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A NUMERICAL EVALUATION OF AN ANALYTICAL REPRESENTATION OF THE COMPONENTS IN A FOURIER DECOMPOSITION OF THE PHASE MATRIX FOR THE SCATTERING OF POLARIZED LIGHT

P. VESTRUCCI and C. E. SIEWERT
Nuclear Engineering and Mathematics Departments, North Carolina State University, Raleigh, NC 27650, U.S.A.

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Abstract—An analytical representation of the components in a Fourier decomposition of the phase matrix for a theory of multiple light scattering that includes the effects of polarization is evaluated, and selected numerical results are given for several particular scattering models.

1. INTRODUCTION

In a recent paper[1] to which we hereafter refer as I, the phase matrix

$$\mathbf{P}(\mu, \mu', \phi - \phi') = \mathbf{L}(\pi - i_2)\mathbf{F}(\cos \Theta)\mathbf{L}(-i_1) \tag{1}$$

that is used in the equation of transfer[2]

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu, \phi) + \mathbf{I}(\tau, \mu, \phi) = \frac{\omega}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \mathbf{P}(\mu, \mu', \phi - \phi') \mathbf{I}(\tau, \mu', \phi') \, d\phi' \, d\mu' \tag{2}$$

was expressed as

$$\mathbf{P}(\mu, \mu', \phi - \phi') = \frac{1}{2} \mathbf{C}^0(\mu, \mu') + \sum_{m=1}^{\infty} \{\mathbf{C}^m(\mu, \mu') \cos [m(\phi - \phi')] + \mathbf{S}^m(\mu, \mu') \sin [m(\phi - \phi')]\}, \tag{3}$$

and explicit analytical expressions were given for the Fourier components $\mathbf{C}^m(\mu, \mu')$ and $\mathbf{S}^m(\mu, \mu')$. We note that the four-vector $\mathbf{I}(\tau, \mu, \phi)$ appearing in Eq. (2) has the four Stokes parameters $I(\tau, \mu, \phi)$, $Q(\tau, \mu, \phi)$, $U(\tau, \mu, \phi)$ and $V(\tau, \mu, \phi)$ as components, and ω is the albedo for single scattering. Here, as in I, the scattering matrix

$$\mathbf{F}(\xi) = \begin{vmatrix} a_1(\xi) & b_1(\xi) & 0 & 0 \\ b_1(\xi) & a_2(\xi) & 0 & 0 \\ 0 & 0 & a_3(\xi) & b_2(\xi) \\ 0 & 0 & -b_2(\xi) & a_4(\xi) \end{vmatrix}, \tag{4}$$

where $\xi = \cos \Theta$, is of the form considered by Hovenier[3], and the six real-valued functions appearing in Eq. (4) are expanded in the forms

$$a_1(\xi) = \sum_{l=0}^{L} \beta_l P_l(\xi), \quad \beta_0 = 1, \tag{5a}$$

$$a_2(\xi) = \sum_{l=2}^{L} \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} \{\alpha_l R_l^2(\xi) + \zeta_l T_l^2(\xi)\}, \tag{5b}$$

$$a_3(\xi) = \sum_{l=2}^{L} \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} \{\zeta_l R_l^2(\xi) + \alpha_l T_l^2(\xi)\}, \tag{5c}$$

$$a_4(\xi) = \sum_{l=0}^{L} \delta_l P_l(\xi), \tag{5d}$$

$$b_1(\xi) = \sum_{l=2}^{L} \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} \gamma_l P_l^2(\xi) \tag{5e}$$

and

$$b_2(\xi) = -\sum_{l=2}^{L} \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} \epsilon_l P_l^2(\xi) \tag{5f}$$

where $P_l(\xi)$ is used to denote the lth order Legendre polynomial. Also

$$P_l^m(\xi) = (1-\xi^2)^{m/2} \frac{d^m}{d\xi^m} P_l(\xi) \tag{6}$$

and the combinations of generalized spherical functions $R_l^m(\xi)$ and $T_l^m(\xi)$ are those defined previously.[1,4]

