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On radiative-transfer problems with reflective boundary conditions and internal emission

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I. Introduction

In a recent work [1] an integral-transform technique and the F_N method [2–5] were used to solve a form of the radiation transport equation basic to the "slowing down" of neutrons or photons. Here we restrict our attention to a one-group (or gray) model, and we extend the analysis of Garcia and Siewert [1] to include the effects of internal emission and reflecting boundaries. We thus consider the equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu, \varphi) + I(\tau, \mu, \varphi) = \frac{\omega}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos \Theta) I(\tau, \mu', \varphi') \, d\varphi' \, d\mu' + S(\tau) \tag{1}$$

and the boundary conditions, for $\mu > 0$ and $\varphi \in [0, 2\pi]$,

$$I(L, \mu, \varphi) = K_1(\mu, \varphi) + \varrho_1^s I(L, -\mu, \varphi) + \frac{1}{\pi} \varrho_1^d \int_0^1 \int_0^{2\pi} \mu' I(L, -\mu', \varphi') \, d\varphi' \, d\mu' \tag{2a}$$

and

$$I(R, -\mu, \varphi) = K_2(\mu, \varphi) + \varrho_2^s I(R, \mu, \varphi) + \frac{1}{\pi} \varrho_2^d \int_0^1 \int_0^{2\pi} \mu' I(R, \mu', \varphi') \, d\varphi' \, d\mu', \tag{2b}$$

where, for $\alpha = 1$ and 2,

$$K_\alpha(\mu, \varphi) = B_\alpha(\mu, \varphi) + \Delta_\alpha \delta(\mu - \mu_0) \delta(\varphi - \varphi_0). \tag{3}$$

Here $\omega < 1$ is the single-scattering albedo, μ is the direction cosine (as measured from the positive τ axis) of the propagating radiation, φ is the azimuthal angle measured with respect to a reference angle φ_r and $\tau \in [L, R]$ is the optical variable. In addition, Θ is the scattering angle, $S(\tau)$ is used to represent internal emission and we consider phase functions that have a Legendre expansion of the form

$$p(\cos \Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos \Theta), \qquad \beta_0 = 1. \tag{4}$$

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In regard to the boundary conditions, we note that ϱ_α^s and ϱ_α^d, $\alpha = 1$ and 2, are coefficients for specular and diffuse reflection, that Δ_1 and Δ_2 are scaling constants and that $B_1(\mu, \varphi)$ and $B_2(\mu, \varphi)$ are assumed to be given. By considering the boundary conditions as given by Eqs. (2) we clearly allow the possibility of solar illumination of a single layer $\tau \in [L, R]$; however, by keeping the terms $B_1(\mu, \varphi)$, $B_2(\mu, \varphi)$ and $\Delta_2 \delta(\mu - \mu_0)\, \delta(\varphi - \varphi_0)$ in Eq. (3) we can use this development to solve multilayer problems by iteration.

Focusing our attention on the azimuthally symmetric component

$$I(\tau, \mu) = \frac{1}{2\pi} \int_0^{2\pi} I(\tau, \mu, \varphi)\, \mathrm{d}\varphi \tag{5}$$

of the intensity $I(\tau, \mu, \varphi)$, we can integrate Eqs. (1) and (2) over φ to find, after making use of Eq. (4) and the addition theorem for the Legendre polynomials, the equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\omega}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau, \mu')\, \mathrm{d}\mu' + S(\tau) \tag{6}$$

and the boundary conditions, for $\mu > 0$,

$$I(L, \mu) = K_1(\mu) + \varrho_1^s I(L, -\mu) + 2\varrho_1^d \int_0^1 I(L, -\mu')\, \mu'\, \mathrm{d}\mu' \tag{7a}$$

and

$$I(R, -\mu) = K_2(\mu) + \varrho_2^s I(R, \mu) + 2\varrho_2^d \int_0^1 I(R, \mu')\, \mu'\, \mathrm{d}\mu' \tag{7b}$$

where, for $\alpha = 1$ and 2,

$$K_\alpha(\mu) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(\mu, \varphi)\, \mathrm{d}\varphi. \tag{8}$$

