Journal of Applied Mathematics and Physics (ZAMP)
Vol. 36, January 1985
0044-2275/85/001070-19 $ 5.30/0


# A generalized spherical harmonics solution basic to the scattering of polarized light

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## I. Introduction

In regard to the diffusion of polarized light in a scattering and absorbing host medium, we let $\boldsymbol{I}(\tau, \mu, \varphi)$ denote the density vector with the four Stokes parameters $I(\tau, \mu, \varphi)$, $Q(\tau, \mu, \varphi)$, $U(\tau, \mu, \varphi)$ and $V(\tau, \mu, \varphi)$ as components and consider the equation of transfer [1–3]

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau, \mu, \varphi) + \boldsymbol{I}(\tau, \mu, \varphi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \boldsymbol{P}(\mu, \mu', \varphi - \varphi') \boldsymbol{I}(\tau, \mu', \varphi') \, \mathrm{d}\mu' \, \mathrm{d}\varphi' \tag{1}$$

where for the phase matrix we use the analytical representation [4]

$$\boldsymbol{P}(\mu, \mu', \varphi - \varphi') = \frac{1}{2} \boldsymbol{C}^0(\mu, \mu') + \sum_{m=1}^{L} [\boldsymbol{C}^m(\mu, \mu') \cos m(\varphi - \varphi') + \boldsymbol{S}^m(\mu, \mu') \sin m(\varphi - \varphi')] \tag{2}$$

with

$$\boldsymbol{C}^m(\mu, \mu') = \boldsymbol{A}^m(\mu, \mu') + \boldsymbol{D} \boldsymbol{A}^m(\mu, \mu') \boldsymbol{D}, \tag{3a}$$

$$\boldsymbol{S}^m(\mu, \mu') = \boldsymbol{A}^m(\mu, \mu') \boldsymbol{D} - \boldsymbol{D} \boldsymbol{A}^m(\mu, \mu'), \tag{3b}$$

$$\boldsymbol{A}^m(\mu, \mu') = \sum_{l=m}^{L} \frac{(l-m)!}{(l+m)!} \boldsymbol{\Pi}_l^m(\mu) \boldsymbol{B}_l \boldsymbol{\Pi}_l^m(\mu'), \tag{3c}$$

and

$$\boldsymbol{D} = \mathrm{diag}\{1, 1, -1, -1\}. \tag{4}$$

In addition

$$\boldsymbol{\Pi}_l^m(\mu) = \begin{vmatrix} P_l^m(\mu) & 0 & 0 & 0 \\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0 \\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0 \\ 0 & 0 & 0 & P_l^m(\mu) \end{vmatrix} \tag{5}$$

where

$$P_l^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu) \tag{6}$$

is used to denote the associated Legendre functions, and the functions $R_l^m(\mu)$ and $T_l^m(\mu)$ are the combinations of generalized spherical functions defined and used in refs. 2 and 4. Here $\tau$ is the optical variable, $\mu$ is the direction cosine of the propagating radiation, $\omega$ is the single-scattering albedo and the scattering law is defined [2–6] by the "Greek constants" $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \varepsilon_l, \zeta_l\}$ so that

$$\boldsymbol{B}_l = \begin{vmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\varepsilon_l \\ 0 & 0 & \varepsilon_l & \delta_l \end{vmatrix}. \tag{7}$$

As we wish to consider that we have a finite layer, $\tau \in [0, \tau_0]$, illuminated on the surface $\tau = 0$ by a solar beam and bounded by a reflecting "ground" at $\tau = \tau_0$, we seek a solution of Eq. (1) subject to the boundary conditions, for $\mu \in [0, 1]$ and $\varphi \in [0, 2\pi]$,

$$\boldsymbol{I}(0, \mu, \varphi) = \pi\delta(\mu - \mu_0)\,\delta(\varphi - \varphi_0)\,\boldsymbol{F} \tag{8a}$$

and

$$\boldsymbol{I}(\tau_0, -\mu, \varphi) = \frac{\lambda_0}{\pi} \boldsymbol{L} \int_0^{2\pi} \int_0^1 \boldsymbol{I}(\tau_0, \mu', \varphi')\,\mu'\,d\mu'\,d\varphi' \tag{8b}$$

where $\lambda_0$ is the Lambert coefficient for reflection, $\boldsymbol{L} = \operatorname{diag}\{1, 0, 0, 0\}$ and the flux vector $\boldsymbol{F}$ has entries $F_I$, $F_Q$, $F_U$ and $F_V$ that are presumed given.

We now separate the density vector into unscattered and diffuse components by writing

$$\boldsymbol{I}(\tau, \mu, \varphi) = \pi\delta(\mu - \mu_0)\,\delta(\varphi - \varphi_0)\, e^{-\tau/\mu}\,\boldsymbol{F} + \boldsymbol{I}_*(\tau, \mu, \varphi). \tag{9}$$

Thus if we substitute Eq. (9) into Eqs. (1) and (8) we see that the diffuse field is defined by

$$\mu \frac{\partial}{\partial\tau} \boldsymbol{I}_*(\tau, \mu, \varphi) + \boldsymbol{I}_*(\tau, \mu, \varphi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \boldsymbol{P}(\mu, \mu', \varphi - \varphi')\,\boldsymbol{I}_*(\tau, \mu', \varphi')\,d\mu'\,d\varphi' + \boldsymbol{F}(\tau, \mu, \varphi), \tag{10}$$

where

$$\boldsymbol{F}(\tau, \mu, \varphi) = \frac{\omega}{4} \boldsymbol{P}(\mu, \mu_0, \varphi - \varphi_0)\, e^{-\tau/\mu_0}\,\boldsymbol{F}, \tag{11}$$

and the boundary conditions, for $\mu \in [0, 1]$ and $\varphi \in [0, 2\pi]$,

$$\boldsymbol{I}_*(0, \mu, \varphi) = \boldsymbol{0} \tag{12a}$$

and

$$\boldsymbol{I}_*(\tau_0, -\mu, \varphi) = \lambda_0 \mu_0 e^{-\tau_0/\mu_0} \boldsymbol{L}\boldsymbol{F} + \frac{\lambda_0}{\pi} \boldsymbol{L} \int_0^{2\pi} \int_0^1 \boldsymbol{I}_*(\tau_0, \mu', \varphi') \mu' \, \mathrm{d}\mu' \, \mathrm{d}\varphi' . \tag{12b}$$

If we integrate Eqs. (10), (11) and (12) over $\varphi$ from 0 to $2\pi$ and note Eq. (2), we conclude that the azimuthally symmetric component

$$\boldsymbol{I}_*(\tau, \mu) = \frac{1}{2\pi} \int_0^{2\pi} \boldsymbol{I}_*(\tau, \mu, \varphi) \, \mathrm{d}\varphi \tag{13}$$

satisfies [7] the equation of transfer

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}_*(\tau, \mu) + \boldsymbol{I}_*(\tau, \mu) = \frac{\omega}{2} \sum_{l=0}^{L} \boldsymbol{\Pi}_l(\mu) \boldsymbol{B}_l \int_{-1}^{1} \boldsymbol{\Pi}_l(\mu') \boldsymbol{I}_*(\tau, \mu') \, \mathrm{d}\mu' + \boldsymbol{F}(\tau, \mu), \tag{14}$$

where

$$\boldsymbol{F}(\tau, \mu) = \frac{\omega}{4} \sum_{l=0}^{L} \boldsymbol{\Pi}_l(\mu) \boldsymbol{B}_l \boldsymbol{\Pi}_l(\mu_0) e^{-\tau/\mu_0} \boldsymbol{F}, \tag{15}$$

and the boundary conditions, for $\mu \in [0, 1]$,

$$\boldsymbol{I}_*(0, \mu) = \boldsymbol{0} \tag{16a}$$

and

$$\boldsymbol{I}_*(\tau_0, -\mu) = \lambda_0 \mu_0 e^{-\tau_0/\mu_0} \boldsymbol{L}\boldsymbol{F} + 2 \lambda_0 \boldsymbol{L} \int_0^1 \boldsymbol{I}_*(\tau_0, \mu') \mu' \, \mathrm{d}\mu' . \tag{16b}$$

Here $\boldsymbol{\Pi}_l(\mu) = \boldsymbol{\Pi}_l^0(\mu) = \mathrm{diag}\{P_l(\mu), R_l(\mu), R_l(\mu), P_l(\mu)\}$ where $P_l(\mu)$ is used to denote the Legendre polynomial and

$$R_l(\mu) = R_l^0(\mu) = \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} (1 - \mu^2) \frac{\mathrm{d}^2}{\mathrm{d}\mu^2} P_l(\mu), \; l \geq 2 . \tag{17}$$

Although $\boldsymbol{I}_*(\tau, \mu)$ is a four-vector, it is apparent from Eqs. (7), (14), (15) and (16) that it is sufficient to investigate two two-vector problems. We thus write [7, 8]