If we now consider that the constants $\{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}$ are given, we can use the expressions given in I and write

$$\mathbf{C}^m(\mu, \mu') = \mathbf{A}^m(\mu, \mu') + \mathbf{D}\,\mathbf{A}^m(\mu, \mu')\mathbf{D} \tag{7a}$$

and

$$\mathbf{S}^m(\mu, \mu') = \mathbf{A}^m(\mu, \mu')\mathbf{D} - \mathbf{D}\,\mathbf{A}^m(\mu, \mu') \tag{7b}$$

where

$$\mathbf{A}^m(\mu, \mu') = \sum_{l=m}^{L} \frac{(l-m)!}{(l+m)!} \mathbf{\Pi}_l^m(\mu)\,\mathbf{B}_l \mathbf{\Pi}_l^m(\mu'), \tag{8}$$

$$\mathbf{D} = \mathrm{diag}\{1, 1, -1, -1\}, \tag{9}$$

$$\mathbf{\Pi}_l^m(\mu) = \begin{vmatrix} P_l^m(\mu) & 0 & 0 & 0 \\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0 \\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0 \\ 0 & 0 & 0 & P_l^m(\mu) \end{vmatrix}. \tag{10}$$

and

$$\mathbf{B}_l = \begin{vmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\epsilon_l \\ 0 & 0 & \epsilon_l & \delta_l \end{vmatrix}. \tag{11}$$

Table 1. The basic constants for problem I.

l	α	β	γ	δ	ϵ	ζ
0	0	1	0	2α	0	0
1	0	$3\left(\alpha + \frac{3}{5}\beta\right)$	0	$\frac{3}{2}$	0	0
2	3	$\frac{1}{2}$	$-\frac{1}{2}(6)^{1/2}$	$\alpha + 3\beta$	0	$6\left(\alpha + \frac{1}{3}\beta\right)$
3	4β	$\frac{6}{5}\beta$	$-\frac{2}{5}(30)^{1/2}\beta$	0	0	0

2. NUMERICAL RESULTS

Our goal now is to compute the Fourier components $\mathbf{C}^m(\mu, \mu')$ and $\mathbf{S}^m(\mu, \mu')$ by using the recursion formulas given in I for the matrices $\mathbf{\Pi}_l^m(\mu)$. To be sure that the recursion formulas can be used without significant loss of accuracy for $\mu \in [-1, 1]$, we used the technique of perturbing the starting value to conclude, using double precision arithmetic and an I.B.M. computer, that we can compute $\mathbf{\Pi}_l^m(\mu)$ accurate to 13 significant figures for all $\mu \in [-1, 1]$ and all appropriate m for $l = 0, 1, 2, \ldots, L = 200$.

As a first test case, we consider the model deduced by Kuščer and Ribarič,[5] from the Mie theory for the scattering of light, with wave number k, by small, absorbing spherical particles of radius a. The basic constants $\{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}$ for this model are given in Table 1 in terms of

$$\alpha = (ka)^2(n^2+2)^2[30(n^2+2)+36(ka)^2(n^2-2)]^{-1} \tag{12a}$$

and

$$\beta = 5(ka)^2(n^2+2)^2(2n^2+3)^{-1}[30(n^2+2)+36(ka)^2(n^2-2)]^{-1}, \tag{12b}$$

where n is the index of refraction of the particle with respect to the surrounding medium. In Table 2 we list some selected values of $a_1(\xi) = a_2(\xi)$, $a_3(\xi) = a_4(\xi)$, $b_1(\xi)$ and $b_2(\xi)$ for the case $n = 1.33$ and $ka = 1/2$. We now condense our notation and report in Table 3 selected results for our computed values of

$$\mathbf{W}^m(\mu, \mu') = \frac{1}{2}[\mathbf{C}^m(\mu, \mu') + \mathbf{S}^m(\mu, \mu')]. \tag{13}$$

Table 2. The elements of $\mathbf{F}(\xi)$ for problem I.

ξ	$a_1(\xi)$	$a_3(\xi)$	$b_1(\xi)$	$b_2(\xi)$
−1	1.331	−1.331	0.0	0.0
−0.8	1.116	−1.088	−2.489(−1)	0.0
−0.6	9.466(−1)	−8.318(−1)	−4.518(−1)	0.0
−0.4	8.269(−1)	−5.634(−1)	−6.053(−1)	0.0
−0.2	7.602(−1)	−2.824(−1)	−7.059(−1)	0.0
0	7.500(−1)	1.129(−2)	−7.500(−1)	0.0
0.2	7.998(−1)	3.176(−1)	−7.341(−1)	0.0
0.4	9.131(−1)	6.366(−1)	−6.547(−1)	0.0
0.6	1.093	9.682(−1)	−5.082(−1)	0.0
0.8	1.344	1.312	−2.911(−1)	0.0
1	1.669	1.669	0.0	0.0

Table 3. The matrix $\mathbf{W}^m(0.2, 0.5)$ for problem I.

