8. The Critical Problem for a Two-Region Spherical Reactor, *O. J. Smith, C. E. Siewert (NC State U)*

We seek the solution to the integral transport equation for the neutron density in a spherical medium consisting of an inner core and a surrounding outer blanket; the two regions are assumed to have the same total cross section, but different multiplication properties are allowed. Within the assumptions of spherical symmetry, the one-speed transport equation takes the form

$$r\rho(r) = \frac{c_1}{2} \int_0^{R_1} t\rho(t)[E_1(|r - t|) - E_1(r + t)]\, dt + \frac{c_2}{2} \int_{R_1}^{R_2} t\rho(t)\, [E_1(|r - t|) - E_1(r + t)]\, dt \quad , \quad 0 \leq r \leq R_2 \quad , \tag{1}$$

where c_i denotes the mean number of secondary neutrons per collision in the i'th region; R_1 and R_2 are the radii of the inner and outer regions, respectively.

As did Mitsis[1] for single-region spheres, we extend Eq. (1) to include negative r, $-R_2 \leq r \leq R_2$, by defining $\rho(-r) \triangleq \rho(r)$; it then follows that the neutron density can be found from the inversion of an appropriately defined transform. Thus,

$$r\rho_1(r) = \tfrac{1}{2} \int_{-1}^{1} \Psi_1(r,\mu)\, d\mu, \quad 0 \leq r \leq R_1 \quad , \tag{2a}$$

and

$$r\rho_2(r) = \tfrac{1}{2} \int_{-1}^{1} \Psi_2(r,\mu)\, d\mu \quad , \quad R_1 \leq r \leq R_2 \quad , \tag{2b}$$

where the transform function $\Psi_i(r,\mu)$ satisfies the equation

$$\mu \frac{\partial}{\partial r} \Psi_i(r,\mu) + \Psi_i(r,\mu) = \frac{c_i}{2} \int_{-1}^{1} \Psi_i(r,\mu)\, d\mu \quad . \tag{3}$$

The boundary conditions, subject to which Eq. (3) must be solved, follow from the definitions of the transforms used to reduce Eq. (1) to Eqs. (2). We find

$$\Psi_1(0,\mu) = -\Psi_1(0,-\mu) \ , \ \mu \in (-1,1) \quad , \tag{4a}$$

$$\Psi_1(R_1,\mu) = \Psi_2(R_1,\mu) \ , \quad \mu \in (-1,1) \quad , \tag{4b}$$

and

$$\Psi_2(R_2,-\mu) = 0 \ , \ \mu \in (0,1) \quad . \tag{4c}$$

Were it not for the antisymmetry condition, Eq. (4a), the solution for $\Psi_1(r,\mu)$ would be that found by Kuszell[2] for a finite two-region slab. However, a similar procedure may be followed.

In terms of the normal modes introduced by Case,[3] we write the solution for the inner region as

$$\Psi_1(r,\mu) = A_+[\phi_+^{(1)}(\mu) \exp(-r/\nu_{01}) - \phi_-^{(1)}(\mu) \exp(r/\nu_{01})] + \int_0^1 A(\nu)\, [\phi_\nu^{(1)}(\mu) \exp(-r/\nu) - \phi_{-\nu}^{(1)}(\mu) \exp(r/\nu)] d\nu \ , \quad 0 \leq r \leq R_1 \quad . \tag{5}$$

Since the notation used here follows that of Case and Zweifel,[4] it is not defined again. Similarly, we write

$$\Psi_2(r,\mu) = B_+ \phi_+^{(2)}(\mu) \exp(-r/\nu_{02}) + B_- \phi_-^{(2)}(\mu) \exp(r/\nu_{02}) + \int_{-1}^{1} B(\nu) \phi_\nu^{(2)}(\mu) \exp(-r/\nu)\, d\nu \quad , \quad R_1 \leq r \leq R_2 \quad . \tag{6}$$

Clearly, one parameter in this problem is arbitrary; we take $B_+ = -1$. Application of the free-surface boundary condition to Eq. (6) yields a valid half-range expansion; B_- and $B(-\nu)$ can thus be found in terms of $B(\nu)$, $\nu > 0$, by taking half-range scalar products.[5]

Considering the final boundary condition, Eq. (4b), we are led to the full-range expansion

$$G(\mu) = A_+[\phi_+^{(1)}(\mu) \exp(-R_1/\nu_{01}) - \phi_-^{(1)}(\mu) \exp(R_1/\nu_{01})] + \int_0^1 A(\nu)\, [\phi_\nu^{(1)}(\mu) \exp(-R_1/\nu) - \phi_{-\nu}^{(1)}(\mu) \exp(R_1/\nu)] d\nu \ , \quad \mu \in (-1,1) \quad , \tag{7}$$

where

$$G(\mu) \triangleq -\phi_+^{(2)}(\mu) \exp(-R_1/\nu_{02}) + B_- \phi_-^{(2)}(\mu) \exp(R_1/\nu_{02}) + \int_{-1}^{1} B(\nu)\, \phi_\nu^{(2)}(\mu) \exp(-R_1/\nu)\, d\nu \quad . \tag{8}$$

For Eq. (7) to be a valid full-range expansion, the following restrictions on the expansion function $G(\mu)$ must be imposed:

$$\int_{-1}^{1} \mu\, G(\mu)\, \phi_+^{(1)}(\mu)\, d\mu = \exp(-2R_1/\nu_{01}) \int_{-1}^{1} \mu\, G(\mu)\, \phi_-^{(1)}(\mu)\, d\mu \tag{9a}$$

and

$$\int_{-1}^{1} \mu\, G(\mu)\, \phi_\nu^{(1)}(\mu)\, d\mu = \exp(-2R_1/\nu) \int_{-1}^{1} \mu\, G(\mu)\, \phi_{-\nu}^{(1)}(\mu)\, d\mu \ , \quad \nu > 0 \quad . \tag{9b}$$

Once the above two conditions on $G(\mu)$ have been satisfied, the expansion coefficients A_+ and $A(\nu)$ are found by taking full-range scalar products of Eq. (7).

The technique given by Muskhelishvili[6] has been used to reduce Eq. (9b) to a Fredholm integral equation for $B(\nu)$, $\nu > 0$. This would require a computer solution if a high degree of accuracy were desired. Equation (9a) is the critical condition from which the allowable values of c_1, c_2, R_1, and R_2 are to be determined.

The reduction of Eq. (9b) to a Fredholm equation for $B(\nu)$, $\nu > 0$, though straightforward, is complex; for this reason, it is not given here.

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