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{\Psi}(\tau, \mu) + \boldsymbol{\Psi}(\tau, \mu) = \frac{\omega}{2} \sum_{l=0}^{L} \boldsymbol{P}_l(\mu) \boldsymbol{C}_l \int_{-1}^{1} \boldsymbol{P}_l(\mu') \boldsymbol{\Psi}(\tau, \mu') \, \mathrm{d}\mu' + \boldsymbol{S}(\tau, \mu), \tag{18}$$

where

$$\boldsymbol{S}(\tau, \mu) = \frac{\omega}{4} \sum_{l=0}^{L} \boldsymbol{P}_l(\mu) \boldsymbol{C}_l \boldsymbol{P}_l(\mu_0) e^{-\tau/\mu_0} \boldsymbol{\Gamma}, \tag{19}$$

and, for $\mu \in [0, 1]$,

$$\boldsymbol{\Psi}(0, \mu) = \boldsymbol{0} \tag{20a}$$

and

$$\boldsymbol{\Psi}(\tau_0, -\mu) = \lambda_0 \mu_0 e^{-\tau_0/\mu_0} \boldsymbol{P} \boldsymbol{\Gamma} + 2\lambda_0 \boldsymbol{P} \int_0^1 \boldsymbol{\Psi}(\tau_0, \mu') \mu' \, d\mu'. \tag{20b}$$

Here $\boldsymbol{P}_l(\mu) = \text{diag}\{P_l(\mu), R_l(\mu)\}$, and we consider two cases:

A: *The* $I - Q$ *Problem*:

$$\boldsymbol{\Psi}(\tau, \mu) = \begin{vmatrix} I_*(\tau, \mu) \\ Q_*(\tau, \mu) \end{vmatrix} \quad \text{and} \quad \boldsymbol{C}_l = \begin{vmatrix} \beta_l & \gamma_l \\ \gamma_l & \alpha_l \end{vmatrix}, \tag{21 a, b}$$

with

$$\boldsymbol{\Gamma} = \begin{vmatrix} F_I \\ F_Q \end{vmatrix} \quad \text{and} \quad \boldsymbol{P} = \text{diag}\{1, 0\}, \tag{21 c, d}$$

and

B: *The* $V - U$ *Problem*:

$$\boldsymbol{\Psi}(\tau, \mu) = \begin{vmatrix} V_*(\tau, \mu) \\ U_*(\tau, \mu) \end{vmatrix} \quad \text{and} \quad \boldsymbol{C}_l = \begin{vmatrix} \delta_l & \varepsilon_l \\ -\varepsilon_l & \zeta_l \end{vmatrix}, \tag{22 a, b}$$

with

$$\boldsymbol{\Gamma} = \begin{vmatrix} F_V \\ F_U \end{vmatrix} \quad \text{and} \quad \boldsymbol{P} = \boldsymbol{0}. \tag{22 c, d}$$

## II. Basic Solution

We now proceed to develop our generalized spherical harmonics solution of Eq. (18) subject to the boundary conditions given by Eqs. (20). Following Chandrasekhar [1], Deuze, Devaux and Herman [9] and a recent paper [10] in which the analytical and computational aspects of the classical spherical harmonics method, relevant to the case where polarization effects can be ignored, were reviewed in detail, we let

$$\boldsymbol{\Psi}(\tau, \mu) = \boldsymbol{\Psi}_h(\tau, \mu) + \boldsymbol{\Psi}_p(\tau, \mu), \tag{23}$$

consider $N$ to be odd and write

$$\boldsymbol{\Psi}_h(\tau, \mu) = \sum_{l=0}^{N} \left(\frac{2l+1}{2}\right) \boldsymbol{P}_l(\mu) \sum_{j=1}^{N} [A_j e^{-\tau/\xi_j} + (-1)^l B_j e^{-(\tau_0-\tau)/\xi_j}] \boldsymbol{G}_l(\xi_j) \boldsymbol{M}(\xi_j) \tag{24a}$$

and

$$\boldsymbol{\Psi}_p(\tau, \mu) = \frac{1}{2} \sum_{l=0}^{N} \left(\frac{2l+1}{2}\right) \boldsymbol{P}_l(\mu) [\boldsymbol{G}_l(\mu_0) \gamma - \boldsymbol{P}_l(\mu_0)] e^{-\tau/\mu_0} \boldsymbol{\Gamma}. \tag{24b}$$

Here the $2 \times 2$ matrices $\boldsymbol{G}_l(\xi)$ are defined [7, 8] by the recursion relation

$$\xi \boldsymbol{h}_l \boldsymbol{G}_l(\xi) = \boldsymbol{J}_{l+1} \boldsymbol{G}_{l+1}(\xi) + \boldsymbol{J}_l \boldsymbol{G}_{l-1}(\xi) \tag{25}$$

and the starting values

$$G_0(\xi) = \text{diag}\,\{1, 0\} \tag{26a}$$

$$G_1(\xi) = \text{diag}\,\{k_0\,\xi, 0\} \tag{26b}$$

and

$$G_2(\xi) = \text{diag}\,\{\tfrac{1}{2}(k_0 k_1 \xi^2 - 1), 1\}\,. \tag{26c}$$

In addition

$$J_l = \text{diag}\,\{l, (1-\delta_{0,l})(1-\delta_{1,l})(l^2-4)^{1/2}\} \tag{27}$$

$$h_l = (2l+1)\,I - \omega\,C_l \tag{28}$$

and

$$k_l = 2l + 1 - \omega\,C_l^{11}\,. \tag{29}$$

We note that $\Psi_h(\tau, \mu)$, with $\{A_j\}$ and $\{B_j\}$ arbitrary, will satisfy the first $N+1$ moments of the homogeneous version of Eq. (18), *i.e.*

$$\int_{-1}^{1} P_\alpha(\mu)\left[\mu\frac{\partial}{\partial\tau}\Psi_h(\tau,\mu) + \Psi_h(\tau,\mu) - \frac{\omega}{2}\sum_{l=0}^{L} P_l(\mu)\,C_l \int_{-1}^{1} P_l(\mu')\,\Psi_h(\tau,\mu')\,\mathrm{d}\mu'\right]\mathrm{d}\mu = \mathbf{0} \tag{30}$$

for $\alpha = 0, 1, \ldots, N$, if the eigenvalues $\{\xi_j\}$ and the vector $M(\xi_j)$ are defined, for $N \geq 3$, by

$$G_{N+1}(\xi_j)\,M(\xi_j) = \mathbf{0}, \qquad j = 1, 2, \ldots, N\,. \tag{31}$$

Recalling a previous work [8] relevant to eigenvalue calculations, we note that det $G_{N+1}(\xi)$ has, for $N$ odd, precisely $2N$ zeros which occur in $\pm$ pairs; we let $\{\xi_j\}$ denote the zeros of det $G_{N+1}(\xi)$ in the right half-plane. To complete the specification of our generalized spherical harmonics solution of Eq. (18) we note that $\Psi_p(\tau, \mu)$ is a particular solution of the first $N+1$ moments of Eq. (18) provided the matrix $\gamma$ is given by

$$\gamma = G_{N+1}^{-1}(\mu_0)\,P_{N+1}(\mu_0)\,. \tag{32}$$

In order to keep our presentation brief, we do not give, as was done previously [10] for the scalar case, the version of $\Psi_p(\tau, \mu)$ required in the event $G_{N+1}(\mu_0)$ is singular or the modifications of $\Psi_h(\tau, \mu)$ required (for the $I - Q$ problem) for the special case $\omega = 1$.

Deferring to the next section our discussion of the computational methods we use to find the eigenvalues $\{\xi_j\}$ and the vectors

$$T_l(\xi_j) = G_l(\xi_j)\,M(\xi_j)\,, \tag{33}$$

we now proceed to use the boundary conditions to find the constants $\{A_j\}$ and $\{B_j\}$ required in Eq. (24a). To obtain a linear system for $\{A_j, B_j\}$ we substitute Eqs. (23) and (24) into Eqs. (20), multiply the resulting equations by

$$\boldsymbol{W}_\alpha(\mu) = \operatorname{diag}\{P_{2\alpha+1}(\mu), (1-\delta_{0,\alpha})\, R_{2\alpha+1}(\mu)\}, \tag{34}$$

for $\alpha = 0, 1, 2, \ldots, (N-1)/2$, and integrate over $\mu$ from 0 to 1 to obtain

$$\sum_{l=0}^{N} (2l+1)\, \boldsymbol{C}_{\alpha,l} \sum_{j=1}^{N} [A_j + (-1)^l B_j\, e^{-\tau_0/\xi_j}]\, \boldsymbol{T}_l(\xi_j) = \boldsymbol{R}_{1,\alpha} \tag{35a}$$

and

$$\sum_{l=0}^{N} (2l+1)\, [\boldsymbol{C}_{\alpha,l} - 2(-1)^l \lambda_0 S_{0,l} S_{\alpha,0} \boldsymbol{P}] \sum_{j=1}^{N} [B_j + (-1)^l A_j\, e^{-\tau_0/\xi_j}]\, \boldsymbol{T}_l(\xi_j) = \boldsymbol{R}_{2,\alpha} \tag{35b}$$

where

$$\boldsymbol{R}_{1,\alpha} = -\frac{1}{2} \sum_{l=0}^{N} (2l+1)\, \boldsymbol{C}_{\alpha,l}\, [\boldsymbol{G}_l(\mu_0)\, \gamma - \boldsymbol{P}_l(\mu_0)]\, \boldsymbol{\Gamma} \tag{36a}$$

and

$$\boldsymbol{R}_{2,\alpha} = \lambda_0 S_{\alpha,0}\, \boldsymbol{R} - \frac{1}{2} \sum_{l=0}^{N} (2l+1)(-1)^l\, \boldsymbol{C}_{\alpha,l}\, [\boldsymbol{G}_l(\mu_0)\, \gamma - \boldsymbol{P}_l(\mu_0)]\, \boldsymbol{\Gamma}\, e^{-\tau_0/\mu_0} \tag{36b}$$

with

$$\boldsymbol{R} = \boldsymbol{P} \left\{ 2\mu_0 + \sum_{l=0}^{N} (2l+1)\, S_{0,l}\, [\boldsymbol{G}_l(\mu_0)\, \gamma - \boldsymbol{P}_l(\mu_0)] \right\} \boldsymbol{\Gamma}\, e^{-\tau_0/\mu_0}. \tag{37}$$

