, \varphi)\,\mathrm{e}^{-au(\mu,\varphi)/\mu} &+ c\,[4\pi u(\mu, \varphi)]^{-1} \Lambda^{-1}[\mu/u(\mu, \varphi)] \\ &\cdot \{\Xi[\mu/u(\mu, \varphi)] + c\,X[\mu/u(\mu, \varphi)]\}\end{aligned} \tag{62a}$$

and

$$\begin{aligned}\Psi(a, \mu, \varphi) = \Psi_1(\mu, \varphi)\,\mathrm{e}^{-au(\mu,\varphi)/\mu} &+ c\,[4\pi u(\mu, \varphi)]^{-1} \Lambda^{-1}[\mu/u(\mu, \varphi)] \\ &\cdot \{T[\mu/u(\mu, \varphi)] + c\,Y[\mu/u(\mu, \varphi)]\}\end{aligned} \tag{62b}$$

where

$$X[\mu/u(\mu, \varphi)] = \sum_{\alpha=0}^{N} \{a_\alpha A_\alpha[-\mu/u(\mu, \varphi)] - \mathrm{e}^{-au(\mu,\varphi)/\mu} b_\alpha A_\alpha[\mu/u(\mu, \varphi)]\} \tag{63a}$$

and

$$Y[\mu/u(\mu, \varphi)] = \sum_{\alpha=0}^{N} \{b_\alpha A_\alpha[-\mu/u(\mu, \varphi)] - \mathrm{e}^{-au(\mu,\varphi)/\mu} a_\alpha A_\alpha[\mu/u(\mu, \varphi)]\}. \tag{63b}$$

We note that in expressing the functions $X(z)$ and $Y(z)$ in terms of $A_\alpha(\pm z)$ we have considered the definition

$$A_\alpha(z) = \int_0^\gamma \tau\,\varphi(\tau)\,P_\alpha(2\tau - 1)\frac{\mathrm{d}\tau}{\tau + z} \tag{64}$$

to be valid for all $z \notin [-\gamma, 0)$. The recursion formula

$$(2\alpha + 1)(2z + 1)A_\alpha(z) + (\alpha + 1)A_{\alpha+1}(z) + \alpha A_{\alpha-1}(z) = 2(2\alpha + 1)\Delta_\alpha \tag{65}$$

is also valid for all $z \notin [-\gamma, 0)$, and thus we can, in principle, compute $A_\alpha(z)$ for all $z \notin [-\gamma, 0)$ from Eq. (65) once the starting value

$$A_0(z) = \frac{1}{k}\mathrm{Tan}^{-1} k + z \int_{-\gamma}^{0} \varphi(\tau)\frac{\mathrm{d}\tau}{\tau - z} \tag{66}$$

has been evaluated. Finally to obtain the desired results for the exiting angular fluxes we must invert the Fourier transform given by Eq. (5). Thus, for $\mu > 0$ and all φ, we find

$$I[0, \boldsymbol{\varrho}, \boldsymbol{\Omega}(-\mu, \varphi)] = (2\pi)^{-2} \iint \Psi(0, -\mu, \varphi) \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{\varrho}} \mathrm{d}\boldsymbol{k} \tag{67a}$$

and

$$I[a, \boldsymbol{\varrho}, \boldsymbol{\Omega}(\mu, \varphi)] = (2\pi)^{-2} \iint \Psi(a, \mu, \varphi) \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{\varrho}} \mathrm{d}\boldsymbol{k}. \tag{67b}$$

V. The searchlight problem

As an application of the developed analysis, we now consider the classical searchlight problem for a half space ($a \to \infty$). We thus write

$$I_1(\boldsymbol{\varrho}, \boldsymbol{\Omega}) = \frac{1}{2\pi\varrho} \delta(\varrho) \delta(\mu - \mu_0) \delta(\varphi - \varphi_0) \tag{68a}$$

and

$$I_2(\boldsymbol{\varrho}, \boldsymbol{\Omega}) = 0 \tag{68b}$$

where we use the polar coordinates $\varrho = |\boldsymbol{\varrho}|$ and α to locate a field point in the $x - y$ plane. Equations (6) yield

$$\Psi_1(\mu, \varphi) = \delta(\mu - \mu_0) \delta(\varphi - \varphi_0) \tag{69a}$$

and

$$\Psi_2(\mu, \varphi) = 0. \tag{69b}$$

It therefore follows from Eq. (62a) that, for $\mu > 0$ and all φ,

$$\Psi(0, -\mu, \varphi) = c[4\pi u(\mu, \varphi)]^{-1} \Lambda^{-1}[\mu/u(\mu, \varphi)] \cdot \left\{ \frac{\mu_0 u(\mu, \varphi)}{\mu_0 u(\mu, \varphi) + \mu u(\mu_0, \varphi_0)} + c \sum_{\alpha=0}^{N} a_\alpha A_\alpha[-\mu/u(\mu, \varphi)] \right\}, \tag{70}$$

where the constants $\{a_\alpha\}$ are to be obtained from the linear algebraic equations

