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THE P_N METHOD FOR RADIATIVE TRANSFER PROBLEMS WITH REFLECTIVE BOUNDARY CONDITIONS

M. Benassi, R. M. Cotta, and C. E. Siewert

Departments of Mathematics and Nuclear Engineering, North Carolina State University, Raleigh, NC 27650 U.S.A.

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Abstract—The P_N method is used to compute the partial heat fluxes relevant to radiative transfer in an anisotropically scattering, plane-parallel medium with specularly and diffusely reflecting boundaries.

1. INTRODUCTION

In a paper published in 1980, Siewert, Maiorino, and Özişik[1] used the F_N method[2] to compute accurately the partial radiative heat fluxes for a general class of radiative heat-transfer problems. Here we make use of exact particular solutions and the spherical harmonics method[3] to compute the partial heat fluxes for the same class of problems.

We consider the equation of transfer[4]

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\omega}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau, \mu')\, d\mu' + (1-\omega)\frac{\sigma}{\pi} T^4(\tau) \tag{1}$$

and boundary conditions

$$I(0, \mu) = \epsilon_1 \frac{\sigma}{\pi} T_1^{\,4} + \rho_1^{\,s} I(0, -\mu) + 2\rho_1^{\,d} \int_0^1 I(0, -\mu')\mu'\, d\mu', \quad \mu > 0, \tag{2a}$$

and

$$I(\tau_0, -\mu) = \epsilon_2 \frac{\sigma}{\pi} T_2^{\,4} + \rho_2^{\,s} I(\tau_0, \mu) + 2\rho_2^{\,d} \int_0^1 I(\tau_0, \mu')\mu'\, d\mu', \quad \mu > 0, \tag{2b}$$

where T_1 and T_2 refer to the two boundary temperatures, $\rho_\alpha^{\,s}$ and $\rho_\alpha^{\,d}$, $\alpha = 1$ and 2, are the coefficients for specular and diffuse reflection and ϵ_1 and ϵ_2 are the emissivities. Though we intend ultimately to solve more general problems,[5] we consider here that the temperature distribution in the medium $T(\tau)$ is known[6].

2. ANALYSIS

We express our P_N approximation to $I(\tau, \mu)$ in the form used by Garcia and Siewert[3], viz.,

$$I(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} [A_j\, e^{-\tau/\xi_j} + (-1)^l B_j\, e^{-(\tau_0-\tau)/\xi_j}] g_l(\xi_j) + I_p(\tau, \mu), \tag{3}$$

where $I_p(\tau, \mu)$ denotes a particular solution of Eq. (1) corresponding to the inhomogeneous source term

$$S(\tau) = (1-\omega)\frac{\sigma}{\pi} T^4(\tau). \tag{4}$$

The polynomials $g_l(\xi)$ are those of Chandrasekhar[4], i.e.,

$$(l+1)g_{l+1}(\xi) = h_l\xi g_l(\xi) - l g_{l-1}(\xi) \tag{5}$$

with $g_0(\xi) = 1$ and $h_l = 2l + 1 - \omega\beta_l$. We consider that N is odd, so that the eigenvalues ξ_j, $j = 1, 2, \ldots, J = (N+1)/2$, are the J positive zeros of $g_{N+1}(\xi)$. Assuming that we can find a particular solution $I_p(\tau, \mu)$, we must simply determine the arbitrary constants $\{A_j\}$ and $\{B_j\}$ so that the approximate solution given by Eq. (3) satisfies, in some approximate way, the boundary conditions given by Eqs. (2). Substituting Eq. (3) into Eqs. (2), we find, for $\mu > 0$,

$$\sum_{l=0}^{N} \frac{2l+1}{2} \{[1-(-1)^l\rho_1^s]P_l(\mu) - 2(-1)^l\rho_1^d S_{0,l}\} \sum_{j=1}^{J} [A_j + (-1)^l B_j\, e^{-\tau_0/\xi_j}]g_l(\xi_j) = K_1(\mu) \tag{6a}$$

and

$$\sum_{l=0}^{N} \frac{2l+1}{2} \{[1-(-1)^l\rho_2^s]P_l(\mu) - 2(-1)^l\rho_2^d S_{0,l}\} \sum_{j=1}^{J} [B_j + (-1)^l A_j\, e^{-\tau_0/\xi_j}]g_l(\xi_j) = K_2(\mu), \tag{6b}$$

where

$$K_1(\mu) = \epsilon_1 \frac{\sigma}{\pi} T_1^4 - I_p(0, \mu) + \rho_1^s I_p(0, -\mu) + 2\rho_1^d \int_0^1 I_p(0, -\mu')\mu'\, d\mu', \tag{7a}$$