II. Analysis

In reference [5] full-range orthogonality properties of appropriate elementary solutions were used with particular solutions of Eq. (6) to deduce a system of singular-integral equations and constraints for the intensity at the surfaces $\tau = L$ and $\tau = R$. Here we use the method reported by Garcia and Siewert [1] to develop equivalent results in a way that circumvents the explicit use of particular solutions of Eq. (6). This more direct development is based on an integral-transform technique and thus does not require familiarity with the elementary solutions of Eq. (6). If we change μ to $-\mu$ in Eq. (6), multiply the resulting equation by $\exp(-\tau/s)$ and integrate over τ from $\tau = a$ *to* $\tau = b$ we find

$$s\mu B(\mu, s) - (\mu - s) \int_a^b e^{-\tau/s} I(\tau, -\mu)\, \mathrm{d}\tau = \frac{\omega s}{2} \sum_{l=0}^{L} (-1)^l \beta_l P_l(\mu) I_l^*(s) + s S^*(s) \tag{9}$$

where

$$B(\mu, s) = I(a, -\mu)\,\mathrm{e}^{-a/s} - I(b, -\mu)\,\mathrm{e}^{-b/s}, \tag{10}$$

$$I_l^*(s) = \int_a^b \mathrm{e}^{-\tau/s} I_l(\tau)\,\mathrm{d}\tau, \tag{11}$$

with

$$I_l(\tau) = \int_1^1 P_l(\mu)\,I(\tau, \mu)\,\mathrm{d}\mu, \tag{12}$$

and

$$S^*(s) = \int_a^b \mathrm{e}^{-\tau/s} S(\tau)\,\mathrm{d}\tau. \tag{13}$$

In order to keep our development general we do not, at this point, specify a and b more precisely than $a, b \in [L, R]$ and $a < b$. We now multiply Eq. (9) by $(\mu - s)^{-1} P_\alpha(\mu)$, $s \notin [-1, 1]$, and integrate over all μ to find

$$(-1)^\alpha I_\alpha^*(s) + \frac{\omega s}{2} \sum_{l=0}^{L} (-1)^l \beta_l I_l^*(s) \int_{-1}^{1} P_l(\mu)\,P_\alpha(\mu)\,\frac{\mathrm{d}\mu}{\mu - s}$$
$$= s \int_{-1}^{1} \mu P_\alpha(\mu)\,B(\mu, s)\,\frac{\mathrm{d}\mu}{\mu - s} - s\,S^*(s) \int_{-1}^{1} P_\alpha(\mu)\,\frac{\mathrm{d}\mu}{\mu - s}. \tag{14}$$

We use $g_l(\xi)$ for the polynomials introduced by Chandrasekhar [6], *i.e.*,

$$h_l\,\xi\,g_l(\xi) = (l + 1)\,g_{l+1}(\xi) + l\,g_{l-1}(\xi) \tag{15}$$

with $g_0(\xi) = 1$ and

$$h_l = 2l + 1 - \omega \beta_l. \tag{16}$$

Thus on multiplying Eq. (14) by $\beta_\alpha g_\alpha(s)$ and summing the resulting equation over α from $\alpha = 0$ to $\alpha = L$, we find

$$\Lambda(s)\,X(s) = s \int_{-1}^{1} \mu G(s, \mu)\,B(\mu, s)\,\frac{\mathrm{d}\mu}{\mu - s} - s\,S^*(s) \int_{-1}^{1} G(s, \mu)\,\frac{\mathrm{d}\mu}{\mu - s} \tag{17}$$

where

$$\Lambda(s) = 1 + s \int_1^1 \psi(\mu)\,\frac{\mathrm{d}\mu}{\mu - s}, \tag{18}$$

$$\psi(\mu) = \frac{\omega}{2}\,G(\mu, \mu), \tag{19}$$

$$G(s, \mu) = \sum_{l=0}^{L} \beta_l\,g_l(s)\,P_l(\mu) \tag{20}$$

and

$$X(s) = \sum_{l=0}^{L} (-1)^l \beta_l\,P_l(s)\,I_l^*(s). \tag{21}$$

We note that the transforms $I_l^*(s)$ and $S^*(s)$ are analytic everywhere in the complex s plane except the origin where both functions have essential singularities. To avoid complications related to these singularities we find it convenient to multiply Eq. (17) by $\exp(a/s)$ and consider the resulting equation only for Re $s \geqslant 0$. In a similar manner we multiply Eq. (17) with s changed to $-s$ by $\exp(-b/s)$ and consider the resulting equation only for Re $s \geqslant 0$. In this way we find, for Re $s \geqslant 0$,

$$\Lambda(s)\, I(s) = \int_{-1}^{1} \mu G(s,\mu)\, C(\mu,s) \frac{\mathrm{d}\mu}{\mu - s} - U(s) \int_{-1}^{1} G(s,\mu) \frac{\mathrm{d}\mu}{\mu - s} \tag{22a}$$

and

$$\Lambda(s)\, J(s) = \int_{-1}^{1} \mu G(s,\mu)\, D(\mu,s) \frac{\mathrm{d}\mu}{\mu - s} - V(s) \int_{-1}^{1} G(s,\mu) \frac{\mathrm{d}\mu}{\mu - s} \tag{22b}$$