Here we have defined

$$\boldsymbol{C}_{\alpha,l} = \int_0^1 \boldsymbol{W}_\alpha(\mu)\, \boldsymbol{P}_l(\mu)\, d\mu = \operatorname{diag}\{S_{\alpha,l}, (1-\delta_{\alpha,0})(1-\delta_{l,0})(1-\delta_{l,1})\, D_{\alpha,l}\}. \tag{38}$$

Considering that we can solve Eqs. (35) to find $\{A_j, B_j\}$, we make a final iteration by substituting Eqs. (23) and (24) into the right-hand side of Eq. (18) and solving the resulting equation to obtain our final result, *viz.*

$$\boldsymbol{\Psi}(\tau, -\mu) = \boldsymbol{K}(\tau, -\mu) + \frac{\omega}{2} [\mu_0\, e^{-\tau/\mu_0}\, \boldsymbol{S}(\tau_0 - \tau : \mu, \mu_0)\, \boldsymbol{\Xi}(-\mu) + \boldsymbol{\Upsilon}(\tau, -\mu)], \tag{39a}$$

and

$$\boldsymbol{\Psi}(\tau, \mu) = \frac{\omega}{2}[\mu_0 C(\tau : \mu, \mu_0) \boldsymbol{\Xi}(\mu) + \boldsymbol{\Upsilon}(\tau, \mu)] \tag{39b}$$

where $S(x : \mu, \mu_0)$ and $C(x : \mu, \mu_0)$ are defined in ref. 10,

$$\boldsymbol{\Xi}(-\mu) = \frac{1}{2}\left[\sum_{l=0}^{N} (-1)^l \boldsymbol{P}_l(\mu) \boldsymbol{C}_l \boldsymbol{G}_l(\mu_0) \boldsymbol{\gamma} + \sum_{l=N+1}^{L} (-1)^l \boldsymbol{P}_l(\mu) \boldsymbol{C}_l \boldsymbol{P}_l(\mu_0)\right] \boldsymbol{\Gamma}, \tag{40a}$$

$$\boldsymbol{\Xi}(\mu) = \frac{1}{2}\left[\sum_{l=0}^{N} \boldsymbol{P}_l(\mu) \boldsymbol{C}_l \boldsymbol{G}_l(\mu_0) \boldsymbol{\gamma} + \sum_{l=N+1}^{L} \boldsymbol{P}_l(\mu) \boldsymbol{C}_l \boldsymbol{P}_l(\mu_0)\right] \boldsymbol{\Gamma}, \tag{40b}$$

$$\boldsymbol{\Upsilon}(\tau, -\mu) = \sum_{l=0}^{N} \boldsymbol{P}_l(\mu) \boldsymbol{C}_l \sum_{j=1}^{N} \xi_j [(-1)^l A_j e^{-\tau/\xi_j} S(\tau_0 - \tau : \mu, \xi_j) + B_j C(\tau_0 - \tau : \mu, \xi_j)] \boldsymbol{T}_l(\xi_j), \tag{41a}$$

$$\boldsymbol{\Upsilon}(\tau, \mu) = \sum_{l=0}^{N} \boldsymbol{P}_l(\mu) \boldsymbol{C}_l \sum_{j=1}^{N} \xi_j [A_j C(\tau : \mu, \xi_j) + (-1)^l B_j e^{-(\tau_0 - \tau)/\xi_j} S(\tau : \mu, \xi_j)] \boldsymbol{T}_l(\xi_j), \tag{41b}$$

and

$$\boldsymbol{K}(\tau, -\mu) = \lambda_0 (\mu_0 \boldsymbol{P} \boldsymbol{\Gamma} e^{-\tau_0/\mu_0} + \boldsymbol{W}) e^{-(\tau_0 - \tau)/\mu}, \tag{42}$$

with

$$\boldsymbol{W} = \sum_{l=0}^{N} (2l+1) S_{0,l} \left\{ \sum_{j=1}^{N} [A_j e^{-\tau_0/\xi_j} + (-1)^l B_j] \mathbf{P} \boldsymbol{T}_l(\xi_j) + \frac{1}{2} \boldsymbol{P} [\boldsymbol{G}_l(\mu_0) \boldsymbol{\gamma} - \boldsymbol{P}_l(\mu_0)] \boldsymbol{\Gamma} e^{-\tau_0/\mu_0} \right\}. \tag{43}$$

As Eqs. (39) are our final analytical results, we proceed to discuss the methods we use to evaluate the developed solution numerically.

## III. Computational methods

We consider the first step in our evaluation of the solution given by Eqs. (39) to be the calculation of the eigenvalues $\{\xi_j\}$ defined as the $N$ zeros, in the right half-plane, of det $\boldsymbol{G}_{N+1}(\xi)$; however, as this calculation has been discussed, in detail, for both the $I - Q$ problem and the $V - U$ problem in a foregoing work [8], we proceed to the determination of the $2 \times 2$ polynomials $\boldsymbol{G}_l(\xi)$, for $|\xi| \leq 1$, and the two-vectors $\boldsymbol{T}_l(\xi_j)$, for $|\xi_j| > 1$. We note [8] that for the $I - Q$ problem all of the eigenvalues $\{\xi_j\}$ are real; however, for the $V - U$ problem some of the eigenvalues $|\xi_j| < 1$ are, in general and for finite $N$, complex. The $V - U$ problem can also have complex eigenvalues $|\xi_j| > 1$ even in the limit $N \to \infty$.

In order to compute the two-vectors $\boldsymbol{T}_l(\xi_j)$ for $|\xi_j| \le 1$ we use the three starting polynomials $\boldsymbol{G}_0(\xi)$, $\boldsymbol{G}_1(\xi)$ and $\boldsymbol{G}_2(\xi)$, for $\xi = \xi_j$, given by Eqs. (26) and deduce $\boldsymbol{G}_l(\xi_j)$, $l = 3, 4, \dots, N+1$, by using the recursion relation, given by Eq. (25), in the forward direction. We then use the normalization

$$\boldsymbol{M}(\xi_j) = \begin{vmatrix} G^{22}_{N+1}(\xi_j) + G^{12}_{N+1}(\xi_j) \\ -G^{21}_{N+1}(\xi_j) - G^{11}_{N+1}(\xi_j) \end{vmatrix}, \tag{44}$$

for $|\xi_j| \le 1$ and take

$$\boldsymbol{T}_l(\xi_j) = \boldsymbol{G}_l(\xi_j)\,\boldsymbol{M}(\xi_j). \tag{45}$$

As we have found that, in general, the foregoing scheme for computing the vectors $\boldsymbol{T}_l(\xi_j)$ is unstable for $|\xi_j| > 1$, we use a generalization of the backward recursion procedure discussed by Gautschi [11]. We let

$$\boldsymbol{T}_{l+1}(\xi_j) = \boldsymbol{R}_l(\xi_j)\,\boldsymbol{T}_l(\xi_j) \tag{46}$$

for $l = 2, 3, \dots, N$, define $\boldsymbol{R}_N(\xi_j) = \boldsymbol{0}$ and deduce from Eqs. (25), (31) and (33) that

$$\boldsymbol{R}_{l-1}(\xi_j) = [\xi_j \boldsymbol{h}_l - \boldsymbol{J}_{l+1}\,\boldsymbol{R}_l(\xi_j)]^{-1}\,\boldsymbol{J}_l \tag{47}$$

for $l = N, N-1, \dots, 3$. Thus we take

$$\boldsymbol{T}_0(\xi_j) = \begin{vmatrix} 1 \\ 0 \end{vmatrix}, \quad \boldsymbol{T}_1(\xi_j) = k_0\,\xi_j \begin{vmatrix} 1 \\ 0 \end{vmatrix} \tag{48a, b}$$

and

$$\boldsymbol{T}_2(\xi_j) = [\xi_j \boldsymbol{h}_2 - \boldsymbol{J}_3\,\boldsymbol{R}_2(\xi_j)]^{-1}\,\boldsymbol{J}_2\,\boldsymbol{T}_1(\xi_j) \tag{48c}$$

and compute the remaining vectors $\boldsymbol{T}_l(\xi_j)$, $l = 3, 4, \dots, N$, from Eq. (46).