$$K_2(\mu) = \epsilon_2 \frac{\sigma}{\pi} T_2^4 - I_p(\tau_0, -\mu) + \rho_2^s I_p(\tau_0, \mu) + 2\rho_2^d \int_0^1 I_p(\tau_0, \mu')\mu'\, d\mu' \tag{7b}$$

and, in general,

$$S_{\alpha,l} = \int_0^1 P_{2\alpha+1}(\mu)P_l(\mu)\, d\mu. \tag{8}$$

To convert Eqs. (6) to a system of linear algebraic equations for the desired constants $\{A_j\}$ and $\{B_j\}$, we use the Marshak projection scheme[7]; thus we multiply Eqs. (6) by $P_{2\alpha+1}(\mu)$ and integrate over μ from zero to one to find, for $\alpha = 0, 1, \ldots, (N-1)/2$,

$$\sum_{l=0}^{N} \frac{2l+1}{2} \sum_{j=1}^{J} \{[1-(-1)^l\rho_1^s]S_{\alpha,l} - 2(-1)^l\rho_1^d S_{0,l}S_{\alpha,0}\}[A_j + (-1)^l B_j\, e^{-\tau_0/\xi_j}]g_l(\xi_j) = R_{1,\alpha} \tag{9a}$$

and

$$\sum_{l=0}^{N} \frac{2l+1}{2} \sum_{j=1}^{J} \{[1-(-1)^l\rho_2^s]S_{\alpha,l} - 2(-1)^l\rho_2^d S_{0,l}S_{\alpha,0}\}[B_j + (-1)^l A_j\, e^{-\tau_0/\xi_j}]g_l(\xi_j) = R_{2,\alpha}. \tag{9b}$$

Here

$$R_{1,\alpha} = \left[\epsilon_1 \frac{\sigma}{\pi} T_1^4 + 2\rho_1^d \int_0^1 I_p(0, -\mu)\mu\, d\mu\right] S_{\alpha,0} - \int_0^1 P_{2\alpha+1}(\mu)[I_p(0, \mu) - \rho_1^s I_p(0, -\mu)]\, d\mu \tag{10a}$$

and

$$R_{2,\alpha} = \left[\epsilon_2 \frac{\sigma}{\pi} T_2^4 + 2\rho_2^d \int_0^1 I_p(\tau_0, \mu)\mu\, d\mu\right] S_{\alpha,0} - \int_0^1 P_{2\alpha+1}(\mu)[I_p(\tau_0, -\mu) - \rho_2^s I_p(\tau_0, \mu)]\, d\mu. \tag{10b}$$

It is apparent that once we have found the constants $\{A_j\}$ and $\{B_j\}$, we can readily compute moments of the intensity from Eq. (3). For example, the net radiative heat flux

$$q(\tau) = \int_{-1}^{1} I(\tau, \mu)\mu\, d\mu \tag{11}$$

and the partial fluxes

$$q^{\pm}(\tau) = \int_0^1 I(\tau, \pm\mu)\mu \, d\mu \tag{12}$$

are given by

$$q(\tau) = (1-\omega)\sum_{j=1}^{J} [A_j e^{-\tau/\xi_j} - B_j e^{-(\tau_0-\tau)/\xi_j}]\xi_j + \int_{-1}^{1} I_p(\tau, \mu)\mu \, d\mu \tag{13}$$

and

$$q^{\pm}(\tau) = \sum_{l=0}^{N} \frac{2l+1}{2}(\pm 1)^l S_{0,l} \sum_{j=1}^{J} [A_j e^{-\tau/\xi_j} + (-1)^l B_j e^{-(\tau_0-\tau)/\xi_j}] g_l(\xi_j) + \int_0^1 I_p(\tau, \pm\mu)\mu \, d\mu. \tag{14}$$

On the other hand, if we wish to compute the intensity we can obtain an improvement to Eq. (3) by substituting Eq. (3) into the r.h.s. of Eq. (1) and solving the resulting equation to obtain, after using Eqs. (2),

$$I(\tau, \mu) = I_p(\tau, \mu) + I_H(0, \mu) e^{-\tau/\mu} + \frac{\omega}{2} \Upsilon(\tau, \mu), \quad \mu > 0, \tag{15a}$$

and

$$I(\tau, -\mu) = I_p(\tau, -\mu) + I_H(\tau_0, -\mu) e^{-(\tau_0-\tau)/\mu} + \frac{\omega}{2} \Upsilon(\tau, -\mu), \quad \mu > 0, \tag{15b}$$