where

$$I(s) = \frac{1}{s} \sum_{l=0}^{L} (-1)^l \beta_l P_l(s) \int_a^b \mathrm{e}^{-(\tau - a)/s} I_l(\tau)\, \mathrm{d}\tau, \tag{23a}$$

$$J(s) = \frac{1}{s} \sum_{l=0}^{L} \beta_l P_l(s) \int_a^b \mathrm{e}^{-(b-\tau)/s} I_l(\tau)\, \mathrm{d}\tau, \tag{23b}$$

$$C(\mu,s) = I(a,-\mu) - I(b,-\mu)\, \mathrm{e}^{-(b-a)/s}, \tag{24a}$$

$$D(\mu,s) = I(b,\mu) - I(a,\mu)\, \mathrm{e}^{-(b-a)/s}, \tag{24b}$$

$$U(s) = \int_a^b \mathrm{e}^{\;(\tau-a)/s} S(\tau)\, \mathrm{d}\tau \tag{25a}$$

and

$$V(s) = \int_a^b \mathrm{e}^{-(b-\tau)/s} S(\tau)\, \mathrm{d}\tau. \tag{25b}$$

We note that the dispersion function $\Lambda(s)$ is analytic in the complex plane cut from -1 to 1 along the real axis and that in general $\Lambda(s)$ has $\varkappa$ pairs of real zeros $\pm v_\beta$, $\beta = 0,1,\ldots,\varkappa - 1$, with $|v_\beta| > 1$. We let $\{v_\beta\}$ denote the positive zeros of $\Lambda(s)$ and observe that Eqs. (22) yield

$$\frac{\omega}{2} v_\beta \int_{-1}^{1} \mu G(v_\beta,\mu)\, C(\mu,v_\beta) \frac{\mathrm{d}\mu}{v_\beta - \mu} = U(v_\beta) \tag{26a}$$

and

$$\frac{\omega}{2} v_\beta \int_{-1}^{1} \mu G(v_\beta,\mu)\, D(\mu,v_\beta) \frac{\mathrm{d}\mu}{v_\beta - \mu} = V(v_\beta). \tag{26b}$$

We can also let $s \to v \in [0,1]$ and use the Plemelj formulas [7] to deduce from Eqs. (22) that

$$\begin{aligned}[\lambda(v) \pm \pi i v \psi(v)]\, I(v) = P \int_{-1}^{1} G(v,\mu)\,[\mu C(\mu,v) - U(v)] \frac{\mathrm{d}\mu}{\mu - v} \\ \pm \pi i G(v,v)\,[v C(v,v) - U(v)]\end{aligned} \tag{27a}$$

and

$$[\lambda(\nu) \pm \pi i \nu \psi(\nu)] J(\nu) = P \int_{-1}^{1} G(\nu, \mu) [\mu D(\mu, \nu) - V(\nu)] \frac{d\mu}{\mu - \nu} \pm \pi i G(\nu, \nu) [\nu D(\nu, \nu) - V(\nu)]. \tag{27b}$$

Here the symbol P is used to indicate that integrals are to be evaluated in the Cauchy principal-value sense and

$$\lambda(\nu) = 1 + \nu P \int_{-1}^{1} \psi(\mu) \frac{d\mu}{\mu - \nu}. \tag{28}$$

From Eqs. (27) we deduce, for $\nu \in [0, 1]$, that

$$\frac{\omega}{2} \nu I(\nu) = \nu C(\nu, \nu) - U(\nu), \tag{29a}$$

$$\frac{\omega}{2} \nu J(\nu) = \nu D(\nu, \nu) - V(\nu), \tag{29b}$$

$$\nu C(\nu, \nu) \lambda(\nu) - \frac{\omega}{2} \nu P \int_{-1}^{1} \mu G(\nu, \mu) C(\mu, \nu) \frac{d\mu}{\mu - \nu} = U(\nu) \tag{30a}$$

and

$$\nu D(\nu, \nu) \lambda(\nu) - \frac{\omega}{2} \nu P \int_{-1}^{1} \mu G(\nu, \mu) D(\mu, \nu) \frac{d\mu}{\mu - \nu} = V(\nu). \tag{30b}$$

Equations (26) and (30) define a system of singular-integral equations and constraints that we must solve to establish $I(a, \mu)$ and $I(b, \mu)$ for $\mu \in [-1, 1]$.