$m = 0$	1.0451005	2.5520807(−1)	0.0	0.0
	9.6166456(−2)	8.2321383(−1)	0.0	0.0
	0.0	0.0	6.5047479(−2)	0.0
	0.0	0.0	0.0	2.1978950(−1)
$m = 1$	1.2211679(−1)	6.5196876(−2)	1.2105016(−1)	0.0
	6.1926618(−2)	1.1760072(−1)	1.4141234(−1)	0.0
	−3.1897668(−1)	−3.2712156(−1)	6.4669760(−1)	0.0
	0.0	0.0	0.0	6.4981213(−1)
$m = 2$	1.3896415(−1)	−2.2455954(−1)	−1.7647631(−1)	0.0
	−1.4503873(−1)	2.4990750(−1)	2.0360533(−1)	0.0
	4.3586742(−2)	−1.1081661(−1)	1.0583070(−1)	0.0
	0.0	0.0	0.0	2.8459693(−2)
$m = 3$	5.6061552(−3)	−9.3435921(−3)	−7.4748737(−3)	0.0
	−6.0733348(−3)	1.0122225(−2)	8.0977798(−3)	0.0
	2.3358980(−3)	−3.8931634(−3)	3.1145307(−3)	0.0
	0.0	0.0	0.0	0.0

For our second test calculation we use a problem suggested by Kawabata[6], and thus we consider the Mie scattering of light, with a wavelength $\lambda = 0.951\ \mu$m, by a gamma distribution[7] of spherical particles with an effective radius r_{eff} = = 0.2 μm, effective variance $v_{eff} = 0.07$ and index of refraction $n = 1.44$. In Table 4, we list the basic constants for this problem that were calculated and estimated to be correct to within ± 1 in the last digits given by van der Stap and de Rooij.[8] For the sake of having a precisely defined test problem we consider the basic constants given in Table 4 to be exact. The functions $a_1(\xi)$, $a_3(\xi)$, $b_1(\xi)$, and $b_2(\xi)$ are given in Table 5, and selected results for $\mathbf{W}^m(\mu, \mu')$ are shown in Table 6.

Table 4. The basic constants for problem II.

l	α	β	γ	δ	ϵ	ζ
0	0.0	1.0	0.0	0.7120634246	0.0	0.0
1	0.0	1.4552931819	0.0	1.7601411931	0.0	0.0
2	3.3091220464	1.0540263128	−0.7552491518	1.0668243107	0.0420726875	2.5773207443
3	0.9633758276	0.3975899378	−0.3619934319	0.3965110389	0.0850671555	0.7574437604
4	0.2474124256	0.1165930161	−0.1155748816	0.0957641237	0.0154318420	0.1638177665
5	0.0452636955	0.0238747702	−0.0249815879	0.0176508810	0.0031534874	0.0278314781
6	0.0068892608	0.0039501033	−0.0041675362	0.0026154886	0.0004010299	0.0038897248
7	0.0008798202	0.0005388807	−0.0005739043	0.0003271332	0.0000460147	0.0004642654
8	0.0000987255	0.0000637172	−0.0000677900	0.0000358314	0.0000042875	0.0000490224
9	0.0000099029	0.0000066697	−0.0000070926	0.0000035142	0.0000003617	0.0000046647
10	0.0000009071	0.0000006329	−0.0000006712	0.0000003148	0.0000000272	0.0000004075
11	0.0000000769	0.0000000553	−0.0000000585	0.0000000261	0.0000000019	0.0000000331
12	0.0000000061	0.0000000045	−0.0000000047	0.0000000020	0.0000000001	0.0000000025
13	0.0000000005	0.0000000003	−0.0000000004	0.0000000001	0.0000000000	0.0000000002

Table 5. The elements of $\mathbf{F}(\xi)$ for problem II.