where

$$\Upsilon(\tau, \mu) = \sum_{l=0}^{N} \beta_l P_l(\mu) \sum_{j=1}^{J} \xi_j [A_j C(\tau: \mu, \xi_j) + (-1)^l B_j e^{-(\tau_0-\tau)/\xi_j} S(\tau: \mu, \xi_j)] g_l(\xi_j) \tag{16a}$$

and

$$\Upsilon(\tau, -\mu) = \sum_{l=0}^{N} \beta_l P_l(\mu) \sum_{j=1}^{J} \xi_j [(-1)^l A_j e^{-\tau/\xi_j} S(\tau_0 - \tau: \mu, \xi_j) + B_j C(\tau_0 - \tau: \mu, \xi_j)] g_l(\xi_j) \tag{16b}$$

with

$$C(\tau: \mu, \xi) = (e^{-\tau/\mu} - e^{-\tau/\xi})/(\mu - \xi) \tag{17a}$$

and

$$S(\tau: \mu, \xi) = (1 - e^{-\tau/\mu} e^{-\tau/\xi})/(\mu + \xi). \tag{17b}$$

To complete the solution given by Eqs. (15), we note that

$$I_H(0, \mu) = (1 - \rho_1{}^s \rho_2{}^s e^{-2\tau_0/\mu})^{-1} \{K_1(\mu) + \rho_1{}^s K_2(\mu) e^{-\tau_0/\mu} + \frac{\omega}{2} \rho_1{}^s [\Upsilon(0, -\mu) + \rho_2{}^s \Upsilon(\tau_0, \mu) e^{-\tau_0/\mu}] + 2[\rho_1{}^d J_1 + \rho_1{}^s \rho_2{}^d J_2 e^{-\tau_0/\mu}]\} \tag{18a}$$

and

$$I_H(\tau_0, -\mu) = (1 - \rho_1{}^s \rho_2{}^s e^{-2\tau_0/\mu})^{-1} \{K_2(\mu) + \rho_2{}^s K_1(\mu) e^{-\tau_0/\mu} + \frac{\omega}{2} \rho_2{}^s [\Upsilon(\tau_0, \mu) + \rho_1{}^s \Upsilon(0, -\mu) e^{-\tau_0/\mu}] + 2[\rho_2{}^d J_2 + \rho_2{}^s \rho_1{}^d J_1 e^{-\tau_0/\mu}]\}, \tag{18b}$$

where

$$J_1 = \frac{\omega}{2}\int_0^1 \Upsilon(0, -\mu)\mu\, \mathrm{d}\mu + \int_0^1 I_H(\tau_0, -\mu)\, \mathrm{e}^{-\tau_0/\mu}\mu\, \mathrm{d}\mu \tag{19a}$$

and

$$J_2 = \frac{\omega}{2}\int_0^1 \Upsilon(\tau_0, \mu)\mu\, \mathrm{d}\mu + \int_0^1 I_H(0, \mu)\, \mathrm{e}^{-\tau_0/\mu}\mu\, \mathrm{d}\mu. \tag{19b}$$

It is apparent that the constants

$$H_1 = \int_0^1 I_H(\tau_0, -\mu)\, \mathrm{e}^{-\tau_0/\mu}\mu\, \mathrm{d}\mu \tag{20a}$$

and

$$H_2 = \int_0^1 I_H(0, \mu)\, \mathrm{e}^{-\tau_0/\mu}\mu\, \mathrm{d}\mu \tag{20b}$$

can be found after we multiply Eqs. (18) by $\mu\exp(-\tau_0/\mu)$, integrate over μ and solve the two resulting linear algebraic equations.

3. NUMERICAL RESULTS

In order to demonstrate the accuracy of our developed P_N solution, we consider the specific problem solved by Siewert, Maiorino and Özişik,[1] viz., the case of constant internal temperature $T(\tau) = T$. The required particular solution

$$I_p(\tau, \mu) = (\sigma/\pi)T^4 \tag{21}$$

is especially simple. We can now express Eqs. (7) as

$$K_1(\mu) = \frac{\sigma}{\pi}[\epsilon_1 T_1^4 - (1 - \rho_1^s - \rho_1^d)T^4] \tag{22a}$$

and

$$K_2(\mu) = \frac{\sigma}{\pi}[\epsilon_2 T_2^4 - (1 - \rho_2^s - \rho_2^d)T^4]. \tag{22b}$$

For this problem, we find that the r.h.s. of Eqs. (9) become

$$R_{1,\alpha} = \frac{\sigma}{\pi}[\epsilon_1 T_1^4 - (1 - \rho_1^s - \rho_1^d)T^4]S_{\alpha,0} \tag{23a}$$

and

$$R_{2,\alpha} = \frac{\sigma}{\pi}[\epsilon_2 T_2^4 - (1 - \rho_2^s - \rho_2^d)T^4]S_{\alpha,0}. \tag{23b}$$