Intending first to establish the distributions $I(L, -\mu)$ and $I(R, \mu)$, for $\mu > 0$, we let $a = L$ and $b = R$ and use the boundary conditions given by Eqs. (7) to deduce from Eqs. (26) and (30) a system of singular-integral equations and constraints the solution of which yields $I(L, -\mu)$ and $I(R, \mu)$ for $\mu > 0$. Proceeding to introduce the F_N method, we let $I_0(\tau, \mu)$ denote the exact solution corresponding to $\omega = 0$ and substitute the approximations, for $\mu > 0$,

$$I(L, -\mu) = I_0(L, -\mu) + \frac{\omega}{2} \sum_{\alpha=0}^{N} a_\alpha P_\alpha(2\mu - 1) \tag{31a}$$

and

$$I(R, \mu) = I_0(R, \mu) + \frac{\omega}{2} \sum_{\alpha=0}^{N} b_\alpha P_\alpha(2\mu - 1) \tag{31b}$$

into the developed system of equations to find, for $\xi \in \{\nu_\beta\} \cup [0, 1]$,

$$\sum_{\alpha=0}^{N} \{a_\alpha [B_\alpha(\xi) - \omega \varrho_1^s A_\alpha(\xi) - 2\omega \varrho_1^d T_{\alpha,0} A_0(\xi)] + e^{-\Delta/\xi} b_\alpha [\omega A_\alpha(\xi) - \varrho_2^s B_\alpha(\xi) - 2\varrho_2^d T_{\alpha,0} B_0(\xi)]\} = 2 T_1(L, \xi) \tag{32a}$$

and

$$\sum_{\alpha=0}^{N} \{b_\alpha [B_\alpha(\xi) - \omega \varrho_2^s A_\alpha(\xi) - 2\omega \varrho_2^d T_{\alpha,0} A_0(\xi)] + e^{-\Delta/\xi} a_\alpha [\omega A_\alpha(\xi) - \varrho_1^s B_\alpha(\xi) - 2\varrho_1^d T_{\alpha,0} B_0(\xi)]\} = 2 T_2(R,\xi) \quad (32\text{b})$$

where $\Delta = R - L$, $2T_{\alpha,0} = \delta_{\alpha,0} + (1/3)\delta_{\alpha,1}$ and the basic functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ are defined by

$$\mathrm{A}_\alpha(\xi) = \int_0^1 \mu P_\alpha(2\mu - 1)\, G(-\xi,\mu) \frac{d\mu}{\mu + \xi} \quad (33)$$

for $\xi \in \{\nu_\beta\} \cup [0, 1]$,

$$B_\alpha(\nu_\beta) = \omega \int_0^1 \mu P_\alpha(2\mu - 1)\, G(\nu_\beta,\mu) \frac{d\mu}{\nu_\beta - \mu} \quad (34\text{a})$$

and

$$B_\alpha(\nu) = 2\lambda(\nu) P_\alpha(2\nu - 1) + \omega P \int_0^1 \mu P_\alpha(2\mu - 1)\, G(\nu,\mu) \frac{d\mu}{\nu - \mu} \quad (34\text{b})$$

for $\nu \in [0, 1]$. We also use, in general,

$$T_1(\tau,\xi) = \int_0^1 [G(-\xi,\mu) I_0(L,\mu)\, e^{-(\tau - L)/\mu} S(R - \tau : \mu,\xi) + G(\xi,\mu) I_0(R, -\mu)\, C(R - \tau : \mu,\xi)]\, \mu\, d\mu + S_1(\tau,\xi) \quad (35\text{a})$$

and

$$T_2(\tau,\xi) = \int_0^1 [G(-\xi,\mu) I_0(R, -\mu)\, e^{-(R-\tau)/\mu} S(\tau - L : \mu,\xi) + G(\xi,\mu) I_0(L,\mu)\, C(\tau - L : \mu,\xi)]\, \mu\, d\mu + S_2(\tau,\xi) \quad (35\text{b})$$

where

$$S_1(\tau,\xi) = \int_0^1 G(\xi,\mu)[U(\tau,\mu) - U(\tau,\xi)] \frac{d\mu}{\mu - \xi} + \int_0^1 G(-\xi,\mu)[U(\tau,\xi) + V(\tau,\mu) - e^{-(R-\tau)/\xi} V(R,\mu)] \frac{d\mu}{\mu + \xi} \quad (36\text{a})$$

and

$$S_2(\tau,\xi) = \int_0^1 G(\xi,\mu)[V(\tau,\mu) - V(\tau,\xi)] \frac{d\mu}{\mu - \xi} + \int_0^1 G(-\xi,\mu)[V(\tau,\xi) + U(\tau,\mu) - e^{-(\tau - L)/\xi} U(L,\mu)] \frac{d\mu}{\mu + \xi}. \quad (36\text{b})$$

