J. Quant. Spectrosc. Radiat. Transfer Vol. 30, No. 6, pp. 523–526, 1983
Printed in Great Britain.

0022–4073/83 $3.00 + .00
Pergamon Press Ltd.

SOLUTIONS TO AN INVERSE PROBLEM IN RADIATIVE TRANSFER WITH POLARIZATION—I

C. E. SIEWERT†
Department of Physics and Astronomy, Free University, Amsterdam, The Netherlands

(Received 12 November 1982)

Abstract—Elementary methods are used to derive from the equation of transfer systems of algebraic equations for a set of basic parameters that defines the phase matrix basic to the scattering of polarized light. The developed inverse solutions require that the four Stokes parameters be measured only on the two surfaces of the considered plane-parallel medium.

1. INTRODUCTION

In previous work concerning the solution of the inverse problem in terms of measurements made only on the surfaces of a finite slab, the effects of polarization were either ignored[1-8] or at best represented by a combination of Rayleigh and isotropic scattering.[9-11] Here for a significantly more general equation of transfer relevant to the scattering of polarized light we seek to deduce the phase matrix from a set of surface measurements.

In a recent paper[12] the equation of transfer, as formulated by Kuščer and Ribarič[13] to describe the diffusion of polarized light in a scattering and absorbing host medium, was converted to a Stokes representation, and equations based entirely on real quantities and utilizing an analytical form for the phase matrix were reported for each of the components in a Fourier decomposition of the density vector. If we let $\mathbf{I}(\tau, \mu, \phi)$ denote the density vector with the four Stokes parameters $I(\tau, \mu, \phi)$, $Q(\tau, \mu, \phi)$, $U(\tau, \mu, \phi)$ and $V(\tau, \mu, \phi)$ as components, then we can consider the equation of transfer[14]

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu, \phi) + \mathbf{I}(\tau, \mu, \phi) = \frac{\omega}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \mathbf{P}(\mu, \mu', \phi - \phi') \mathbf{I}(\tau, \mu', \phi') \, d\phi' \, d\mu' \tag{1}$$

and the boundary conditions

$$\mathbf{I}(0, \mu, \phi) = \mathbf{F}_1(\mu, \phi) \tag{2a}$$

and

$$\mathbf{I}(\tau_0, -\mu, \phi) = \mathbf{F}_2(\mu, \phi), \tag{2b}$$

for $\mu > 0$ and $\phi \in [0, 2\pi]$, where $\mathbf{F}_1(\mu, \phi)$ and $\mathbf{F}_2(\mu, \phi)$ are considered given. For the phase matrix we use the analytical representation[15]

$$\mathbf{P}(\mu, \mu', \phi - \phi') = \frac{1}{2} \mathbf{C}^0(\mu, \mu') + \sum_{m=1}^{L} [\mathbf{C}^m(\mu, \mu') \cos m(\phi - \phi') + \mathbf{S}^m(\mu, \mu') \sin m(\phi - \phi')] \tag{3}$$

where

$$\mathbf{C}^m(\mu, \mu') = \mathbf{A}^m(\mu, \mu') + \mathbf{D}\mathbf{A}^m(\mu, \mu')\mathbf{D}, \tag{4a}$$

$$\mathbf{S}^m(\mu, \mu') = \mathbf{A}^m(\mu, \mu')\mathbf{D} - \mathbf{D}\mathbf{A}^m(\mu, \mu'), \tag{4b}$$

$$\mathbf{A}^m(\mu, \mu') = \sum_{l=m}^{L} \frac{(l-m)!}{(l+m)!} \mathbf{\Pi}_l^m(\mu) \mathbf{B}_l \mathbf{\Pi}_l^m(\mu'), \tag{5}$$

$$\mathbf{D} = \mathrm{diag}\{1, 1, -1, -1\} \tag{6}$$

†Permanent address: Mathematics Department, North Carolina State University, Raleigh, NC 27650, U.S.A.

and

$$\mathbf{B}_l = \begin{vmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\epsilon_l \\ 0 & 0 & \epsilon_l & \delta_l \end{vmatrix} \tag{7}$$

We note[15] that

$$\mathbf{\Pi}_l^m(\mu) = \begin{vmatrix} P_l^m(\mu) & 0 & 0 & 0 \\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0 \\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0 \\ 0 & 0 & 0 & P_l^m(\mu) \end{vmatrix} \tag{8}$$

where

$$P_l^m(\mu) = (1-\mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_l(\mu) \tag{9}$$

is used to denote the associated Legendre function, and the functions $R_l^m(\mu)$ and $T_l^m(\mu)$ are the combinations of generalized spherical functions used in Refs. 12 and 15.