ξ	$a_1(\xi)$	$a_3(\xi)$	$b_1(\xi)$	$b_2(\xi)$
−1	2.973(−1)	−2.973(−1)	0.0	0.0
−0.8	2.698(−1)	−2.532(−1)	−7.035(−2)	1.801(−2)
−0.6	2.687(−1)	−1.946(−1)	−1.479(−1)	2.257(−2)
−0.4	3.003(−1)	−1.097(−1)	−2.333(−1)	1.600(−2)
−0.2	3.760(−1)	1.833(−2)	−3.252(−1)	8.548(−4)
0	5.155(−1)	2.138(−1)	−4.181(−1)	−1.979(−2)
0.2	7.499(−1)	5.110(−1)	−5.005(−1)	−4.197(−2)
0.4	1.128	9.590(−1)	−5.506(−1)	−6.029(−2)
0.6	1.722	1.627	−5.309(−1)	−6.721(−2)
0.8	2.643	2.612	−3.800(−1)	−5.212(−2)
1	4.052	4.052	0.0	0.0

Table 6. The matrix $\mathbf{W}^m(0.2, 0.5)$ for problem II.

$m = 0$	1.2449077	1.9986371(−1)	0.0	0.0
	8.8519123(−2)	1.0105471	0.0	0.0
	0.0	0.0	7.8946210(−1)	1.1727873(−2)
	0.0	0.0	−2.0087779(−2)	9.8930129(−1)
$m = 4$	1.9317355(−2)	−1.8555909(−2)	−1.4095999(−2)	0.0
	−1.1855813(−2)	3.1235603(−2)	2.4754226(−2)	−4.9077507(−5)
	3.5914737(−5)	−2.0122333(−2)	2.5387166(−2)	−1.5758997(−3)
	0.0	−1.8706455(−3)	2.4693005(−3)	1.5657710(−2)
$m = 9$	3.3182069(−7)	−4.5428805(−7)	−3.5346401(−7)	0.0
	−2.9248043(−7)	5.9407165(−7)	4.8875163(−7)	−3.7593779(−9)
	6.5442673(−8)	−3.4430635(−7)	3.7977795(−7)	−1.4607213(−8)
	0.0	−1.7708187(−8)	2.2652773(−8)	1.7340491(−7)
$m = 13$	5.4961805(−12)	−1.0526915(−11)	−8.4215322(−12)	0.0
	−6.8424949(−12)	1.3798612(−11)	1.1719369(−11)	0.0
	2.6317288(−12)	−8.6572113(−12)	8.6950157(−12)	0.0
	0.0	0.0	0.0	1.8320602(−12)

Our third test problem was also suggested by Kawabata[6] and corresponds to a gamma distribution with $r_{\text{eff}} = 1.05\ \mu\text{m}$ and $v_{\text{eff}} = 0.07$ and is for $\lambda = 0.782\ \mu\text{m}$ and $n = 1.43$. In Table 7, we list the basic constants (assumed exact) computed and again estimated to be correct to within ± 1 in the last digits given by de Rooij;[9] in Table 8 we show $a_1(\xi)$, $a_3(\xi)$, $b_1(\xi)$, and $b_2(\xi)$, and in Table 9 we report some results for $\mathbf{W}^m(\mu, \mu')$.

It is clear that the matrix $\mathbf{W}^m(\mu, \mu')$ has 16 elements for given values of m, μ and μ', and

Table 7. The basic constants for problem III.

