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Neutron Transport Calculations in Cylindrical Geometry

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Abstract—*An integral transformation technique and the F_N method are used to solve, for the case of isotropic scattering, the albedo problem and the internal-source problem relevant to neutron transport theory for a bare cylinder of infinite length. The neutron flux distribution and the neutron current distributions are found with five or six figures of accuracy and are tabulated for selected illustrative cases. In addition, selected results for the albedo are reported with an accuracy of eight to ten significant figures.*

Calculs du transport de neutrons dans une géométrie cylindrique

Résumé—*Une technique de transformation intégrale ainsi que la méthode F_N sont utilisées afin de résoudre pour le cas de la diffusion isotrope les problèmes de l'albédo et de la source interne pertinents à la théorie de transport pour un cylindre nu de longueur infinie. La distribution du flux neutronique et les distributions du courant neutronique sont déterminées avec une précision de 5 ou 6 chiffres et portées en tableau pour des cas sélectionnés illustratifs. En outre, des résultats sélectionnés sur l'albédo sont indiqués avec une précision de 8 à 10 chiffres significatifs.*

Berechnungen des Neutronentransports in zylindrischer Geometrie

Zusammenfassung—*Mit Hilfe eines Integral-Transformationsverfahrens und der F_N-Methode werden für den Fall der isotropen Streuung das Problem der Albedo und das Problem der inneren Quelle in Zusammenhang mit der Neutronentransporttheorie für einen nackten Zylinder unendlicher Länge gelöst. Die Verteilung des Neutronenflusses und des Neutronenstroms wird mit einer Genauigkeit von fünf bzw. sechs Stellen ermittelt, und die Werte werden für ausgewählte aussagekräftige Fälle tabellarisch dargestellt. Außerdem werden ausgewählte Ergebnisse für die Albedo mit einer Genauigkeit von acht bis zehn signifikanten Ziffern angegeben.*

INTRODUCTION

In a recent paper,[1] we used the transformation idea developed by Mitsis,[2] the elementary solutions of Case,[3] and the F_N method[4] to solve the critical problem in neutron transport theory for an infinite bare cylinder. Here we extend our analysis to solve a class of neutron transport problems in cylindrical geometry. First, we consider the transport equation written for $r \in (0,R)$ as

$$T\Psi(r,\mu,\phi) = \frac{c}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} \Psi(r,\mu,\phi)\,d\phi\,d\mu + \frac{1}{4\pi}(1-c)Q(r) \ , \tag{1}$$

where

$$T\Psi(r,\mu,\phi) = \left[(1-\mu^2)^{1/2}\left(\cos\phi \frac{\partial}{\partial r} - \frac{1}{r}\sin\phi \frac{\partial}{\partial \phi}\right) + 1\right]\Psi(r,\mu,\phi) \quad . \quad (2)$$

Here $\theta = \cos^{-1}\mu$ and ϕ are the angles used to define the direction of propagation $\mathbf{\Omega}$ with respect to transverse and radial axes, $Q(r)$ is used to denote an internal source of neutrons, and c is the mean number of neutrons per collision. For the boundary condition we write, for $\mu \in [-1,1]$ and $\phi \in [\pi/2, 3\pi/2]$,

$$\Psi(R,\mu,\phi) = \frac{1}{4\pi} F \ , \quad (3)$$

where the constant F is assumed given. We note that the integral form of the transport equation for the neutron flux distribution $\phi(r)$ in a bare homogeneous right circular cylinder of infinite length and radius R was expressed by Mitsis[2] for the case of no inhomogeneous source term and no incident neutrons as

$$\phi(r) = c(K\phi)(r) \ , \quad (4)$$

where the integral operator K is defined by

$$(Kf)(r) = \int_0^1 \left[K_0(r/\mu)\int_0^r I_0(t/\mu)f(t)t\,dt + I_0(r/\mu)\int_r^R K_0(t/\mu)f(t)t\,dt\right]\frac{d\mu}{\mu^2} \ , \quad (5)$$

where $I_0(x)$ and $K_0(x)$ denote modified Bessel functions.[5] Generalizing Mitsis' work[2] to include internal sources and incident radiation, we write, for all r, μ, and ϕ,