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi_\beta) = \frac{\mu_0}{\mu_0 + \xi_\beta u(\mu_0, \varphi_0)}, \quad \beta = 0, 1, 2, \ldots, N. \tag{71}$$

To be specific we use here a collocation strategy based on one that has proved successful [8] in previous F_N calculations, *viz.* we take $\xi_0 = s_0$ and

$$\xi_\beta = \zeta_\beta (1 + k^2 \zeta_\beta^2)^{-1/2}, \quad \beta = 1, 2, \ldots, N, \tag{72}$$

where

$$\zeta_\beta = \frac{1}{2} + \frac{1}{2} \cos[(2\beta - 1)\pi/(2N)] \tag{73}$$

are the zeroes of the Chebyshev polynomial of the first kind $T_N(2x - 1)$.

As a first numerical test of the developed formalism we consider the case of normal incidence $\mu_0 = 1$ and note that Eq. (71) reduces to

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\xi_\beta) = \frac{1}{1+\xi_\beta}. \tag{74}$$

Thus for this case the constants $\{a_\alpha\}$ are real, and Eq. (70) reduces to

$$\Psi(0, -\mu, \varphi) = c\,[4\pi u(\mu, \varphi)]^{-1} \Lambda^{-1}[\mu/u(\mu, \varphi)] \cdot \left\{\frac{u(\mu, \varphi)}{u(\mu, \varphi)+\mu} + c\sum_{\alpha=0}^{N} a_\alpha A_\alpha[-\mu/u(\mu, \varphi)]\right\}. \tag{75}$$

Equation (75) is valid for $\mu > 0$ and all φ and therefore provides the Fourier transform of the exiting angular flux.

We now express the dispersion function $\Lambda(s)$ as

$$\Lambda(s) = 1 - c\,\Gamma(s), \tag{76}$$

where

$$\Gamma(s) = -s^2(1-k^2 s^2)^{-1} \int_0^1 \frac{d\mu}{\mu^2 - s^2(1-k^2 s^2)^{-1}} \tag{77}$$

is independent of c, so that we can write Eq. (75), for $\mu > 0$ and all φ, as

$$\Psi(0, -\mu, \varphi) = \frac{c}{4\pi}\left[\frac{1}{\mu + u(\mu, \varphi)}\right] + \Psi_2(0, -\mu, \varphi) \tag{78}$$

where

$$\Psi_2(0, -\mu, \varphi) = \frac{c^2}{4\pi\Lambda[\mu/u(\mu, \varphi)]} \cdot \left\{\frac{\Gamma[\mu/u(\mu, \varphi)]}{\mu + u(\mu, \varphi)} + \frac{1}{u(\mu, \varphi)}\sum_{\alpha=0}^{N} a_\alpha A_\alpha[-\mu/u(\mu, \varphi)]\right\}. \tag{79}$$

We have observed that

$$\Psi_1(0, -\mu, \varphi) = \frac{c}{4\pi}\left[\frac{1}{\mu + u(\mu, \varphi)}\right] \tag{80}$$

is the two-dimensional Fourier transform of

$$I_1[0, \varrho, \boldsymbol{\Omega}(-\mu, \varphi)] = \frac{c}{4\pi\varrho(1-\mu^2)^{1/2}}\,\delta(\varphi - \alpha)\,e^{-(1+\mu)\varrho/(1-\mu^2)^{1/2}} \tag{81}$$

so that the distribution of particles exiting the $z = 0$ plane can be written as

$$I[0, \varrho, \boldsymbol{\Omega}(-\mu, \varphi)] = I_1[0, \varrho, \boldsymbol{\Omega}(-\mu, \varphi)] + \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} \Psi_2(0, -\mu, \varphi)\,e^{-ik\varrho\cos(\alpha-\psi)}\,k\,dk\,d\psi, \tag{82}$$

$\mu > 0$ and $\varphi \in [0, 2\pi]$. It is apparent that $I_1[0, \varrho, \boldsymbol{\Omega}(-\mu, \varphi)]$ is the distribution of particles that escape after having a single collision in the medium. Now in order to complete the desired solution, we clearly must evaluate the second term (that describes the exiting particles that have had more than one collision in the medium) in Eq. (82). This can, in principle, be done once we have solved Eq. (74) to find, for selected values of k, the constants $\{a_\alpha\}$ required in order to compute $\Psi_2(0, -\mu, \varphi)$ from Eq. (79).