l	α	β	γ	δ	ϵ	ζ
0	0.0	1.0	0.0	0.88243540	0.0	0.0
1	0.0	2.03225848	0.0	2.11458547	0.0	0.0
2	3.90532118	2.85899972	0.10361497	2.71662098	−0.13771642	3.61704456
3	3.28964455	2.78417438	0.05954641	2.85256106	0.06047831	3.34430795
4	3.44546122	3.11726886	0.09601171	3.02881712	−0.16044948	3.29212056
5	3.23886530	3.08999502	0.01522133	3.10612898	0.17225131	3.23028280
6	3.46535723	3.21713816	0.02792890	3.21129828	−0.12477189	3.40342052
7	3.31711392	3.29940214	−0.00000846	3.28837307	0.25119953	3.25193914
8	3.48453721	3.19604192	0.02016006	3.22794414	−0.02931777	3.45150900
9	3.19297542	3.23712261	0.00840113	3.23778481	0.29866990	3.11907166
10	3.25637456	2.92646590	0.04402004	2.96573706	0.08307330	3.21622259
11	2.78026344	2.85698504	0.01159773	2.88267523	0.30733240	2.71679542
12	2.77753256	2.44191856	0.06918755	2.46757706	0.17571286	2.71353934
13	2.17496710	2.26281163	0.00907065	2.31055486	0.27252805	2.12850756
14	2.16701139	1.86533710	0.07833823	1.86669562	0.22884734	2.07782100
15	1.53303542	1.61325728	0.00570925	1.67423917	0.20684244	1.50489356
16	1.55710655	1.31567429	0.07020785	1.29162978	0.23730745	1.45055305
17	0.98001501	1.04111467	0.00421235	1.10436328	0.13435821	0.96704459
18	1.03779027	0.86404721	0.05285268	0.82173502	0.20937922	0.92718905
19	0.57317386	0.61283907	0.00397545	0.66847518	0.07501055	0.56959387
20	0.64561140	0.53201130	0.03470754	0.48265140	0.16148739	0.54479649
21	0.30981057	0.33212972	0.00367739	0.37437307	0.03614636	0.31016876
22	0.37693168	0.30876489	0.02042517	0.26268481	0.11044265	0.29553976
23	0.15640031	0.16747541	0.00289075	0.19552657	0.01505217	0.15743888
24	0.20751996	0.16962840	0.01100688	0.13307703	0.06764847	0.14886888
25	0.07448234	0.07943324	0.00190769	0.09593859	0.00538795	0.07505984
26	0.10820852	0.08853101	0.00553330	0.06316772	0.03745875	0.07018746
27	0.03374910	0.03579940	0.00108225	0.04452407	0.00162729	0.03384374
28	0.05366033	0.04404051	0.00263623	0.02834567	0.01894993	0.03129080
29	0.01464497	0.01545903	0.00054434	0.01966222	0.00039411	0.01452037
30	0.02540622	0.02094874	0.00120532	0.01214888	0.00886644	0.01334009
31	0.00611210	0.00643295	0.00025108	0.00830619	0.00006478	0.00595916
32	0.01152804	0.00955853	0.00053348	0.00502193	0.00388843	0.00549477
33	0.00245916	0.00258771	0.00011009	0.00337188	0.00000078	0.00234964
34	0.00503099	0.00419683	0.00022961	0.00201716	0.00161974	0.00220330
35	0.00095482	0.00100739	0.00004744	0.00132043	−0.00000376	0.00089331
36	0.00211878	0.00177860	0.00009612	0.00079087	0.00064866	0.00086367
37	0.00035792	0.00037958	0.00002052	0.00050047	−0.00000106	0.00032851
38	0.00086368	0.00072962	0.00003897	0.00030320	0.00025228	0.00033135
39	0.00012959	0.00013842	0.00000893	0.00018413	0.00000024	0.00011719
40	0.00034165	0.00029044	0.00001514	0.00011364	0.00009602	0.00012432
41	0.00004537	0.00004886	0.00000386	0.00006593	0.00000040	0.00004067
42	0.00013140	0.00011241	0.00000554	0.00004158	0.00003594	0.00004553
43	0.00001538	0.00001671	0.00000163	0.00002302	0.00000024	0.00001376
44	0.00004921	0.00004235	0.00000184	0.00001483	0.00001325	0.00001625
45	0.00000506	0.00000555	0.00000066	0.00000785	0.00000009	0.00000455
46	·0.00001796	0.00001555	0.00000051	0.00000515	0.00000481	0.00000564
47	0.00000162	0.00000179	0.00000026	0.00000262	0.00000003	0.00000147
48	0.00000640	0.00000557	0.00000010	0.00000174	0.00000171	0.00000191
49	0.00000050	0.00000056	0.00000010	0.00000086	0.00000000	0.00000047
50	0.00000223	0.00000195	0.00000000	0.00000057	0.00000060	0.00000063
51	0.00000015	0.00000017	0.00000004	0.00000028	0.00000000	0.00000015
52	0.00000077	0.00000067	−0.00000001	0.00000018	0.00000021	0.00000020
53	0.00000005	0.00000005	0.00000001	0.00000009	0.00000000	0.00000004
54	0.00000026	0.00000023	−0.00000001	0.00000006	0.00000007	0.00000006
55	0.00000001	0.00000002	0.00000000	0.00000003	0.00000000	0.00000001
56	0.00000009	0.00000008	0.00000000	0.00000002	0.00000002	0.00000002
57	0.00000000	0.00000000	0.00000000	0.00000001	0.00000000	0.00000000
58	0.00000003	0.00000003	0.00000000	0.00000001	0.00000001	0.00000001
59	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
60	0.00000001	0.00000001	0.00000000	0.00000000	0.00000000	0.00000000

Table 8. The elements of $\mathbf{F}(\xi)$ for problem III.