In addition

$$C(x : \mu,\xi) = \frac{e^{-x/\mu} - e^{-x/\xi}}{\mu - \xi} \quad (37\text{a})$$

and

$$S(x : \mu,\xi) = \frac{I - e^{-x/\mu} - e^{-x/\xi}}{\mu + \xi} \quad (37\text{b})$$

and to be more specific we now use

$$U(\tau, s) = \int_{\tau}^{R} e^{-(x-\tau)/s} S(x)\,dx \tag{38a}$$

and

$$V(\tau, s) = \int_{L}^{\tau} e^{-(\tau-x)/s} S(x)\,dx. \tag{38b}$$

We note that in obtaining Eqs. (32) we have made use of some basic relationships concerning the $\omega = 0$ result $I_0(\tau, \mu)$. First of all we can solve Eq. (6) with $\omega = 0$ to obtain, for $\mu > 0$,

$$I_0(\tau, \mu) = I_0(L, \mu)\, e^{-(\tau-L)/\mu} + \frac{1}{\mu} V(\tau, \mu) \tag{39a}$$

and

$$I_0(\tau, -\mu) = I_0(R, -\mu)\, e^{-(R-\tau)/\mu} + \frac{1}{\mu} U(\tau, \mu) \tag{39b}$$

where, for $\mu > 0$,

$$I_0(L, \mu) = K_1(\mu) + \varrho_1^s I_0(L, -\mu) + 2\varrho_1^d \int_0^1 I_0(L, -\mu')\, \mu'\, d\mu' \tag{40a}$$

and

$$I_0(R, -\mu) = K_2(\mu) + \varrho_2^s I_0(R, \mu) + 2\varrho_2^d \int_0^1 I_0(R, \mu')\, \mu'\, d\mu'. \tag{40b}$$

We have also used, for $\mu > 0$,

$$I(L, \mu) = I_0(L, \mu) + \frac{\omega}{2}\left[\varrho_1^s \sum_{\alpha=0}^{N} a_\alpha P_\alpha(2\mu - 1) + \varrho_1^d \left(a_0 + \frac{1}{3} a_1\right)\right] \tag{41a}$$

and

$$I(R, -\mu) = I_0(R, -\mu) + \frac{\omega}{2}\left[\varrho_2^s \sum_{\alpha=0}^{N} b_\alpha P_\alpha(2\mu - 1) + \varrho_2^d \left(b_0 + \frac{1}{3} b_1\right)\right] \tag{41b}$$

which follow from Eqs. (7), (31) and (40).

In a manner similar to the foregoing development we can use, for $\tau \in (L, R)$ and $\mu > 0$, the approximations

$$I(\tau, -\mu) = I_0(\tau, -\mu) + \frac{\omega}{2} \sum_{\alpha=0}^{N} c_\alpha(\tau) P_\alpha(2\mu - 1) \tag{42a}$$

and

$$I(\tau, \mu) = I_0(\tau, \mu) + \frac{\omega}{2} \sum_{\alpha=0}^{N} d_\alpha(\tau) P_\alpha(2\mu - 1) \tag{42b}$$

in Eqs. (26a) and (30a) with $a = \tau$ and $b = R$ and in Eqs. (26b) and (30b) with $a = L$ and $b = \tau$ to find, for $\xi \in \{\nu_\beta\} \cup [0, 1]$,

$$\sum_{\alpha=0}^{N} [c_\alpha(\tau) B_\alpha(\xi) - \omega\, d_\alpha(\tau) A_\alpha(\xi)] = 2 W_1(\tau, \xi) \tag{43a}$$

and

$$\sum_{\alpha=0}^{N} [d_\alpha(\tau) B_\alpha(\xi) - \omega c_\alpha(\tau) A_\alpha(\xi)] = 2 W_2(\tau, \xi) \tag{43b}$$

where

$$W_1(\tau, \xi) = T_1(\tau, \xi) - \frac{1}{2} e^{-(R-\tau)/\xi} \sum_{\alpha=0}^{N} b_\alpha [\omega A_\alpha(\xi) - \varrho_2^s B_\alpha(\xi) - 2 \varrho_2^d T_{\alpha,0} B_0(\xi)] \tag{44a}$$

and

$$W_2(\tau, \xi) = T_2(\tau, \xi) - \frac{1}{2} e^{-(\tau-L)/\xi} \sum_{\alpha=0}^{N} a_\alpha [\omega A_\alpha(\xi) - \varrho_1^s B_\alpha(\xi) - 2 \varrho_1^d T_{\alpha,0} B_0(\xi)]. \tag{44b}$$