To start our numerical work, we first compute the eigenvalues $\{\xi_j\}$. To this end, we multiply Eq. (5) by ξ and rewrite the resulting equation, for $l = 0, 2, 4, \ldots,$ as

$$\frac{l(l-1)}{h_l h_{l-1}} g_{l-2}(\xi) + \frac{1}{h_l}\left[\frac{(l+1)^2}{h_{l+1}} + \frac{l^2}{h_{l-1}}\right] g_l(\xi) + \frac{(l+2)(l+1)}{h_{l+1}h_l} g_{l+2}(\xi) = \xi^2 g_l(\xi). \tag{24}$$

It follows that the squares of the J positive zeros of $g_{N+1}(\xi)$ are the J eigenvalues ξ^2 of the $J \times J$

tridiagonal matrix **A** obtained from Eq. (24), for $l = 0, 2, 4, \ldots, (N-1)$, and the truncation condition $g_{N+1}(\xi) = 0$. We have used a FORTRAN program in the EISPACK program package[8] to find the required eigenvalues of **A**. Having found the eigenvalues, we compute the polynomials $g_l(\xi)$ by either forward ($\xi \leq 1$) or backward ($\xi > 1$) recursion.

We note that Dave[9] has reported recursion relations that establish a very convenient and accurate way to evaluate the constants $S_{\alpha,l}$ defined by Eq. (8); we express our version of the formulas required to compute all non-zero values of these constants as

$$S_{\alpha,2\alpha+1} = 1/(4\alpha + 3), \tag{25a}$$

$$S_{\alpha,l+2} = \left(\frac{1-l+2\alpha}{4+l+2\alpha}\right)\left(\frac{l+1}{l+1-2\alpha}\right)\left(\frac{l+2+2\alpha}{l+2}\right)S_{\alpha,l} \tag{25b}$$

and

$$S_{\alpha+1,0} = -\frac{1}{2}\left(\frac{2\alpha+1}{\alpha+2}\right)S_{\alpha,0} \tag{25c}$$

with $S_{0,0} = 1/2$ and $\alpha, l = 0, 1, 2, \ldots$.

We are now ready to prescribe the data and solve Eqs. (9) to find the required constants $\{A_j\}$ and $\{B_j\}$. We use here a scattering law, given in Table 1, that is the result of a Mie scattering calculation[5,10] for size parameter = 3 and index of refraction = 1.2. In Tables 2 and 3, we list partial heat fluxes found from our P_N solution with $N \leq 99$ and correct, we believe, to within ± 1 in the last digit given. In Tables 4 and 5, we compare our results for the partial heat fluxes at the boundaries to the exact results of Ref. 1. It is clear from Tables 4 and 5 that for the considered problems our P_N solution yields for $\omega = 0.8$ and $\omega = 0.95$ boundary results that are accurate to four significant figures for modest values of N. On the other hand, the convergence

Table 1. The scattering law.

l	β_l	β_{l+4}	β_{l+8}
0	1	1.24514	0.00667
1	2.35789	0.51215	0.00081
2	2.76628	0.16096	
3	2.20142	0.03778	

Table 2. Partial heat fluxes for $\rho_1^s = \rho_1^d = \rho_2^s = \rho_2^d = 0.25$, $\sigma T^4 = \pi$, $T_1 = T_2 = 0$ and $\tau_0 = 1$.

	$\omega = 0.2$		$\omega = 0.8$		$\omega = 0.95$	
τ	$q^-(\tau)$	$q^+(\tau)$	$q^-(\tau)$	$q^+(\tau)$	$q^-(\tau)$	$q^+(\tau)$
0	0.4153	0.2076	0.2333	0.1167	0.8480(−1)	0.4240(−1)
0.1	0.4060	0.2485	0.2266	0.1343	0.8220(−1)	0.4837(−1)
0.2	0.3950	0.2811	0.2186	0.1499	0.7910(−1)	0.5378(−1)
0.3	0.3823	0.3082	0.2096	0.1641	0.7566(−1)	0.5880(−1)
0.4	0.3677	0.3311	0.1997	0.1770	0.7190(−1)	0.6347(−1)
0.5	0.3508	0.3508	0.1888	0.1888	0.6783(−1)	0.6783(−1)

of the method for the case of strong absorption, $\omega = 0.2$, is somewhat less rapid. We have observed similar trends for the other cases discussed in Ref. 1, i.e., the case of purely specular reflection with $\rho_1^s = \rho_2^s = 0.5$ and the case of purely diffuse reflection with $\rho_1^d = \rho_2^d = 0.5$. It is also apparent from the results quoted in Tables 4 and 5 that the partial heat flux $q^-(0)$ for the case $\omega = 0.2$, no internal heat generation and $T_2 = 0$ is the least accurate in this set of calculations.