Continuing with the case of normal incidence, we focus our attention on the result for $\mu = 1$ and deduce from Eqs. (79), (81), and (82) that

$$I[0, \varrho, \boldsymbol{\Omega}(-1, \varphi)] = \frac{c}{8\pi\varrho}\delta(\varrho)\delta(\varphi - \alpha) + \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} \Psi_2(0, -1, \varphi)\,\mathrm{e}^{-ik\varrho\cos(\alpha-\psi)}\,k\,\mathrm{d}k\,\mathrm{d}\psi \tag{83}$$

where

$$\Psi_2(0, -1, \varphi) = \frac{c^2}{8\pi\Lambda(1)}\left[\Gamma(1) + 2\sum_{\alpha=0}^{N} a_\alpha A_\alpha(-1)\right]. \tag{84}$$

We can integrate Eq. (57) to find

$$\Lambda(1) = 1 - \frac{c}{2}(1 - k^2)^{-1/2}\ln\left[\frac{1 + (1 - k^2)^{1/2}}{1 - (1 - k^2)^{1/2}}\right], \quad k < 1, \tag{85a}$$

$$\Lambda(1) = 1 - c, \quad k = 1, \tag{85b}$$

and

$$\Lambda(1) = 1 - c(k^2 - 1)^{-1/2}\,\mathrm{Tan}^{-1}(k^2 - 1)^{1/2}, \quad k > 1, \tag{85c}$$

and we can integrate Eq. (66) to obtain

$$A_0(-1) = \frac{1}{k}\mathrm{Tan}^{-1}k + (1 - k^2)^{-1/2}\ln\left[\frac{(1 + k^2)^{1/2} - (1 - k^2)^{1/2}}{(1 + k^2)^{1/2} + (1 - k^2)^{1/2}}\right], \quad k < 1, \tag{86a}$$

$$A_0(-1) = \frac{\pi}{4} - \sqrt{2}, \quad k = 1, \tag{86b}$$

and

$$A_0(-1) = \frac{1}{k}\mathrm{Tan}^{-1}k - 2(k^2 - 1)^{-1/2}\,\mathrm{Tan}^{-1}\left[\frac{k^2 - 1}{k^2 + 1}\right]^{1/2}, \quad k > 1, \tag{86c}$$

so that $A_\alpha(-1)$ can be readily deduced from Eq. (65). We note from Eq. (84) that $\Psi_2(0, -1, \varphi)$ is independent of ψ, and thus we can carry out the integration over ψ in Eq. (83) to obtain

$$I[0, \varrho, \boldsymbol{\Omega}(-1, \varphi)] = \frac{c}{8\pi\varrho}\delta(\varrho)\delta(\varphi - \alpha) + \frac{1}{2\pi}\int_0^{\infty}\Psi_2(0, -1, \varphi)J_0(k\varrho)\,k\,\mathrm{d}k \tag{87}$$

where $J_0(x)$ is used to denote the zero-th-order Bessel function of the first kind [9]. Continuing, we substitute Eq. (84) into Eq. (87) to find

$$I[0, \varrho, \boldsymbol{\Omega}(-1, \varphi)] = \frac{c}{8\pi\varrho}\delta(\varrho)\delta(\varphi - \alpha) + \frac{c^2}{32\pi\varrho}[1 - D(\varrho)] \tag{88}$$

where

$$D(\varrho) = \int_0^\infty \left[1 - \frac{2}{\pi} M(x/\varrho)\right] J_0(x)\,\mathrm{d}x \tag{89}$$

with

$$M(k) = \frac{k}{\Lambda(1)}\left[\Gamma(1) + 2\sum_{\alpha=0}^{N} a_\alpha A_\alpha(-1)\right]. \tag{90}$$

We have, with modest computational effort, solved Eq. (74) for various values of c and k and evaluated Eq. (90) to obtain results that converged to several significant figures with $N < 20$. We have also used the Monte Carlo method to provide evidence that the F_N results do in fact converge toward the correct values. In principle, we thus can use the method of Longman [10] to evaluate the integral in Eq. (89) in order to establish $D(\varrho)$. More complete numerical studies are therefore the subjects of our continuing work on this class of problems.

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Abstract

Fourier and other integral transform techniques are used to reduce the problem of radiation transport in plane-parallel media with non-uniform surface illumination to a convenient computational form, and the F_N method is used to provide a basis for approximate solutions.

Zusammenfassung

Fourier- und andere Integral-Transformationen werden benützt, um das Problem des Strahlungstransportes in planparallelen Medien mit ungleichförmiger Flächenbeleuchtung der numerischen Rechnung zugänglich zu machen. Die F_N-Methode wird als Grundlage einer praktischen Näherung benützt.

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