To complete our analysis here we must deduce $I_0(L, \mu)$ and $I_0(R, -\mu)$, for $\mu > 0$, so that Eqs. (39) will yield $I_0(\tau, \mu)$ for all τ and all μ. We can use Eqs. (39) to rewrite Eqs. (40) as

$$I_0(L, \mu) = T_1(\mu) + \varrho_1^s I_0(R, -\mu) e^{-\Delta/\mu} + 2 \varrho_1^d \Phi_1 \tag{45a}$$

and

$$I_0(R, -\mu) = T_2(\mu) + \varrho_2^s I_0(L, \mu) e^{-\Delta/\mu} + 2 \varrho_2^d \Phi_2 \tag{45b}$$

where

$$T_1(\mu) = K_1(\mu) + \frac{1}{\mu} \varrho_1^s U(L, \mu) + 2 \varrho_1^d U_0, \tag{46a}$$

$$T_2(\mu) = K_2(\mu) + \frac{1}{\mu} \varrho_2^s V(R, \mu) + 2 \varrho_2^d V_0, \tag{46b}$$

$$U_0 = \int_0^1 U(L, \mu) \, d\mu, \tag{47a}$$

$$V_0 = \int_0^1 V(R, \mu) \, d\mu, \tag{47b}$$

$$\Phi_1 = \int_0^1 I_0(R, -\mu) e^{-\Delta/\mu} \mu \, d\mu \tag{48a}$$

and

$$\Phi_2 = \int_0^1 I_0(L, \mu) e^{-\Delta/\mu} \mu \, d\mu. \tag{48b}$$

We can now solve Eqs. (45) to find, for $\mu > 0$,

$$I_0(L, \mu) = D(\mu) \{T_1(\mu) + 2 \varrho_1^d \Phi_1 + \varrho_1^s [T_2(\mu) + 2 \varrho_2^d \Phi_2] e^{-\Delta/\mu}\} \tag{49a}$$

and

$$I_0(R, -\mu) = D(\mu) \{T_2(\mu) + 2 \varrho_2^d \Phi_2 + \varrho_2^s [T_1(\mu) + 2 \varrho_1^d \Phi_1] e^{-\Delta/\mu}\}, \tag{49b}$$

where

$$D(\mu) = (1 - \varrho_1^s \varrho_2^s \, e^{-2\Delta/\mu})^{-1}. \tag{50}$$

Finally we can complete our solution for $I_0(\tau, \mu)$ by multiplying Eqs. (49) by $\mu \exp(-\Delta/\mu)$, integrating over μ and solving the resulting two equations to obtain

$$\Phi_1 = M[(1 - 2\varrho_2^d \varrho_1^s J) R_2 + 2\varrho_2^d K R_1] \tag{51a}$$

and

$$\Phi_2 = M[2\varrho_1^d K R_2 + (1 - 2\varrho_1^d \varrho_2^s J) R_1] \tag{51b}$$

where

$$R_1 = \int_0^1 e^{-\Delta/\mu} D(\mu) [T_1(\mu) + \varrho_1^s T_2(\mu) e^{-\Delta/\mu}] \mu \, d\mu, \tag{52a}$$

$$R_2 = \int_0^1 e^{-\Delta/\mu} D(\mu) [T_2(\mu) + \varrho_2^s T_1(\mu) e^{-\Delta/\mu}] \mu \, d\mu, \tag{52b}$$

$$J = \int_0^1 e^{-2\Delta/\mu} D(\mu) \mu \, d\mu, \tag{52c}$$

$$K = \int_0^1 e^{-\Delta/\mu} D(\mu) \mu \, d\mu \tag{52d}$$

and

$$M = [1 - 2(\varrho_1^d \varrho_2^s + \varrho_2^d \varrho_1^s) J + 4\varrho_1^d \varrho_2^d (\varrho_1^s \varrho_2^s J^2 - K^2)]^{-1}. \tag{52e}$$

As we now have explicit results for the functions $T_\alpha(\tau, \xi)$ and $W_\alpha(\tau, \xi)$, $\alpha = 1$ and 2, we can consider Eqs. (32) at $N + 1$ values of $\xi \in \{\nu_\beta\} \cup [0, 1]$ and solve the resulting linear algebraic equations to obtain the constants $\{a_\alpha\}$ and $\{b_\alpha\}$ required to define, by way of Eqs. (31) and (41), the desired intensities at the boundaries $\tau = L$ and $\tau = R$. In a similar manner we can, for any value of $\tau \in (L, R)$, consider Eqs. (43) at $N + 1$ values of $\xi \in \{\nu_\beta\} \cup [0, 1]$ and solve the resulting system of equations to obtain $\{c_\alpha(\tau)\}$ and $\{d_\alpha(\tau)\}$ which are required in Eqs. (42) to define the intensity within the medium.