$$\Psi(r,\mu,\phi) = I(r,\mu,\phi) + \frac{1}{4\pi} F \quad (6)$$

and deduce that

$$I(r) = \int_{-1}^1 \int_0^{2\pi} I(r,\mu,\phi)\,d\phi\,d\mu \quad (7)$$

satisfies the integral equation

$$I(r) = (1-c)(K[Q-F])(r) + c(KI)(r) \ . \quad (8)$$

Proceeding to derive a pseudo-problem we can solve to obtain $I(r)$, we let

$$W(r) = cI(r) + (1-c)[Q(r) - F] \quad (9)$$

and

$$\Phi(r,\mu) = K_0(r/\mu)\int_0^r I_0(t/\mu)W(t)t\,dt + I_0(r/\mu)\int_r^R K_0(t/\mu)W(t)t\,dt \ , \quad (10)$$

so that we can write Eq. (8) as

$$I(r) = \int_0^1 \Phi(r,\mu)\frac{d\mu}{\mu^2} \ . \quad (11)$$

Thus if we can find $\Phi(r,\mu)$, then $I(r)$ will follow from Eq. (11). The desired

$$\Psi(r) = \int_{-1}^1 \int_0^{2\pi} \Psi(r,\mu,\phi)\,d\phi\,d\mu \quad (12)$$

will then be available from

$$\Psi(r) = I(r) + F \ . \quad (13)$$

Following Mitsis,[2] we can now differentiate Eq. (10) to obtain, for $\mu \in [0,1]$ and $r \in (0,R)$,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{\mu^2}\right)\Phi(r,\mu) = -W(r) \ , \quad (14)$$

or, after we use Eqs. (9) and (11),

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{\mu^2}\right)\Phi(r,\mu) = -c\int_0^1 \Phi(r,\mu)\frac{d\mu}{\mu^2} - (1-c)[Q(r) - F] \ . \quad (15)$$

We can also deduce from Eq. (10) the boundary condition, for $\mu \in [0,1]$:

$$K_1(R/\mu)\Phi(R,\mu) + \mu K_0(R/\mu)\frac{\partial}{\partial r}\Phi(r,\mu)\bigg|_{r=R} = 0 \ . \quad (16)$$

We now express our solution to Eq. (15) as[1,2]

$$\Phi(r,\mu) = \mu^2\left\{A(\nu_0)[\phi(\nu_0,\mu) + \phi(-\nu_0,\mu)]I_0(r/\nu_0) + \int_0^1 A(\nu)[\phi(\nu,\mu) + \phi(-\nu,\mu)]I_0(r/\nu)\,d\nu\right\} + \Phi_p(r,\mu) \ , \quad (17)$$

where

$$\phi(\pm\nu,\mu) = \frac{c}{2}\nu P v\left(\frac{1}{\nu \mp \mu}\right) + (1 - c\nu\tanh^{-1}\nu)\delta(\nu \mp \mu) \quad (18)$$

and

$$\phi(\pm\nu_0,\mu) = \frac{c}{2}\nu_0\left(\frac{1}{\nu_0 \mp \mu}\right) \quad (19)$$

are the familiar (generalized) functions[3,6] appropriate to plane geometry problems. In addition, ν_0 is the positive solution of

$$1 + \frac{c}{2}\nu_0 \int_{-1}^{1} \frac{d\mu}{\mu - \nu_0} = 0 \ , \tag{20}$$

and $\Phi_p(r,\mu)$ denotes a particular solution of Eq. (15) corresponding to the inhomogeneous source term

$$S(r) = -(1-c)[Q(r) - F] \ . \tag{21}$$

Following our previous work,[1] we now use the full-range orthogonality condition[3]

$$(\xi - \xi')\int_{-1}^{1} \mu\phi(\xi,\mu)\phi(\xi',\mu)\, d\mu = 0 \tag{22}$$

to deduce from Eq. (17) that, for $\xi \in P = \nu_0 \cup [0,1]$,

$$\int_0^1 [\phi(\xi,\mu) - \phi(-\xi,\mu)]\left[1 - \xi\Upsilon(r/\xi)\frac{\partial}{\partial r}\right] \times [\Phi(r,\mu) - \Phi_p(r,\mu)]\frac{d\mu}{\mu} = 0 \ , \tag{23}$$