For the inverse problem considered here we seek to express ω and the $6L-3$ Greek constants δ_0, β_1, δ_1 and $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \epsilon_l, \zeta_l\}$, $l = 2, 3, \ldots, L$, in terms of $\mathbf{I}(0, \mu, \phi)$ and $\mathbf{I}(\tau_0, \mu, \phi)$. We note that $\beta_0 = 1$ and that $\alpha_0 = \alpha_1 = \gamma_0 = \gamma_1 = \epsilon_0 = \epsilon_1 = \zeta_0 = \zeta_1 = 0$.

2. INVERSE EQUATIONS

If we substitute Eq. (3) into Eq. (1) and integrate the resulting equation over ϕ from 0 to 2π we find that

$$\mathbf{I}(\tau, \mu) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{I}(\tau, \mu, \phi)\,\mathrm{d}\phi \tag{10}$$

satisfies the equation of transfer

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \frac{\omega}{2} \sum_{l=0}^{L} \mathbf{\Pi}_l(\mu) \mathbf{B}_l \int_{-1}^{1} \mathbf{\Pi}_l(\mu') \mathbf{I}(\tau, \mu')\,\mathrm{d}\mu' \tag{11}$$

where

$$\mathbf{\Pi}_l(\mu) = \mathrm{diag}\{P_l(\mu), R_l(\mu), R_l(\mu), P_l(\mu)\}. \tag{12}$$

In addition $R_0(\mu) = R_1(\mu) = 0$ and, for $l \geqslant 2$,

$$R_l(\mu) = \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} (1-\mu^2) \frac{\mathrm{d}^2}{\mathrm{d}\mu^2} P_l(\mu). \tag{13}$$

Here we consider that $\mathbf{I}(\tau, \mu)$ can be measured experimentally for all $\mu \in [-1, 1]$ at the two surfaces $\tau = 0$ and $\tau = \tau_0$, and we seek to compute ω and the $\mathbf{B}_l$ from the measurements.

Although $\mathbf{I}(\tau, \mu)$ is a four vector, it is clear from Eq. (11) and boundary conditions of the form

$$\mathbf{I}(0, \mu) = \mathbf{I}_1(\mu), \quad \mu > 0, \tag{14a}$$

and

$$\mathbf{I}(\tau_0, -\mu) = \mathbf{I}_2(\mu), \quad \mu > 0, \tag{14b}$$

where $\mathbf{I}_1(\mu)$ and $\mathbf{I}_2(\mu)$ are considered known, that the coupling is not complete. Thus we study, as have Siewert and Pinheiro,[16] the two-vector problem

$$\mu \frac{\partial}{\partial \tau} \mathbf{\Psi}(\tau, \mu) + \mathbf{\Psi}(\tau, \mu) = \frac{\omega}{2} \sum_{l=0}^{L} \mathbf{P}_l(\mu)\mathbf{C}_l \int_{-1}^{1} \mathbf{P}_l(\mu')\mathbf{\Psi}(\tau, \mu')\, \mathrm{d}\mu' \tag{15}$$

where

$$\mathbf{P}_l(\mu) = \mathrm{diag}\,\{P_l(\mu), R_l(\mu)\}. \tag{16}$$

Clearly if the two components of $\mathbf{\Psi}(\tau, \mu)$ are considered to be I and Q then

$$\mathbf{C}_l = \begin{vmatrix} \beta_l & \gamma_l \\ \gamma_l & \alpha_l \end{vmatrix}; \tag{17}$$

whereas if the two components of $\mathbf{\Psi}(\tau, \mu)$ are considered to be V and U then

$$\mathbf{C}_l = \begin{vmatrix} \delta_l & \epsilon_l \\ -\epsilon_l & \zeta_l \end{vmatrix}. \tag{18}$$

We note that Larsen[17] has used the adjoint equation to define a solution to a multigroup inverse problem defined in terms of an equation of transfer similar to our Eq. (15); however, as will be seen, we do not require an adjoint equation here.