As noted in the foregoing discussion we now have available a complete formalism that yields the intensity $I(\tau, \mu)$ for all $\mu \in [-1, 1]$ and all $\tau \in [L, R]$; however, we can use here a refinement discussed previously [1, 8] to improve the intensity calculation. From the definitions given by Eqs. (33) and (34) we note [1] that, for $\mu \in [0, 1]$,

$$B_\alpha(\mu) = 2 P_\alpha(2\mu - 1) - \omega \{[2 - A_0(\mu)] P_\alpha(2\mu - 1) + G_\alpha(\mu)\}. \tag{53}$$

Here the polynomials $G_\alpha(\mu)$, with $G_0(\mu) = 0$, satisfy, for $\alpha \geqslant 0$,

$$(\alpha + 1) G_{\alpha+1}(\mu) = (2\alpha + 1)(2\mu - 1) G_\alpha(\mu) - \alpha G_{\alpha-1}(\mu) + 2(2\alpha + 1) \sum_{l=0}^{L} \beta_l T_{\alpha, l} \, g_l(\mu) \tag{54}$$

where

$$T_{\alpha,l} = \int_0^1 \mu P_\alpha(2\mu - 1) P_l(\mu)\, d\mu. \tag{55}$$

We can now substitute Eq. (53) into the version of Eqs. (32) valid for $\xi = \mu \in [0, 1]$ to obtain

$$\sum_{\alpha=0}^{N} (a_\alpha - \varrho_2^s e^{-\Delta/\mu} b_\alpha) P_\alpha(2\mu - 1) = X_1(\mu) + \varrho_2^d (b_0 + \tfrac{1}{3} b_1) e^{-\Delta/\mu} \tag{56a}$$

and

$$\sum_{\alpha=0}^{N} (b_\alpha - \varrho_1^s e^{-\Delta/\mu} a_\alpha) P_\alpha(2\mu - 1) = X_2(\mu) + \varrho_1^d (a_0 + \tfrac{1}{3} a_1) e^{-\Delta/\mu} \tag{56b}$$

where

$$X_1(\mu) = T_1(L, \mu) + \frac{\omega}{2} \sum_{\alpha=0}^{N} [a_\alpha C_{1,\alpha}(\mu) - e^{-\Delta/\mu} b_\alpha A_{2,\alpha}(\mu)] \tag{57a}$$

and

$$X_2(\mu) = T_2(R, \mu) + \frac{\omega}{2} \sum_{\alpha=0}^{N} [b_\alpha C_{2,\alpha}(\mu) - e^{-\Delta/\mu} a_\alpha A_{1,\alpha}(\mu)]. \tag{57b}$$

Here, for $\beta = 1$ and 2,

$$A_{\beta,\alpha}(\mu) = A_\alpha(\mu) + \varrho_\beta^s B_\alpha^*(\mu) + \varrho_\beta^d (\delta_{\alpha,0} + \tfrac{1}{3}\delta_{\alpha,1}) [2 - A_0(\mu)] \tag{58a}$$

and

$$C_{\beta,\alpha}(\mu) = B_\alpha^*(\mu) + \varrho_\beta^s A_\alpha(\mu) + \varrho_\beta^d (\delta_{\alpha,0} + \tfrac{1}{3}\delta_{\alpha,1}) A_0(\mu) \tag{58b}$$

where

$$B_\alpha^*(\mu) = [2 - A_0(\mu)] P_\alpha(2\mu - 1) + G_\alpha(\mu). \tag{58c}$$

We can readily solve Eqs. (56) to find

$$\sum_{\alpha=0}^{N} a_\alpha P_\alpha(2\mu - 1) = D(\mu) [Y_1(\mu) + \varrho_2^s e^{-\Delta/\mu} Y_2(\mu)] \tag{59a}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha P_\alpha(2\mu - 1) = D(\mu) [Y_2(\mu) + \varrho_1^s e^{-\Delta/\mu} Y_1(\mu)] \tag{59b}$$

where $D(\mu)$ is given by Eq. (50),

$$A = \tfrac{1}{2}(a_0 + \tfrac{1}{3} a_1), \tag{60a}$$

$$B = \tfrac{1}{2}(b_0 + \tfrac{1}{3} b_1), \tag{60b}$$

$$Y_1(\mu) = X_1(\mu) + 2\varrho_2^d B e^{-\Delta/\mu} \tag{61a}$$

and

$$Y_2(\mu) = X_2(\mu) + 2\varrho_1^d A e^{-\Delta/\mu}. \tag{61b}$$

We note that expressions for A and B alternative to those given by Eqs. (60) can be found from Eqs. (59). Thus we can multiply Eqs. (59) by μ, integrate over μ from zero to one and solve the resulting equations to obtain

$$A = M\,[(1 - 2\varrho_2^d \varrho_1^s J)\,S_1 + 2\varrho_2^d\,K\,S_2] \tag{62a}$$

and

$$B = M\,[(1 - 2\varrho_1^d \varrho_2^s J)\,S_2 + 2\varrho_1^d\,K\,S_1] \tag{62b}$$

where

$$S_1 = \int_0^1 \mu\,D(\mu)\,[X_1(\mu) + \varrho_2^s\,e^{-\Delta/\mu}\,X_2(\mu)]\,d\mu \tag{63a}$$

and

$$S_2 = \int_0^1 \mu\,D(\mu)\,[X_2(\mu) + \varrho_1^s\,e^{-\Delta/\mu}\,X_1(\mu)]\,d\mu\,. \tag{63b}$$

It is apparent that the exact values of A and B for the case $\omega = 0$ can be readily obtained from Eqs. (62) and (63).