where

$$\Upsilon(x) = I_0(x)/I_1(x) \ . \tag{24}$$

We can now set $r = R$ in Eq. (23) and use Eq. (16) to obtain, for $\xi \in P$,

$$\int_0^1 [\phi(\xi,\mu) - \phi(-\xi,\mu)] \times [\mu + \xi\Upsilon(R/\xi)\Xi(R/\mu)]\Phi(R,\mu)\frac{d\mu}{\mu^2} = \xi R(\xi) \ , \tag{25}$$

where

$$\Xi(x) = K_1(x)/K_0(x) \tag{26}$$

and

$$\xi R(\xi) = \int_0^1 [\phi(\xi,\mu) - \phi(-\xi,\mu)]\left[\Phi_p(R,\mu) - \xi\Upsilon(R/\xi)\frac{\partial}{\partial r}\Phi_p(r,\mu)\bigg|_{r=R}\right]\frac{d\mu}{\mu} \ . \tag{27}$$

We consider that we can find exactly the required particular solution $\Phi_p(r,\mu)$ so that Eq. (25) represents a singular integral equation and a constraint for the unknown function $\Phi(R,\mu)$.

THE F_N SOLUTION

We now substitute the approximation

$$\Phi(R,\mu) = \mu^2 \sum_{\alpha=0}^{N} a_\alpha \mu^\alpha \tag{28}$$

into Eq. (25) to find, for $\xi \in P$,

$$\sum_{\alpha=0}^{N} a_\alpha [E_\alpha(\xi) + \Upsilon(R/\xi)D_\alpha(\xi)] = R(\xi) \ , \tag{29}$$

where

$$E_\alpha(\xi) = \frac{1}{\xi}\int_0^1 \mu^{\alpha+1}[\phi(\xi,\mu) - \phi(-\xi,\mu)]\, d\mu \tag{30}$$

and

$$D_\alpha(\xi) = \int_0^1 \mu^\alpha[\phi(\xi,\mu) - \phi(-\xi,\mu)]\Xi(R/\mu)\, d\mu \tag{31}$$

are the functions discussed previously.[1] To find the desired coefficients $\{a_\alpha\}$, we consider Eq. (29) at a set of collocation points $\{\xi_\beta\}$ and solve the resulting system of linear algebraic equations. In this work we use the collocation scheme defined[7] by $\xi_0 = \nu_0$ and

$$\xi_\beta = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\beta - 1}{2N}\pi\right) \ , \quad \beta = 1, 2, \ldots, N \ . \tag{32}$$

It is apparent that once we have found the coefficients $\{a_\alpha\}$ we can enter Eq. (28) into Eq. (17) evaluated at $r = R$ and then use the full-range orthogonality relation to find

$$A(\xi) = [N(\xi)I_0(R/\xi)]^{-1}\left\{\xi\sum_{\alpha=0}^{N} a_\alpha E_\alpha(\xi) - \int_0^1 [\phi(\xi,\mu) - \phi(-\xi,\mu)]\Phi_p(R,\mu)\frac{d\mu}{\mu}\right\} \ , \tag{33}$$

where the normalization factors are

$$N(\nu_0) = \frac{c}{2}\nu_0^3\left(\frac{c}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right) \tag{34}$$

and

$$N(\nu) = \nu\left[(1 - c\nu\tanh^{-1}\nu)^2 + \frac{c^2\nu^2\pi^2}{4}\right] \ . \tag{35}$$

We can substitute Eq. (17) into Eq. (11) and use Eq. (13) to obtain

$$\Psi(r) = A(\nu_0)I_0(r/\nu_0) + \int_0^1 A(\nu)I_0(r/\nu)\, d\nu + \int_0^1 \Phi_p(r,\mu)\frac{d\mu}{\mu^2} + F \ . \tag{36}$$

Continuing, we next integrate Eq. (1) over μ from -1 to 1 and over ϕ from 0 to 2π to find

$$\frac{d}{dr}[rj(r)] = (1-c)r[Q(r) - \Psi(r)] \ , \tag{37}$$

where

$$j(r) = \int_{-1}^{1}\int_0^{2\pi} \Psi(r,\mu,\phi)(1-\mu^2)^{1/2}\cos\phi\, d\phi\, d\mu \tag{38}$$

is the neutron current. Thus we can use Eq. (36) in Eq. (37) and integrate to obtain a general result for $j(r)$, namely,

$$j(r) = -(1-c)\left[\nu_0 A(\nu_0) I_1(r/\nu_0) + \int_0^1 \nu A(\nu) I_1(r/\nu)\, d\nu + \frac{1}{r}\int_0^r x \int_0^1 \Phi_p(x,\mu)\,\frac{d\mu}{\mu^2}\, dx - \frac{1}{r}\int_0^r xQ(x)\, dx + \frac{r}{2}F\right] . \tag{39}$$