In order to complete this section we note that the constants $S_{\alpha,l}$ and $D_{\alpha,l}$ required to establish, by way of Eq. (38), the matrices $\boldsymbol{C}_{\alpha,l}$ can be computed conveniently from recursion formulas. First of all, the non-zero elements of $\{S_{\alpha,l}\}$, for $\alpha, l = 0, 1, 2, \dots$ can be found from [10]

$$S_{\alpha,2\alpha+1} = \frac{1}{4\alpha+3} \tag{49a}$$

$$S_{\alpha,l+2} = \left(\frac{1-l+2\alpha}{4+l+2\alpha}\right)\left(\frac{l+1}{l+1-2\alpha}\right)\left(\frac{l+2+2\alpha}{l+2}\right) S_{\alpha,l} \tag{49b}$$

$$S_{\alpha+1,0} = -\frac{1}{2}\left(\frac{2\alpha+1}{\alpha+2}\right) S_{\alpha,0} \tag{49c}$$

and

$$S_{0,0} = \frac{1}{2}. \tag{49d}$$

Since

$$D_{\alpha,l} = \left[\frac{(2\alpha-1)!\,(l-2)!}{(2\alpha+3)!\,(l+2)!}\right]^{1/2} \int_0^1 P_{2\alpha+1}^2(\mu)\, P_l^2(\mu)\,\mathrm{d}\mu, \tag{50}$$

we find that ref. 10 provides a set of formulas that is adequate to compute all non-zero elements of $\{D_{\alpha,l}\}$ for $\alpha = 1, 2, \ldots$ and $l = 2, 3, \ldots$:

$$D_{\alpha,2\alpha+1} = \frac{1}{4\alpha+3}, \tag{51 a}$$

$$D_{\alpha,l+2} = \left(\frac{1-l+2\alpha}{4+l+2\alpha}\right)\left(\frac{l+2+2\alpha}{l+1-2\alpha}\right)\left[\frac{(l+3)(l-1)}{l(l+4)}\right]^{1/2} D_{\alpha,l} \tag{51 b}$$

$$D_{\alpha+1,2} = -\frac{1}{2}\left(\frac{2\alpha-1}{\alpha+3}\right)\left[\frac{(\alpha+2)(2\alpha+5)}{\alpha(2\alpha+1)}\right]^{1/2} D_{\alpha,2} \tag{51 c}$$

and

$$D_{1,2} = \frac{\sqrt{5}}{16}. \tag{51 d}$$

Since we now are able to compute accurately all quantities required to define the system of linear algebraic equations given by Eqs. (35), we can solve that system to find the constants $\{A_j, B_j\}$ that are needed to complete the solution given by Eqs. (39).

## IV. Numerical Results

In order to demonstrate the accuracy of the developed solution we now wish to report some numerical results for several selected cases. We consider here two phase matrices the "Greek constants" for which were computed by de Rooij and van der Stap [5, 12] and reported by Vestrucci and Siewert [6]. In order to avoid retabulating these constants $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \varepsilon_l, \zeta_l\}$ we consider that ref. 6 is available and note that

phase matrix I $\Rightarrow$ problem II of ref. 6 ($L = 13$)
phase matrix II $\Rightarrow$ problem III of ref. 6 ($L = 60$).

Rather than consider a flux vector

$$\boldsymbol{F} = \begin{vmatrix} F_I \\ F_Q \\ F_U \\ F_V \end{vmatrix}, \tag{52}$$

we prefer to follow the suggestion of de Haan and Hovenier [13] and consider the following four cases that span the space $\mathbb{R}^4$:

$$\boldsymbol{F}_1 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix}, \quad \boldsymbol{F}_2 = \begin{vmatrix} 1 \\ 1 \\ 0 \\ 0 \end{vmatrix}, \quad \boldsymbol{F}_3 = \begin{vmatrix} 1 \\ 0 \\ 1 \\ 0 \end{vmatrix}, \quad \text{and} \quad \boldsymbol{F}_4 = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 1 \end{vmatrix}. \tag{53}$$

Thus for the $I - Q$ problem, we consider

$$\boldsymbol{\Gamma}_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \quad \text{and} \quad \boldsymbol{\Gamma}_2 = \begin{vmatrix} 1 \\ 1 \end{vmatrix}, \tag{54a}$$

whereas for the $V - U$ problem we consider

$$\boldsymbol{\Gamma}_1 = \begin{vmatrix} 1 \\ 0 \end{vmatrix} \quad \text{and} \quad \boldsymbol{\Gamma}_2 = \begin{vmatrix} 0 \\ 1 \end{vmatrix}. \tag{54b}$$

Now upon reviewing the formulation given in section I and utilizing the explicit expressions for the matrices $\boldsymbol{\Pi}_l^m(\mu)$ given in ref. 4, we find for the case of a normally incident beam, $\mu_0 = 1$, that there are two cases when the solution for the diffuse field $\boldsymbol{I}_*(\tau, \mu, \varphi)$ is independent of $\varphi$ so that the result developed herein for the azimuthally symmetric component $\boldsymbol{I}_*(\tau, \mu)$ is adequate to construct the complete diffuse field. These two cases correspond to $\boldsymbol{F} = \boldsymbol{F}_1$ and $\boldsymbol{F} = \boldsymbol{F}_4$ or alternatively to $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_1$ for both the $I - Q$ problem and the $V - U$ problem. These cases therefore are the ones we consider here.

The "Greek constants" reported in ref. 6 were calculated for the case of nonabsorbing particles, $\omega = 1$; however to allow some absorption and to avoid recomputing the constants, we compromise slightly the physics of our problem and use the constants of ref. 6 with $\omega = 0.99$.

In Tables 1–8 we report, for the case $\lambda_0 = 0.1$, results for the $I - Q$ problem relevant to the two considered phase matrices and for two layer thicknesses, $\tau_0$. Analogous results are reported in Tables 9–16 for the $V - U$ problem. We note that the results in Tables 1–16 were obtained with a maximum value of $N = 399$ and that they are thought to be correct to within $\pm 1$ in the last digits reported.

Finally we note that for the $V - U$ problem where some of the eigenvalues $\{\xi_j\}$ and associated vectors $\boldsymbol{T}_l(\xi_j)$ are complex, we chose to arrange our calculation in such a way that everything could be done in the real mode (on an IBM-3081) rather than to use the code in a complex mode.

Table 1
The diffuse intensity $I_*(\tau, \mu, \varphi)$ for phase matrix $I$ with $F_I = 1$, $F_Q = 0$, $\omega = 0.99$, $\lambda_0 = 0.1$ and $\tau_0 = 1$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 2.0256(−1) | 1.9821(−1) | 1.9326(−1) | 1.8219(−1) | 1.4409(−1) | 1.1159(−1) | 8.3790(−2) |
| −0.9 | 2.0836(−1) | 2.0451(−1) | 1.9993(−1) | 1.8927(−1) | 1.5032(−1) | 1.1522(−1) | 8.3790(−2) |
| −0.8 | 2.1545(−1) | 2.1224(−1) | 2.0814(−1) | 1.9805(−1) | 1.5825(−1) | 1.1996(−1) | 8.3790(−2) |
| −0.7 | 2.2399(−1) | 2.2161(−1) | 2.1814(−1) | 2.0886(−1) | 1.6837(−1) | 1.2624(−1) | 8.3790(−2) |
| −0.6 | 2.3405(−1) | 2.3278(−1) | 2.3020(−1) | 2.2212(−1) | 1.8140(−1) | 1.3469(−1) | 8.3790(−2) |
| −0.5 | 2.4550(−1) | 2.4577(−1) | 2.4444(−1) | 2.3823(−1) | 1.9834(−1) | 1.4634(−1) | 8.3790(−2) |
| −0.4 | 2.5771(−1) | 2.6013(−1) | 2.6063(−1) | 2.5732(−1) | 2.2055(−1) | 1.6297(−1) | 8.3790(−2) |
| −0.3 | 2.6883(−1) | 2.7430(−1) | 2.7746(−1) | 2.7868(−1) | 2.4960(−1) | 1.8775(−1) | 8.3790(−2) |
| −0.2 | 2.7478(−1) | 2.8441(−1) | 2.9124(−1) | 2.9908(−1) | 2.8587(−1) | 2.2654(−1) | 8.3790(−2) |
| −0.1 | 2.6972(−1) | 2.8466(−1) | 2.9590(−1) | 3.1156(−1) | 3.2198(−1) | 2.8648(−1) | 8.3790(−2) |
| −0.0 | 2.44 (−1) | 2.7074(−1) | 2.8835(−1) | 3.1317(−1) | 3.4336(−1) | 3.3437(−1) | 8.3790(−2) |
| 0.0 | | 2.7074(−1) | 2.8835(−1) | 3.1317(−1) | 3.4336(−1) | 3.3437(−1) | 2.85 (−1) |
| 0.1 | | 1.0899(−1) | 1.8344(−1) | 2.7098(−1) | 3.5449(−1) | 3.6060(−1) | 3.3254(−1) |
| 0.2 | | 6.6517(−2) | 1.2308(−1) | 2.1061(−1) | 3.4333(−1) | 3.7520(−1) | 3.6512(−1) |
| 0.3 | | 5.1224(−2) | 9.7852(−2) | 1.7719(−1) | 3.2569(−1) | 3.7875(−1) | 3.8722(−1) |
| 0.4 | | 4.4370(−2) | 8.5936(−2) | 1.5978(−1) | 3.1364(−1) | 3.8043(−1) | 4.0388(−1) |
| 0.5 | | 4.1390(−2) | 8.0659(−2) | 1.5191(−1) | 3.0933(−1) | 3.8550(−1) | 4.2037(−1) |
| 0.6 | | 4.0669(−2) | 7.9439(−2) | 1.5050(−1) | 3.1260(−1) | 3.9620(−1) | 4.4005(−1) |
| 0.7 | | 4.1525(−2) | 8.1138(−2) | 1.5402(−1) | 3.2308(−1) | 4.1360(−1) | 4.6501(−1) |
| 0.8 | | 4.3652(−2) | 8.5237(−2) | 1.6179(−1) | 3.4061(−1) | 4.3843(−1) | 4.9677(−1) |
| 0.9 | | 4.6928(−2) | 9.1531(−2) | 1.7353(−1) | 3.6535(−1) | 4.7142(−1) | 5.3665(−1) |
| 1 | | 5.1333(−2) | 9.9995(−2) | 1.8926(−1) | 3.9772(−1) | 5.1345(−1) | 5.8596(−1) |