In conclusion we note that, in contrast to the findings of Garcia and Siewert[3] in regard to the problem of a finite slab illuminated by a solar beam, we found for the considered problems that the Marshak projection scheme[7] yielded better results than the projection scheme based on the functions $\mu P_\alpha(2\mu - 1)$. We also found that Eqs. (15) yield (for the considered problems and with $N \leq 99$) results for the intensity that are in general accurate, we believe, to four significant figures.

Table 3. Partial heat fluxes for $\rho_1^s = \rho_1^d = \rho_2^s = \rho_2^d = 0.25$, $\epsilon_1\sigma T_1^4 = \pi$, $T = T_2 = 0$ and $\tau_0 = 1$.

τ	$\omega = 0.2$ $q^-(\tau)$	$q^+(\tau)$	$\omega = 0.8$ $q^-(\tau)$	$q^+(\tau)$	$\omega = 0.95$ $q^-(\tau)$	$q^+(\tau)$
0	0.2655(−1)	0.5133	0.1666	0.5833	0.3006	0.6503
0.1	0.2782(−1)	0.4402	0.1645	0.5520	0.2950	0.6351
0.2	0.2990(−1)	0.3823	0.1639	0.5245	0.2905	0.6213
0.3	0.3259(−1)	0.3342	0.1644	0.4995	0.2866	0.6084
0.4	0.3584(−1)	0.2936	0.1655	0.4764	0.2831	0.5962
0.5	0.3969(−1)	0.2588	0.1673	0.4551	0.2799	0.5844
0.6	0.4419(−1)	0.2288	0.1696	0.4351	0.2769	0.5731
0.7	0.4943(−1)	0.2028	0.1724	0.4164	0.2740	0.5621
0.8	0.5556(−1)	0.1801	0.1757	0.3989	0.2711	0.5513
0.9	0.6277(−1)	0.1603	0.1793	0.3824	0.2682	0.5406
1	0.7145(−1)	0.1429	0.1834	0.3668	0.2649	0.5298

Table 4. The partial heat flux $q^-(0) = q^+(\tau_0)$ for $\rho_1^s = \rho_1^d = \rho_2^s = \rho_2^d = 0.25$, $\sigma T^4 = \pi$, $T_1 = T_2 = 0$ and $\tau_0 = 1$.

ω	N = 9	N = 19	N = 39	N = 69	N = 99	Exact
0.2	0.4160	0.4155	0.4153	0.4153	0.4153	0.4153
0.8	0.2335	0.2334	0.2333	0.2333	0.2333	0.2333
0.95	0.8483(−1)	0.8481(−1)	0.8480(−1)	0.8480(−1)	0.8480(−1)	0.8480(−1)

Table 5. Partial heat fluxes $q^-(0)$ and $q^+(\tau_0)$ for $\rho_1^s = \rho_1^d = \rho_2^s = \rho_2^d = 0.25$, $\epsilon_1\sigma T_1^4 = \pi$, $T = T_2 = 0$ and $\tau_0 = 1$.

ω	Partial Heat Fluxes	N = 19	N = 39	N = 69	N = 99	Exact
0.2	$q^-(0)$	0.2621(−1)	0.2647(−1)	0.2653(−1)	0.2655(−1)	0.2656(−1)
	$q^+(\tau_0)$	0.1429	0.1429	0.1429	0.1429	0.1429
0.8	$q^-(0)$	0.1665	0.1666	0.1666	0.1666	0.1666
	$q^+(\tau_0)$	0.3668	0.3668	0.3668	0.3668	0.3668
0.95	$q^-(0)$	0.3006	0.3006	0.3006	0.3006	0.3006
	$q^+(\tau_0)$	0.5298	0.5298	0.5298	0.5298	0.5298

In addition, we have observed that the P_N formulation used in this work does not deteriorate as the slab thickness τ_0 increases, and we have found, in general, that the computational methods used remain valid when $1-\omega$ is very small, e.g., 10^{-12}. Of course, some elementary modifications are required for the special case $\omega = 1$.

For recent reviews of various semi-analytical and computational methods basic to radiative transfer we refer to papers by Sanchez and McCormick[11] and Mengüç and Viskanta,[12] and finally we remark that the developed P_N solution is particularly easy to use and that it is adequately accurate for many applications.

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