NUMERICAL RESULTS

For our numerical work we consider here the special case $Q(r) = Q$, a constant, and note that a relevant particular solution is

$$\Phi_p(r,\mu) = \mu^2(Q-F) . \tag{40}$$

Since

$$\int_{-1}^1 \phi(\xi,\mu)\mu\, d\mu = \xi(1-c) , \tag{41}$$

we deduce from Eq. (27) that for this example

$$R(\xi) = (1-c)(Q-F) . \tag{42}$$

Continuing, we observe that Eq. (33) yields

$$A(\xi) = \xi[N(\xi)I_0(R/\xi)]^{-1} \times \left[\sum_{\alpha=0}^{N} a_\alpha E_\alpha(\xi) - (1-c)(Q-F)\right] , \tag{43}$$

that Eq. (36) yields

$$\Psi(r) = A(\nu_0)I_0(r/\nu_0) + \int_0^1 A(\nu)I_0(r/\nu)\, d\nu + Q , \tag{44}$$

and that Eq. (39) yields

$$j(r) = -(1-c)\left[\nu_0 A(\nu_0) I_1(r/\nu_0) + \int_0^1 \nu A(\nu) I_1(r/\nu)\, d\nu\right] . \tag{45}$$

We note from Eqs. (29), (42), (43), and (44) that the neutron flux $\Psi(r)$ can be expressed as

$$\Psi(r) = Q - (Q-F)F(r) , \tag{46}$$

where $F(r)$ is independent of Q and F. Similarly, we see from Eqs. (29), (42), (43), and (45) that the neutron current $j(r)$ can be written as

$$j(r) = (Q-F)C(r) , \tag{47}$$

where $C(r)$ is independent of Q and F. We therefore list in Tables I through IV selected results for $F(r)$

TABLE I

The Function $F(r)$ for $R = 1$

r/R	$c = 0.3$	$c = 0.5$	$c = 0.7$	$c = 0.9$
0.0	0.364405	0.457065	0.596607	0.824677
0.1	0.366325	0.458897	0.598138	0.825431
0.2	0.372157	0.464449	0.602766	0.827704
0.3	0.382121	0.473895	0.610603	0.831533
0.4	0.396622	0.487552	0.621852	0.836986
0.5	0.416309	0.505927	0.636839	0.844173
0.6	0.442206	0.529816	0.656077	0.853271
0.7	0.475976	0.560510	0.680397	0.864571
0.8	0.520597	0.600306	0.711290	0.878607
0.9	0.582596	0.654245	0.752048	0.896589
1.0	0.694369	0.747538	0.819470	0.924929

TABLE II

The Function $F(r)$ for $R = 10$

r/R	$c = 0.3$	$c = 0.5$	$c = 0.7$	$c = 0.9$
0.0	0.460882(−4)[a]	0.138859(−3)	0.801829(−3)	0.201898(−1)
0.1	0.595325(−4)	0.173680(−3)	0.946293(−3)	0.216079(−1)
0.2	0.112121(−3)	0.305085(−3)	0.145897(−2)	0.261624(−1)
0.3	0.253579(−3)	0.637744(−3)	0.262685(−2)	0.348252(−1)
0.4	0.625301(−3)	0.144881(−2)	0.512494(−2)	0.494721(−1)
0.5	0.162250(−2)	0.344884(−2)	0.104520(−1)	0.733358(−1)
0.6	0.437880(−2)	0.848704(−2)	0.219360(−1)	0.111811
0.7	0.122887(−1)	0.215284(−1)	0.471009(−1)	0.173872
0.8	0.362559(−1)	0.566444(−1)	0.103616	0.274762
0.9	0.116748	0.158763	0.237138	0.442664
1.0	0.558361	0.600996	0.663331	0.781243

[a]Read as 0.460882×10^{-4}.