Table 2
The Stokes parameter $Q(\tau, \mu, \varphi)$ for phase matrix $I$ with $F_I = 1$, $F_Q = 0$, $\omega = 0.99$, $\lambda_0 = 0.1$ and $\tau_0 = 1$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | |
| −0.9 | −1.0667(−2) | −1.0347(−2) | −9.9643(−3) | −9.0819(−3) | −5.9125(−3) | −2.9697(−3) | |
| −0.8 | −2.2413(−2) | −2.1791(−2) | −2.1033(−2) | −1.9258(−2) | −1.2728(−2) | −6.4943(−3) | |
| −0.7 | −3.5399(−2) | −3.4511(−2) | −3.3400(−2) | −3.0746(−2) | −2.0691(−2) | −1.0759(−2) | |
| −0.6 | −4.9778(−2) | −4.8689(−2) | −4.7272(−2) | −4.3798(−2) | −3.0132(−2) | −1.6043(−2) | |
| −0.5 | −6.5631(−2) | −6.4454(−2) | −6.2825(−2) | −5.8672(−2) | −4.1507(−2) | −2.2791(−2) | |
| −0.4 | −8.2839(−2) | −8.1767(−2) | −8.0089(−2) | −7.5545(−2) | −5.5419(−2) | −3.1745(−2) | |
| −0.3 | −1.0079(−1) | −1.0013(−1) | −9.8669(−2) | −9.4237(−2) | −7.2555(−2) | −4.4220(−2) | |
| −0.2 | −1.1795(−1) | −1.1809(−1) | −1.1720(−1) | −1.1357(−1) | −9.3095(−2) | −6.2596(−2) | |
| −0.1 | −1.3182(−1) | −1.3330(−1) | −1.3329(−1) | −1.3086(−1) | −1.1407(−1) | −8.9614(−2) | |
| −0.0 | −1.37 (−1) | −1.4324(−1) | −1.4515(−1) | −1.4474(−1) | −1.2993(−1) | −1.1134(−1) | |
| 0.0 | | −1.4324(−1) | −1.4515(−1) | −1.4474(−1) | −1.2993(−1) | −1.1134(−1) | −8.60 (−2) |
| 0.1 | | −5.9370(−2) | −9.6530(−2) | −1.3291(−1) | −1.4183(−1) | −1.2498(−1) | −1.0243(−1) |
| 0.2 | | −3.5069(−2) | −6.2939(−2) | −1.0144(−1) | −1.3866(−1) | −1.3206(−1) | −1.1364(−1) |
| 0.3 | | −2.5115(−2) | −4.6675(−2) | −8.0121(−2) | −1.2599(−1) | −1.2931(−1) | −1.1759(−1) |
| 0.4 | | −1.9395(−2) | −3.6666(−2) | −6.5015(−2) | −1.1103(−1) | −1.2027(−1) | −1.1454(−1) |
| 0.5 | | −1.5388(−2) | −2.9372(−2) | −5.3084(−2) | −9.5540(−2) | −1.0752(−1) | −1.0611(−1) |
| 0.6 | | −1.2148(−2) | −2.3325(−2) | −4.2668(−2) | −7.9558(−2) | −9.2047(−2) | −9.3337(−2) |
| 0.7 | | −9.2214(−3) | −1.7772(−2) | −3.2770(−2) | −6.2639(−2) | −7.3975(−2) | −7.6602(−2) |
| 0.8 | | −6.3403(−3) | −1.2248(−2) | −2.2709(−2) | −4.4192(−2) | −5.3009(−2) | −5.5803(−2) |
| 0.9 | | −3.3176(−3) | −6.4190(−3) | −1.1946(−2) | −2.3555(−2) | −2.8598(−2) | −3.0503(−2) |
| 1 | | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

Table 3
The diffuse intensity $I_*(\tau, \mu, \varphi)$ for phase matrix $I$ with $F_I = 1$, $F_Q = 0$, $\omega = 0.99$, $\lambda_0 = 0.1$ and $\tau_0 = 10$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 6.7974(−1) | 6.8078(−1) | 6.5529(−1) | 5.7798(−1) | 3.2977(−1) | 1.5208(−1) | 1.9432(−2) |
| −0.9 | 6.7161(−1) | 6.8334(−1) | 6.6367(−1) | 5.9147(−1) | 3.4370(−1) | 1.6316(−1) | 1.9432(−2) |
| −0.8 | 6.6135(−1) | 6.8488(−1) | 6.7160(−1) | 6.0503(−1) | 3.5793(−1) | 1.7495(−1) | 1.9432(−2) |
| −0.7 | 6.4859(−1) | 6.8524(−1) | 6.7899(−1) | 6.1861(−1) | 3.7243(−1) | 1.8739(−1) | 1.9432(−2) |
| −0.6 | 6.3288(−1) | 6.8421(−1) | 6.8573(−1) | 6.3221(−1) | 3.8719(−1) | 2.0039(−1) | 1.9432(−2) |
| −0.5 | 6.1367(−1) | 6.8155(−1) | 6.9171(−1) | 6.4580(−1) | 4.0220(−1) | 2.1381(−1) | 1.9432(−2) |
| −0.4 | 5.9023(−1) | 6.7698(−1) | 6.9677(−1) | 6.5933(−1) | 4.1747(−1) | 2.2749(−1) | 1.9432(−2) |
| −0.3 | 5.6154(−1) | 6.7017(−1) | 7.0078(−1) | 6.7279(−1) | 4.3301(−1) | 2.4131(−1) | 1.9432(−2) |
| −0.2 | 5.2595(−1) | 6.6068(−1) | 7.0352(−1) | 6.8612(−1) | 4.4885(−1) | 2.5523(−1) | 1.9432(−2) |
| −0.1 | 4.8006(−1) | 6.4790(−1) | 7.0477(−1) | 6.9930(−1) | 4.6501(−1) | 2.6931(−1) | 1.9432(−2) |
| −0.0 | 4.07 (−1) | 6.3084(−1) | 7.0419(−1) | 7.1226(−1) | 4.8151(−1) | 2.8359(−1) | 1.9432(−2) |
| 0.0 | | 6.3084(−1) | 7.0419(−1) | 7.1226(−1) | 4.8151(−1) | 2.8359(−1) | 8.80 (−2) |
| 0.1 | | 6.0439(−1) | 7.0118(−1) | 7.2494(−1) | 4.9837(−1) | 2.9810(−1) | 1.0773(−1) |
| 0.2 | | 5.4862(−1) | 6.9228(−1) | 7.3722(−1) | 5.1562(−1) | 3.1286(−1) | 1.2393(−1) |
| 0.3 | | 4.9020(−1) | 6.7400(−1) | 7.4851(−1) | 5.3328(−1) | 3.2788(−1) | 1.3934(−1) |
| 0.4 | | 4.4589(−1) | 6.5290(−1) | 7.5812(−1) | 5.5139(−1) | 3.4320(−1) | 1.5442(−1) |
| 0.5 | | 4.1664(−1) | 6.3575(−1) | 7.6679(−1) | 5.6998(−1) | 3.5883(−1) | 1.6935(−1) |
| 0.6 | | 4.0050(−1) | 6.2620(−1) | 7.7646(−1) | 5.8909(−1) | 3.7481(−1) | 1.8425(−1) |
| 0.7 | | 3.9569(−1) | 6.2605(−1) | 7.8945(−1) | 6.0884(−1) | 3.9117(−1) | 1.9918(−1) |
| 0.8 | | 4.0103(−1) | 6.3630(−1) | 8.0799(−1) | 6.2946(−1) | 4.0796(−1) | 2.1421(−1) |
| 0.9 | | 4.1593(−1) | 6.5776(−1) | 8.3416(−1) | 6.5139(−1) | 4.2524(−1) | 2.2938(−1) |
| 1 | | 4.4028(−1) | 6.9127(−1) | 8.6995(−1) | 6.7525(−1) | 4.4317(−1) | 2.4475(−1) |