ξ	$a_1(\xi)$	$a_3(\xi)$	$b_1(\xi)$	$b_2(\xi)$
−1	5.127(−1)	−5.127(−1)	0.0	0.0
−0.95	7.326(−1)	2.540(−1)	5.550(−2)	5.796(−1)
−0.90	4.683(−1)	8.998(−2)	1.271(−2)	3.993(−1)
−0.85	3.059(−1)	3.174(−2)	3.781(−2)	2.524(−1)
−0.8	2.383(−1)	2.588(−2)	5.005(−2)	1.865(−1)
−0.6	1.404(−1)	3.182(−2)	3.780(−2)	8.122(−2)
−0.4	1.304(−1)	4.539(−2)	2.852(−2)	5.434(−2)
−0.2	1.542(−1)	7.585(−2)	2.706(−2)	4.799(−2)
0	2.060(−1)	1.316(−1)	3.268(−2)	4.861(−2)
0.2	3.010(−1)	2.307(−1)	4.754(−2)	5.284(−2)
0.4	4.837(−1)	4.161(−1)	7.340(−2)	5.972(−2)
0.6	8.803(−1)	8.099(−1)	1.054(−1)	7.038(−2)
0.8	1.909	1.825	1.043(−1)	9.567(−2)
1	4.745(+1)	4.745(+1)	0.0	0.0

Table 9. The matrix $\mathbf{W}^m(0.4, 0.8)$ for problem III.

$m = 0$	8.1508313(−1)	−1.9039394(−2)	0.0	0.0
	2.8263546(−2)	4.1939027(−1)	0.0	0.0
	0.0	0.0	3.9908947(−1)	3.5208836(−2)
	0.0	0.0	6.1401956(−3)	7.3804970(−1)
$m = 20$	3.6598425(−6)	1.0706722(−6)	1.0873596(−6)	0.0
	5.0854133(−7)	9.2048273(−6)	7.4980610(−6)	2.6951380(−6)
	5.4081692(−7)	1.9210294(−6)	−2.2944878(−7)	−2.5782571(−6)
	0.0	−5.3140774(−6)	5.2344540(−6)	3.5946744(−6)
$m = 40$	1.5606385(−11)	−1.0952955(−12)	−9.4649768(−13)	0.0
	−1.3801833(−13)	1.3391565(−11)	1.5610645(−11)	−7.6559361(−12)
	−1.3322365(−12)	2.3226081(−12)	−2.0658091(−12)	−3.6526685(−13)
	0.0	−1.0714166(−12)	9.6122479(−13)	−5.9211138(−12)
$m = 60$	1.9006330(−25)	0.0	0.0	0.0
	0.0	1.1191803(−24)	1.0918832(−24)	0.0
	0.0	−7.7184849(−25)	7.5302292(−25)	0.0
	0.0	0.0	0.0	0.0

therefore to limit the quantity of our tabulated results we have listed $\mathbf{W}^m(\mu, \mu')$ only for two values of μ and μ'. We have also limited our tabulations to eight significant figures even though testing with a perturbation procedure suggested that we had at least ten stable figures for all μ and μ'. We have made some comparisons with calculations provided by de Haan[10] who used numerical integration to find $\mathbf{W}^m(\mu, \mu')$. We found, in general, excellent agreement with de Haan's results as long as the elements of $\mathbf{W}^m(\mu, \mu')$ were in absolute value greater than 10^{-13}. Although it may not be a limitation for practical applications, it is clear that the numerical integration method cannot yield really small numbers by the addition and subtraction (even in double-precision arithmetic) of numbers on the order of unity.

We are confident that the analytical representation provides an accurate and economical way to compute the Fourier components of the phase matrix.

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