Finally we use Eqs. (59) in Eqs. (31) and (41) to find our improved results for the intensity at the boundaries; thus for $\mu \in [0, 1]$ we write

$$I(L, -\mu) = I_0(L, -\mu) + \frac{\omega}{2}\,D(\mu)\,[Y_1(\mu) + \varrho_2^s\,e^{-\Delta/\mu}\,Y_2(\mu)]\,, \tag{64a}$$

$$I(R, \mu) = I_0(R, \mu) + \frac{\omega}{2}\,D(\mu)\,[Y_2(\mu) + \varrho_1^s\,e^{-\Delta/\mu}\,Y_1(\mu)]\,, \tag{64b}$$

$$I(L, \mu) = I_0(L, \mu) + \frac{\omega}{2}\,\{\varrho_1^s\,D(\mu)\,[Y_1(\mu) + \varrho_2^s\,e^{-\Delta/\mu}\,Y_2(\mu)] + 2\varrho_1^d\,A\} \tag{64c}$$

and

$$I(R, -\mu) = I_0(R, -\mu) + \frac{\omega}{2}\,\{\varrho_2^s\,D(\mu)\,[Y_2(\mu) + \varrho_1^s\,e^{-\Delta/\mu}\,Y_1(\mu)] + 2\varrho_2^d\,B\}\,. \tag{64d}$$

To conclude this work we use Eqs. (43) and (53) to rewrite Eqs. (42) as

$$I(\tau, -\mu) = I_0(\tau, -\mu) + \frac{\omega}{2}\left\{W_1(\tau, \mu) + \frac{\omega}{2}\sum_{\alpha=0}^{N}[c_\alpha(\tau)\,B_\alpha^*(\mu) + d_\alpha(\tau)\,A_\alpha(\mu)]\right\} \tag{65a}$$

and

$$I(\tau, \mu) = I_0(\tau, \mu) + \frac{\omega}{2}\left\{W_2(\tau, \mu) + \frac{\omega}{2}\sum_{\alpha=0}^{N}[d_\alpha(\tau)\,B_\alpha^*(\mu) + c_\alpha(\tau)\,A_\alpha(\mu)]\right\}. \tag{65b}$$

We note that one of the principal merits of the way our analysis was carried out here is that we do not require particular solutions of Eq. (6); and thus source terms $S(\tau)$ given as numerical data (as could result in an iterative solution [9] of problems involving, for example, both conductive and radiative heat transfer) can be considered.

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References

[1] R. D. M. Garcia and C. E. Siewert, J. Comp. Phys. *46*, 237 (1982).
[2] C. E. Siewert and P. Benoist, Nucl. Sci. Eng. *69*, 156 (1979).
[3] C. E. Siewert, Astrophys. Space Sci. *58*, 131 (1978).
[4] N. J. McCormick and R. Sanchez, J. Math. Phys. *22*, 199 (1981).
[5] C. Devaux, C. E. Siewert and Y. L. Yuan, Ap. J. *253*, 773 (1982).
[6] S. Chandrasekhar, *Radiative transfer*. Oxford University Press, London 1950.
[7] N. I. Muskhelishvili, *Singular integral equations*. Noordhoff, Groningen, The Netherlands 1953.
[8] R. D. M. Garcia and C. E. Siewert, Nucl. Sci. Eng. *81*, 474 (1982).
[9] M. N. Özişik, *Radiative transfer and interaction with conduction and convection*. Wiley, New York 1973.

Abstract

An integral-transform technique and the F_N method are used to solve basic radiative-transfer problems. Boundary conditions that account for specular and diffuse reflection as well as external illumination are used, and internal emission is allowed. Emphasis is given to the azimuthally symmetric component of the complete radiation field.

Zusammenfassung

Eine Integral-Transformation und die F_N-Methode werden zur Lösung von grundlegenden Problemen der Strahlungsübertragung herangezogen. Es werden Randbedingungen mit spekulärer und diffuser Reflexion wie auch mit äußerer Bestrahlung benützt, und innere Emission ist zugelassen. Besonders betrachtet werden die Komponenten des gesamten Strahlungsfeldes, welches azimutale Symmetrie besitzt.

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