Table 4
The Stokes parameter $Q(\tau, \mu, \varphi)$ for phase matrix $I$ with $F_I = 1$, $F_Q = 0$, $\omega = 0.99$, $\lambda_0 = 0.1$ and $\tau_0 = 10$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | |
| −0.9 | −1.4668(−2) | −1.1782(−2) | −8.6955(−3) | −4.4601(−3) | −8.4495(−4) | −5.3608(−4) | |
| −0.8 | −2.9844(−2) | −2.4061(−2) | −1.7763(−2) | −9.0772(−3) | −1.6620(−3) | −1.0111(−3) | |
| −0.7 | −4.5536(−2) | −3.6873(−2) | −2.7232(−2) | −1.3867(−2) | −2.4533(−3) | −1.4227(−3) | |
| −0.6 | −6.1731(−2) | −5.0246(−2) | −3.7133(−2) | −1.8844(−2) | −3.2207(−3) | −1.7700(−3) | |
| −0.5 | −7.8369(−2) | −6.4197(−2) | −4.7492(−2) | −2.4024(−2) | −3.9652(−3) | −2.0547(−3) | |
| −0.4 | −9.5316(−2) | −7.8722(−2) | −5.8325(−2) | −2.9421(−2) | −4.6868(−3) | −2.2832(−3) | |
| −0.3 | −1.1229(−1) | −9.3773(−2) | −6.9631(−2) | −3.5043(−2) | −5.3839(−3) | −2.4671(−3) | |
| −0.2 | −1.2870(−1) | −1.0923(−1) | −8.1379(−2) | −4.0893(−2) | −6.0537(−3) | −2.6191(−3) | |
| −0.1 | −1.4327(−1) | −1.2482(−1) | −9.3477(−2) | −4.6960(−2) | −6.6930(−3) | −2.7445(−3) | |
| −0.0 | −1.51 (−1) | −1.3997(−1) | −1.0572(−1) | −5.3206(−2) | −7.2975(−3) | −2.8399(−3) | |
| 0.0 | | −1.3997(−1) | −1.0572(−1) | −5.3206(−2) | −7.2975(−3) | −2.8399(−3) | −5.79 (−3) |
| 0.1 | | −1.5210(−1) | −1.1765(−1) | −5.9543(−2) | −7.8614(−3) | −2.9009(−3) | −4.8455(−3) |
| 0.2 | | −1.4836(−1) | −1.2700(−1) | −6.5765(−2) | −8.3761(−3) | −2.9225(−3) | −4.2147(−3) |
| 0.3 | | −1.3446(−1) | −1.2945(−1) | −7.1212(−2) | −8.8275(−3) | −2.8995(−3) | −3.6575(−3) |
| 0.4 | | −1.1816(−1) | −1.2485(−1) | −7.4522(−2) | −9.1884(−3) | −2.8256(−3) | −3.1376(−3) |
| 0.5 | | −1.0136(−1) | −1.1479(−1) | −7.4406(−2) | −9.3976(−3) | −2.6926(−3) | −2.6381(−3) |
| 0.6 | | −8.4122(−2) | −1.0033(−1) | −7.0057(−2) | −9.3217(−3) | −2.4874(−3) | −2.1469(−3) |
| 0.7 | | −6.6007(−2) | −8.1882(−2) | −6.0955(−2) | −8.7276(−3) | −2.1844(−3) | −1.6534(−3) |
| 0.8 | | −4.6407(−2) | −5.9341(−2) | −4.6640(−2) | −7.2787(−3) | −1.7342(−3) | −1.1450(−3) |
| 0.9 | | −2.4650(−2) | −3.2280(−2) | −2.6561(−2) | −4.5434(−3) | −1.0507(−3) | −6.0353(−4) |
| 1 | | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

Table 5
The diffuse intensity $I_*(\tau, \mu, \varphi)$ for phase matrix $II$ with $F_I = 1$, $F_Q = 0$, $\omega$ 0.99, $\lambda_0 = 0.1$ and $\tau_0 = 1$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 1.9113(−1) | 1.8617(−1) | 1.8114(−1) | 1.7094(−1) | 1.3995(−1) | 1.1434(−1) | 8.9714(−2) |
| −0.9 | 1.9304(−1) | 1.8830(−1) | 1.8343(−1) | 1.7343(−1) | 1.4217(−1) | 1.1562(−1) | 8.9714(−2) |
| −0.8 | 1.6634(−1) | 1.6347(−1) | 1.6039(−1) | 1.5372(−1) | 1.3093(−1) | 1.1022(−1) | 8.9714(−2) |
| −0.7 | 1.6159(−1) | 1.5948(−1) | 1.5706(−1) | 1.5150(−1) | 1.3065(−1) | 1.1035(−1) | 8.9714(−2) |
| −0.6 | 1.6347(−1) | 1.6198(−1) | 1.6006(−1) | 1.5522(−1) | 1.3469(−1) | 1.1287(−1) | 8.9714(−2) |
| −0.5 | 1.6974(−1) | 1.6898(−1) | 1.6763(−1) | 1.6355(−1) | 1.4269(−1) | 1.1786(−1) | 8.9714(−2) |
| −0.4 | 1.7894(−1) | 1.7929(−1) | 1.7883(−1) | 1.7603(−1) | 1.5537(−1) | 1.2622(−1) | 8.9714(−2) |
| −0.3 | 1.8881(−1) | 1.9104(−1) | 1.9217(−1) | 1.9193(−1) | 1.7424(−1) | 1.4026(−1) | 8.9714(−2) |
| −0.2 | 1.9392(−1) | 1.9937(−1) | 2.0326(−1) | 2.0785(−1) | 2.0031(−1) | 1.6499(−1) | 8.9714(−2) |
| −0.1 | 1.8393(−1) | 1.9488(−1) | 2.0300(−1) | 2.1483(−1) | 2.2737(−1) | 2.0720(−1) | 8.9714(−2) |
| −0.0 | 1.39 (−1) | 1.6629(−1) | 1.8215(−1) | 2.0435(−1) | 2.3856(−1) | 2.4326(−1) | 8.9714(−2) |
| 0.0 | | 1.6629(−1) | 1.8215(−1) | 2.0435(−1) | 2.3856(−1) | 2.4326(−1) | 2.08 (−1) |
| 0.1 | | 5.6469(−2) | 9.9377(−2) | 1.5675(−1) | 2.3208(−1) | 2.5439(−1) | 2.5109(−1) |
| 0.2 | | 3.0836(−2) | 5.9582(−2) | 1.0931(−1) | 2.0711(−1) | 2.4931(−1) | 2.6566(−1) |
| 0.3 | | 2.2653(−2) | 4.4766(−2) | 8.6121(−2) | 1.8373(−1) | 2.3712(−1) | 2.6779(−1) |
| 0.4 | | 1.9655(−2) | 3.9032(−2) | 7.6177(−2) | 1.7085(−1) | 2.2925(−1) | 2.6879(−1) |
| 0.5 | | 1.9102(−2) | 3.7918(−2) | 7.4174(−2) | 1.6914(−1) | 2.3105(−1) | 2.7657(−1) |
| 0.6 | | 2.0416(−2) | 4.0402(−2) | 7.8757(−2) | 1.7934(−1) | 2.4600(−1) | 2.9693(−1) |
| 0.7 | | 2.3861(−2) | 4.7015(−2) | 9.1046(−2) | 2.0481(−1) | 2.7953(−1) | 3.3710(−1) |
| 0.8 | | 3.0399(−2) | 5.9629(−2) | 1.1459(−1) | 2.5324(−1) | 3.4196(−1) | 4.0938(−1) |
| 0.9 | | 4.0948(−2) | 8.0182(−2) | 1.5363(−1) | 3.3693(−1) | 4.5240(−1) | 5.3904(−1) |
| 1 | | 5.6305(−1) | 1.0779 | 1.9755 | 3.8049 | 4.5997 | 4.9502 |

Table 6
The Stokes parameter $Q(\tau, \mu, \varphi)$ for phase matrix $II$ with $F_I = 1$, $F_Q = 0$, $\omega = 0.99$, $\lambda_0 = 0.1$ and $\tau_0 = 1$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | |
| −0.9 | 3.5700(−3) | 3.4140(−3) | 3.2520(−3) | 2.9152(−3) | 1.8400(−3) | 9.1061(−4) | |
| −0.8 | 9.8484(−3) | 9.3633(−3) | 8.8738(−3) | 7.8893(−3) | 4.9241(−3) | 2.4511(−3) | |
| −0.7 | 1.1213(−2) | 1.0718(−2) | 1.0208(−2) | 9.1595(−3) | 5.8506(−3) | 2.9548(−3) | |
| −0.6 | 1.1694(−2) | 1.1253(−2) | 1.0784(−2) | 9.7871(−3) | 6.4353(−3) | 3.3115(−3) | |
| −0.5 | 1.2514(−2) | 1.2134(−2) | 1.1713(−2) | 1.0775(−2) | 7.3461(−3) | 3.8806(−3) | |
| −0.4 | 1.3788(−2) | 1.3495(−2) | 1.3144(−2) | 1.2299(−2) | 8.8014(−3) | 4.8351(−3) | |
| −0.3 | 1.5395(−2) | 1.5255(−2) | 1.5029(−2) | 1.4381(−2) | 1.1032(−2) | 6.4680(−3) | |
| −0.2 | 1.6888(−2) | 1.7019(−2) | 1.7022(−2) | 1.6769(−2) | 1.4219(−2) | 9.3787(−3) | |
| −0.1 | 1.7320(−2) | 1.7941(−2) | 1.8312(−2) | 1.8667(−2) | 1.7817(−2) | 1.4400(−2) | |
| −0.0 | 1.47 (−2) | 1.6903(−2) | 1.8011(−2) | 1.9314(−2) | 2.0218(−2) | 1.9014(−2) | |
| 0.0 | | 1.6903(−2) | 1.8011(−2) | 1.9314(−2) | 2.0218(−2) | 1.9014(−2) | 1.43 (−2) |
| 0.1 | | 6.2553(−3) | 1.0736(−2) | 1.6194(−2) | 2.1328(−2) | 2.1488(−2) | 1.9558(−2) |
| 0.2 | | 3.6711(−3) | 6.9096(−3) | 1.2115(−2) | 2.0436(−2) | 2.2578(−2) | 2.2247(−2) |
| 0.3 | | 2.8472(−3) | 5.4808(−3) | 1.0071(−2) | 1.9109(−2) | 2.2616(−2) | 2.3598(−2) |
| 0.4 | | 2.5212(−3) | 4.8846(−3) | 9.1199(−3) | 1.8215(−2) | 2.2415(−2) | 2.4263(−2) |
| 0.5 | | 2.3704(−3) | 4.6032(−3) | 8.6471(−3) | 1.7669(−2) | 2.2193(−2) | 2.4553(−2) |
| 0.6 | | 2.2622(−3) | 4.3961(−3) | 8.2771(−3) | 1.7087(−2) | 2.1692(−2) | 2.4301(−2) |
| 0.7 | | 2.0831(−3) | 4.0477(−3) | 7.6257(−3) | 1.5814(−2) | 2.0189(−2) | 2.2785(−2) |
| 0.8 | | 1.6282(−3) | 3.1661(−3) | 5.9761(−3) | 1.2484(−2) | 1.6053(−2) | 1.8267(−2) |
| 0.9 | | 7.3951(−4) | 1.4526(−3) | 2.7949(−3) | 6.1454(−3) | 8.1984(−3) | 9.6403(−3) |
| 1 | | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