TABLE III

The Function $C(r)$ for $R = 1$

r/R	$c = 0.3$	$c = 0.5$	$c = 0.7$	$c = 0.9$
0.0	0.0	0.0	0.0	0.0
0.1	0.127878(−1)[a]	0.114495(−1)	0.896058(−2)	0.412527(−2)
0.2	0.257786(−1)	0.230372(−1)	0.179904(−1)	0.826188(−2)
0.3	0.391839(−1)	0.349062(−1)	0.271603(−1)	0.124214(−1)
0.4	0.532335(−1)	0.472094(−1)	0.365455(−1)	0.166157(−1)
0.5	0.681876(−1)	0.601172(−1)	0.462273(−1)	0.208576(−1)
0.6	0.843554(−1)	0.738269(−1)	0.562979(−1)	0.251610(−1)
0.7	0.102125	0.885799(−1)	0.668660(−1)	0.295418(−1)
0.8	0.122020	0.104692	0.780700(−1)	0.340192(−1)
0.9	0.144842	0.122633	0.901073(−1)	0.386199(−1)
1.0	0.172259	0.143351	0.103366	0.433942(−1)

[a]Read as 0.127878×10^{-1}.

TABLE IV

The Function $C(r)$ for $R = 10$

r/R	$c = 0.3$	$c = 0.5$	$c = 0.7$	$c = 0.9$
0.0	0.0	0.0	0.0	0.0
0.1	0.184303(−4)[a]	0.389822(−4)	0.130955(−3)	0.104474(−2)
0.2	0.534070(−4)	0.107912(−3)	0.333733(−3)	0.231088(−2)
0.3	0.138456(−3)	0.262732(−3)	0.724214(−3)	0.407205(−2)
0.4	0.360162(−3)	0.635751(−3)	0.153521(−2)	0.671926(−2)
0.5	0.956164(−3)	0.155958(−2)	0.326708(−2)	0.108557(−1)
0.6	0.259910(−2)	0.389208(−2)	0.702139(−2)	0.174461(−1)
0.7	0.725771(−2)	0.990046(−2)	0.152642(−1)	0.280614(−1)
0.8	0.209862(−1)	0.257960(−1)	0.336452(−1)	0.453015(−1)
0.9	0.641922(−1)	0.698261(−1)	0.757463(−1)	0.736022(−1)
1.0	0.229733	0.210537	0.181187	0.121792

[a]Read as 0.184303×10^{-4}.

and $C(r)$ that we believe correct to within ± 1 in the last digits given. The results given in Tables I through IV were obtained and found to be stable as $N \rightarrow 15$. Also, we list in Table V, for the case $Q = 0$, the albedo

$$A^* = -\frac{\int_0^1 \int_0^{\pi/2} \Psi(R,\mu,\phi)(1-\mu^2)^{1/2}\cos\phi \, d\phi \, d\mu}{\int_0^1 \int_{\pi/2}^{\pi} \Psi(R,\mu,\phi)(1-\mu^2)^{1/2}\cos\phi \, d\phi \, d\mu} , \tag{48}$$

which we express as

$$A^* = 1 - 4C(R) , \tag{49}$$

and compare our results to the definitive results of Sanchez.[8]

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TABLE V

The Albedo

R	c	F_N	Sanchez (Ref. 8)
0.5	0.2	0.48189977	0.4818997813
	0.4	0.57430099	0.5743009836
	0.6	0.686013116	0.6860131149
	0.8	0.824174364	0.8241743632
1	0.2	0.264086136	0.2640861372
	0.4	0.364560116	0.3645601177
	0.6	0.4994505861	0.4994505863
	0.8	0.6929081089	0.6929081089
3	0.2	0.080127580	0.0801275814
	0.4	0.15642266	0.1564226671
	0.6	0.267036813	0.2670368137
	0.8	0.4570249841	0.4570249840
7	0.2	0.054390794	0.0543907951
	0.4	0.121182634	0.1211826341
	0.6	0.217007651	0.2170076512
	0.8	0.3792969096	0.3792969095
10	0.2	0.051078235	0.0510782365
	0.4	0.11607835	0.1160783551
	0.6	0.209223391	0.2092233953
	0.8	0.366521528	0.3665215281

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