Changing μ to $-\mu$ in Eq. (15) and letting

$$\mathbf{F}(\tau, \mu) = \mu \frac{\partial}{\partial \tau} \mathbf{\Psi}(\tau, \mu), \tag{19}$$

we rewrite Eq. (15) as

$$\mathbf{F}(\tau, -\mu) + \mathbf{\Psi}(\tau, -\mu) = \frac{\omega}{2} \sum_{l=0}^{L} (-1)^l \mathbf{P}_l(\mu)\mathbf{C}_l \mathbf{\Psi}_l(\tau) \tag{20}$$

where

$$\mathbf{\Psi}_l(\tau) = \int_{-1}^{1} \mathbf{P}_l(\mu)\mathbf{\Psi}(\tau, \mu)\, \mathrm{d}\mu. \tag{21}$$

We can now multiply Eq. (20) by $\mathbf{\Psi}^T(\tau, \mu)\mathbf{E}$, where $\mathbf{E} = \mathbf{I}$ (the unit matrix) if the components of $\mathbf{\Psi}(\tau, \mu)$ are I and Q, or $\mathbf{E} = \mathrm{diag}\{1, -1\}$ if $\mathbf{\Psi}(\tau, \mu)$ has components V and U, to find

$$T_0(\tau) + \int_{-1}^{1} \mathbf{\Psi}^T(\tau, \mu)\mathbf{E}\mathbf{\Psi}(\tau, -\mu)\, \mathrm{d}\mu = \frac{\omega}{2} \sum_{l=0}^{L} (-1)^l \mathbf{\Psi}_l^T(\tau)\mathbf{W}_l \mathbf{\Psi}_l(\tau) \tag{22}$$

where $\mathbf{W}_l = \mathbf{E}\mathbf{C}_l$ is symmetric and

$$T_0(\tau) = \int_{-1}^{1} \mathbf{\Psi}^T(\tau, \mu)\mathbf{E}\mathbf{F}(\tau, -\mu)\, \mathrm{d}\mu. \tag{23}$$

Next, we differentiate Eq. (23) and use Eqs. (19)–(21) to find

$$\frac{\mathrm{d}}{\mathrm{d}\tau} T_0(\tau) + \int_{-1}^{1} \mathbf{\Psi}^T(\tau, \mu)\mathbf{E} \frac{\partial}{\partial \tau} \mathbf{\Psi}(\tau, -\mu)\, \mathrm{d}\mu = \frac{\omega}{2} \sum_{l=0}^{L} (-1)^l\, \mathbf{\Psi}_l^T(\tau)\mathbf{W}_l \frac{\mathrm{d}}{\mathrm{d}\tau} \mathbf{\Psi}_l(\tau). \tag{24}$$

Upon differentiating Eq. (22), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} T_0(\tau) + 2 \int_{-1}^{1} \mathbf{\Psi}^T(\tau, \mu)\mathbf{E} \frac{\partial}{\partial \tau} \mathbf{\Psi}(\tau, -\mu)\, \mathrm{d}\mu = \omega \sum_{l=0}^{L} (-1)^l\, \mathbf{\Psi}_l^T(\tau)\mathbf{W}_l \frac{\mathrm{d}}{\mathrm{d}\tau} \mathbf{\Psi}_l(\tau). \tag{25}$$

Equations (24) and (25) clearly yield the fact that $T_0(\tau)$ is a constant, and thus we can integrate Eq. (24) to find

$$\omega \sum_{l=0}^{L} (-1)^l [\boldsymbol{\Psi}_l^T(\tau_0)\mathbf{W}_l \boldsymbol{\Psi}_l(\tau_0) - \boldsymbol{\Psi}_l^T(0)\mathbf{W}_l \boldsymbol{\Psi}_l(0)]$$

$$= 4 \int_0^1 [\boldsymbol{\Psi}^T(\tau_0, \mu)\mathbf{E}\boldsymbol{\Psi}(\tau_0, -\mu) - \boldsymbol{\Psi}^T(0, \mu)\mathbf{E}\boldsymbol{\Psi}(0, -\mu)]\mathrm{d}\mu. \quad (26)$$