Table 7
The diffuse intensity $I_*(\tau, \mu, \varphi)$ for phase matrix *II* with $F_I = 1$, $F_Q = 0$, $\omega = 0.99$, $\lambda_0 = 0.1$ and $\tau_0 = 10$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 6.0804(−1) | 5.8851(−1) | 5.6262(−1) | 5.0121(−1) | 2.9906(−1) | 1.4422(−1) | 2.9884(−2) |
| −0.9 | 6.0352(−1) | 5.9134(−1) | 5.7031(−1) | 5.1465(−1) | 3.1606(−1) | 1.5652(−1) | 2.9884(−2) |
| −0.8 | 5.6042(−1) | 5.6765(−1) | 5.5995(−1) | 5.2005(−1) | 3.3323(−1) | 1.7045(−1) | 2.9884(−2) |
| −0.7 | 5.4100(−1) | 5.6045(−1) | 5.6100(−1) | 5.3101(−1) | 3.5167(−1) | 1.8623(−1) | 2.9884(−2) |
| −0.6 | 5.2519(−1) | 5.5648(−1) | 5.6468(−1) | 5.4367(−1) | 3.7092(−1) | 2.0378(−1) | 2.9884(−2) |
| −0.5 | 5.0856(−1) | 5.5253(−1) | 5.6863(−1) | 5.5679(−1) | 3.9070(−1) | 2.2286(−1) | 2.9884(−2) |
| −0.4 | 4.8817(−1) | 5.4659(−1) | 5.7144(−1) | 5.6966(−1) | 4.1086(−1) | 2.4300(−1) | 2.9884(−2) |
| −0.3 | 4.6136(−1) | 5.3706(−1) | 5.7199(−1) | 5.8171(−1) | 4.3129(−1) | 2.6366(−1) | 2.9884(−2) |
| −0.2 | 4.2428(−1) | 5.2236(−1) | 5.6924(−1) | 5.9242(−1) | 4.5195(−1) | 2.8447(−1) | 2.9884(−2) |
| −0.1 | 3.6857(−1) | 5.0061(−1) | 5.6208(−1) | 6.0126(−1) | 4.7280(−1) | 3.0538(−1) | 2.9884(−2) |
| −0.0 | 2.62 (−1) | 4.6888(−1) | 5.4914(−1) | 6.0768(−1) | 4.9382(−1) | 3.2648(−1) | 2.9884(−2) |
| 0.0 | | 4.6888(−1) | 5.4914(−1) | 6.0768(−1) | 4.9382(−1) | 3.2648(−1) | 1.14 (−1) |
| 0.1 | | 4.2037(−1) | 5.2854(−1) | 6.1103(−1) | 5.1495(−1) | 3.4784(−1) | 1.5563(−1) |
| 0.2 | | 3.4951(−1) | 4.9717(−1) | 6.1070(−1) | 5.3618(−1) | 3.6951(−1) | 1.8404(−1) |
| 0.3 | | 2.8894(−1) | 4.5834(−1) | 6.0636(−1) | 5.5752(−1) | 3.9157(−1) | 2.0921(−1) |
| 0.4 | | 2.5002(−1) | 4.2434(−1) | 5.9986(−1) | 5.7910(−1) | 4.1408(−1) | 2.3330(−1) |
| 0.5 | | 2.3030(−1) | 4.0374(−1) | 5.9645(−1) | 6.0139(−1) | 4.3718(−1) | 2.5704(−1) |
| 0.6 | | 2.2771(−1) | 4.0158(−1) | 6.0355(−1) | 6.2555(−1) | 4.6115(−1) | 2.8078(−1) |
| 0.7 | | 2.4385(−1) | 4.2396(−1) | 6.3072(−1) | 6.5419(−1) | 4.8658(−1) | 3.0483(−1) |
| 0.8 | | 2.8533(−1) | 4.8214(−1) | 6.9257(−1) | 6.9280(−1) | 5.1494(−1) | 3.2961(−1) |
| 0.9 | | 3.6374(−1) | 6.0054(−1) | 8.2655(−1) | 7.5902(−1) | 5.5111(−1) | 3.5635(−1) |
| 1 | | 3.8277 | 5.0028 | 4.4029 | 1.4034 | 6.8349(−1) | 3.9826(−1) |

Table 8
The Stokes parameter $Q(\tau, \mu, \varphi)$ for phase matrix *II* with $F_I = 1$, $F_Q = 0$, $\omega = 0.99$, $\lambda_0 = 0.1$ and $\tau_0 = 10$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | |
| −0.9 | 6.3858(−3) | 5.4602(−3) | 4.4955(−3) | 2.8856(−3) | 7.2498(−4) | 3.4629(−4) | |
| −0.8 | 1.4777(−2) | 1.1678(−2) | 9.1146(−3) | 5.4901(−3) | 1.3162(−3) | 6.4591(−4) | |
| −0.7 | 1.7039(−2) | 1.4007(−2) | 1.1253(−2) | 7.0648(−3) | 1.8012(−3) | 9.0668(−4) | |
| −0.6 | 1.7836(−2) | 1.5293(−2) | 1.2654(−2) | 8.2712(−3) | 2.2268(−3) | 1.1312(−3) | |
| −0.5 | 1.8623(−2) | 1.6572(−2) | 1.4045(−2) | 9.4664(−3) | 2.6376(−3) | 1.3229(−3) | |
| −0.4 | 1.9514(−2) | 1.7988(−2) | 1.5562(−2) | 1.0749(−2) | 3.0570(−3) | 1.4846(−3) | |
| −0.3 | 2.0418(−2) | 1.9529(−2) | 1.7229(−2) | 1.2158(−2) | 3.4995(−3) | 1.6233(−3) | |
| −0.2 | 2.1101(−2) | 2.1154(−2) | 1.9051(−2) | 1.3719(−2) | 3.9759(−3) | 1.7520(−3) | |
| −0.1 | 2.0969(−2) | 2.2781(−2) | 2.1019(−2) | 1.5456(−2) | 4.4955(−3) | 1.8830(−3) | |
| −0.0 | 1.79 (−2) | 2.4207(−2) | 2.3081(−2) | 1.7379(−2) | 5.0670(−3) | 2.0224(−3) | |
| 0.0 | | 2.4207(−2) | 2.3081(−2) | 1.7379(−2) | 5.0670(−3) | 2.0224(−3) | 1.8521(−3) |
| 0.1 | | 2.4789(−2) | 2.5072(−2) | 1.9472(−2) | 5.6966(−3) | 2.1723(−3) | 1.8879(−3) |
| 0.2 | | 2.3268(−2) | 2.6533(−2) | 2.1655(−2) | 6.3851(−3) | 2.3324(−3) | 1.7954(−3) |
| 0.3 | | 2.1307(−2) | 2.7093(−2) | 2.3726(−2) | 7.1210(−3) | 2.4994(−3) | 1.6724(−3) |
| 0.4 | | 1.9892(−2) | 2.7095(−2) | 2.5411(−2) | 7.8665(−3) | 2.6648(−3) | 1.5531(−3) |
| 0.5 | | 1.8937(−2) | 2.6795(−2) | 2.6478(−2) | 8.5361(−3) | 2.8090(−3) | 1.4412(−3) |
| 0.6 | | 1.8022(−2) | 2.6016(−2) | 2.6623(−2) | 8.9632(−3) | 2.8911(−3) | 1.3226(−3) |
| 0.7 | | 1.6468(−2) | 2.4026(−2) | 2.5157(−2) | 8.8491(−3) | 2.8309(−3) | 1.1731(−3) |
| 0.8 | | 1.2895(−2) | 1.9072(−2) | 2.0534(−2) | 7.6911(−3) | 2.4864(−3) | 9.5370(−4) |
| 0.9 | | 6.3371(−3) | 1.0029(−2) | 1.1875(−2) | 5.0844(−3) | 1.6881(−3) | 6.0737(−4) |
| 1 | | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