In a similar manner and following a previous study,[3] we can multiply Eq. (20) by $\mu^2\boldsymbol{\Psi}^T(\tau, \mu)\mathbf{E}$, integrate over μ, differentiate with respect to τ and find

$$\omega \sum_{l=0}^{L} (-1)^l (2l+1)[\mathbf{D}_l^T(\tau_0)\mathbf{W}_l \mathbf{h}_l^{-1}\mathbf{D}_l(\tau_0) - \mathbf{D}_l^T(0)\mathbf{W}_l \mathbf{h}_l^{-1}\mathbf{D}_l(0)] =$$

$$4 \int_0^1 \mu^2[\boldsymbol{\Psi}^T(\tau_0, \mu)\mathbf{E}\boldsymbol{\Psi}(\tau_0, -\mu) - \boldsymbol{\Psi}^T(0, \mu)\mathbf{E}\boldsymbol{\Psi}(0, -\mu)]\,\mathrm{d}\mu, \quad (27)$$

where

$$\mathbf{D}_l(\tau) = \int_{-1}^1 \mu \mathbf{P}_l(\mu)\boldsymbol{\Psi}(\tau, \mu)\,\mathrm{d}\mu \quad (28)$$

and

$$\mathbf{h}_l = (2l+1)\mathbf{I} - \omega\mathbf{C}_l. \quad (29)$$

It is apparent that Eq. (26) represents an equation that is linear in all of the $3L-1$ unknown components of $\omega\mathbf{W}_l$, $l = 0, 1, 2, \ldots, L$. Thus if we have available the surface measurements from $3L-1$ independent experiments then all of the desired unknowns can be found, at least in principle, by solving $3L-1$ versions of Eq. (26). Furthermore, since

$$\omega\mathbf{W}_l = (2l+1)[\mathbf{E} + (\omega\mathbf{W}_l\mathbf{h}_l^{-1})^{-1}]^{-1} \quad (30)$$

we can also find the desired $3L-1$ unknowns by solving the linear algebraic system defined by $3L-1$ independent versions of Eq. (27). It is also possible that the required number of experiments may be reduced by solving Eqs. (26) and (27) simultaneously. This, however, could require an iterative procedure which may or may not prove convergent.

In the accompanying Part II of this work McCormick and Sanchez[18] make use of higher-order Fourier projections to develop additional solutions to the considered inverse problem.

Acknowledgements—The author wishes to express his gratitude to J. W. Hovenier and the Free University of Amsterdam for their kind hospitality during recent visits to The Netherlands. The author would also like to express his appreciation for helpful discussions concerning this investigation held with N. J. McCormick and R. Sanchez. This work was supported in part by the U.S. National Science Foundation.

REFERENCES

1. C. E. Siewert, *J. Math. Phys.* **19**, 1587 (1978).
2. C. E. Siewert, *J. Math. Phys.* **19**, 2619 (1978).
3. C. E. Siewert, *Zs. angew. Math. Phys.* **30**, 522 (1979).
4. C. E. Siewert, *JQSRT* **22**, 441 (1979).
5. N. J. McCormick, *J. Math. Phys.* **20**, 1504 (1979).
6. N. J. McCormick and R. Sanchez, *J. Math. Phys.* **22**, 199 (1981).
7. R. Sanchez and N. J. McCormick, *J. Math. Phys.* **22**, 847 (1981).
8. R. Sanchez and N. J. McCormick, *JQSRT* **28**, 169 (1982).
9. C. E. Siewert, *Astrophys. Space Sci.* **60**, 237 (1979).
10. C. E. Siewert and J. R. Maiorino, *Zs. angew. Math. Phys.* **31**, 767 (1980).
11. N. J. McCormick, *Astrophys. Space Sci.* **71**, 235 (1980).
12. C. E. Siewert, *Astrophys. J.* **245**, 1080 (1981).
13. I. Kuščer and M. Ribarič, *Optica Acta* **6**, 42 (1959).
14. S. Chandrasekhar, *Radiative Transfer*. Oxford University Press, London (1950).
15. C. E. Siewert, *Astronomy and Astrophys.* **109**, 195 (1982).
16. C. E. Siewert and F. J. V. Pinheiro, *Zs. angew. Math. Phys.* **33**, 807 (1982).
17. E. W. Larsen, *J. Math. Phys.* **22**, 158 (1981).
18. N. J. McCormick and R. Sanchez, *JQSRT*, **30**, 527 (1983).