Table 9
The Stokes parameter $V_*(\tau, \mu, \varphi)$ for phase matrix $I$ with $F_V = 1$, $F_U = 0$, $\omega = 0.99$ and $\tau_0 = 1$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | −4.4001(−2) | −4.1685(−2) | −3.9427(−2) | −3.5033(−2) | −2.2338(−2) | −1.1658(−2) | |
| −0.9 | −3.9735(−2) | −3.7412(−2) | −3.5195(−2) | −3.0993(−2) | −1.9501(−2) | −1.0246(−2) | |
| −0.8 | −3.4466(−2) | −3.2095(−2) | −2.9890(−2) | −2.5856(−2) | −1.5721(−2) | −8.2637(−3) | |
| −0.7 | −2.7928(−2) | −2.5448(−2) | −2.3210(−2) | −1.9295(−2) | −1.0664(−2) | −5.4765(−3) | |
| −0.6 | −1.9804(−2) | −1.7121(−2) | −1.4777(−2) | −1.0882(−2) | −3.8460(−3) | −1.5170(−3) | |
| −0.5 | −9.7591(−3) | −6.7263(−3) | −4.1502(−3) | −8.42 (−5) | 5.4396(−3) | 4.2150(−3) | |
| −0.4 | 2.4433(−3) | 6.0637(−3) | 9.0829(−3) | 1.3681(−2) | 1.8219(−2) | 1.2756(−2) | |
| −0.3 | 1.6638(−2) | 2.1212(−2) | 2.5015(−2) | 3.0788(−2) | 3.5878(−2) | 2.6006(−2) | |
| −0.2 | 3.1684(−2) | 3.7692(−2) | 4.2736(−2) | 5.0613(−2) | 5.9640(−2) | 4.7586(−2) | |
| −0.1 | 4.5456(−2) | 5.3353(−2) | 5.9986(−2) | 7.0600(−2) | 8.7344(−2) | 8.2747(−2) | |
| −0.0 | 5.53 (−2) | 6.6827(−2) | 7.5692(−2) | 8.9563(−2) | 1.1249(−1) | 1.1730(−1) | |
| 0.0 | | 6.6827(−2) | 7.5692(−2) | 8.9563(−2) | 1.1249(−1) | 1.1730(−1) | 1.06 (−1) |
| 0.1 | | 3.7587(−2) | 6.4215(−2) | 9.7815(−2) | 1.3752(−1) | 1.4605(−1) | 1.4110(−1) |
| 0.2 | | 3.0105(−2) | 5.5619(−2) | 9.5311(−2) | 1.5729(−1) | 1.7438(−1) | 1.7404(−1) |
| 0.3 | | 2.8502(−2) | 5.3934(−2) | 9.6421(−2) | 1.7352(−1) | 2.0045(−1) | 2.0636(−1) |
| 0.4 | | 2.8905(−2) | 5.5251(−2) | 1.0071(−1) | 1.9004(−1) | 2.2597(−1) | 2.3849(−1) |
| 0.5 | | 3.0391(−2) | 5.8375(−2) | 1.0747(−1) | 2.0849(−1) | 2.5280(−1) | 2.7173(−1) |
| 0.6 | | 3.2657(−2) | 6.2887(−2) | 1.1642(−1) | 2.2968(−1) | 2.8220(−1) | 3.0738(−1) |
| 0.7 | | 3.5600(−2) | 6.8648(−2) | 1.2749(−1) | 2.5420(−1) | 3.1516(−1) | 3.4660(−1) |
| 0.8 | | 3.9204(−2) | 7.5654(−2) | 1.4076(−1) | 2.8259(−1) | 3.5257(−1) | 3.9044(−1) |
| 0.9 | | 4.3496(−2) | 8.3971(−2) | 1.5641(−1) | 3.1543(−1) | 3.9527(−1) | 4.3997(−1) |
| 1 | | 4.8534(−2) | 9.3718(−2) | 1.7469(−1) | 3.5335(−1) | 4.4420(−1) | 4.9629(−1) |

Table 10
The Stokes parameter $U(\tau, \mu, \varphi)$ for phase matrix $I$ with $F_V = 1$, $F_U = 0$, $\omega = 0.99$ and $\tau_0 = 1$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | |
| −0.9 | 1.0190(−3) | 9.2512(−4) | 8.3943(−4) | 6.8789(−4) | 3.4818(−4) | 1.5223(−4) | |
| −0.8 | 1.7568(−3) | 1.5851(−3) | 1.4298(−3) | 1.1579(−3) | 5.6580(−4) | 2.4086(−4) | |
| −0.7 | 2.1885(−3) | 1.9553(−3) | 1.7463(−3) | 1.3859(−3) | 6.3290(−4) | 2.5332(−4) | |
| −0.6 | 2.2840(−3) | 2.0048(−3) | 1.7580(−3) | 1.3406(−3) | 5.2112(−4) | 1.7028(−4) | |
| −0.5 | 2.0093(−3) | 1.6981(−3) | 1.4280(−3) | 9.8300(−4) | 1.9004(−4) | −3.9099(−5) | |
| −0.4 | 1.3323(−3) | 9.9984(−4) | 7.1765(−4) | 2.6875(−4) | −4.1774(−4) | −4.2683(−4) | |
| −0.3 | 2.3918(−4) | −1.1005(−4) | −3.9858(−4) | −8.3936(−4) | −1.3790(−3) | −1.0863(−3) | |
| −0.2 | −1.2332(−3) | −1.6019(−3) | −1.8977(−3) | −2.3328(−3) | −2.7671(−3) | −2.1927(−3) | |
| −0.1 | −2.9764(−3) | −3.3742(−3) | −3.6783(−3) | −4.1037(−3) | −4.5021(−3) | −4.0019(−3) | |
| −0.0 | −4.75 (−3) | −5.2930(−3) | −5.6385(−3) | −6.0659(−3) | −6.2903(−3) | −5.8810(−3) | |
| 0.0 | | −5.2930(−3) | −5.6385(−3) | −6.0659(−3) | −6.2903(−3) | −5.8810(−3) | −5.00 (−3) |
| 0.1 | | −3.0777(−3) | −5.0577(−3) | −7.1304(−3) | −8.1580(−3) | −7.5796(−3) | −6.6102(−3) |
| 0.2 | | −2.3235(−3) | −4.1668(−3) | −6.7302(−3) | −9.3614(−3) | −9.1186(−3) | −8.1263(−3) |
| 0.3 | | −1.9889(−3) | −3.6742(−3) | −6.2603(−3) | −9.7689(−3) | −1.0064(−2) | −9.2994(−3) |
| 0.4 | | −1.7580(−3) | −3.2948(−3) | −5.7707(−3) | −9.6515(−3) | −1.0385(−2) | −9.9331(−3) |
| 0.5 | | −1.5486(−3) | −2.9267(−3) | −5.2115(−3) | −9.1210(−3) | −1.0136(−2) | −9.9747(−3) |
| 0.6 | | −1.3270(−3) | −2.5216(−3) | −4.5395(−3) | −8.2008(−3) | −9.3371(−3) | −9.4008(−3) |
| 0.7 | | −1.0744(−3) | −2.0492(−3) | −3.7177(−3) | −6.8745(−3) | −7.9750(−3) | −8.1781(−3) |
| 0.8 | | −7.7699(−4) | −1.4860(−3) | −2.7114(−3) | −5.1035(−3) | −6.0089(−3) | −6.2545(−3) |
| 0.9 | | −4.2277(−4) | −8.1021(−4) | −1.4849(−3) | −2.8342(−3) | −3.3772(−3) | −3.5589(−3) |
| 1 | | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

Table 11
The Stokes parameter $V_*(\tau, \mu, \varphi)$ for phase matrix $I$ with $F_V = 1$, $F_U = 0$, $\omega = 0.99$ and $\tau_0 = 10$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | −5.5358(−2) | −4.0464(−2) | −3.0082(−2) | −1.7035(−2) | −3.3573(−3) | −9.0663(−4) | |
| −0.9 | −4.6350(−2) | −3.0844(−2) | −2.1259(−2) | −1.0740(−2) | −1.8140(−3) | −4.9241(−4) | |
| −0.8 | −3.6749(−2) | −2.0244(−2) | −1.1408(−2) | −3.6305(−3) | −5.5359(−5) | −1.3140(−5) | |
| −0.7 | −2.6522(−2) | −8.5537(−3) | −3.9888(−4) | 4.4035(−3) | 1.9482(−3) | 5.3957(−4) | |
| −0.6 | −1.5643(−2) | 4.3538(−3) | 1.1917(−2) | 1.3489(−2) | 4.2316(−3) | 1.1749(−3) | |
| −0.5 | −4.0992(−3) | 1.8619(−2) | 2.5713(−2) | 2.3776(−2) | 6.8358(−3) | 1.9029(−3) | |
| −0.4 | 8.0892(−3) | 3.4397(−2) | 4.1187(−2) | 3.5438(−2) | 9.8095(−3) | 2.7351(−3) | |
| −0.3 | 2.0834(−2) | 5.1854(−2) | 5.8566(−2) | 4.8681(−2) | 1.3210(−2) | 3.6858(−3) | |
| −0.2 | 3.3894(−2) | 7.1152(−2) | 7.8107(−2) | 6.3746(−2) | 1.7108(−2) | 4.7740(−3) | |
| −0.1 | 4.6620(−2) | 9.2420(−2) | 1.0009(−1) | 8.0918(−2) | 2.1584(−2) | 6.0239(−3) | |
| −0.0 | 5.61 (−2) | 1.1566(−1) | 1.2480(−1) | 1.0053(−1) | 2.6741(−2) | 7.4645(−3) | |
| 0.0 | | 1.1566(−1) | 1.2480(−1) | 1.0053(−1) | 2.6741(−2) | 7.4645(−3) | 1.69 (−3) |
| 0.1 | | 1.3991(−1) | 1.5246(−1) | 1.2296(−1) | 3.2700(−2) | 9.1307(−3) | 2.2592(−3) |
| 0.2 | | 1.5910(−1) | 1.8241(−1) | 1.4862(−1) | 3.9613(−2) | 1.1066(−2) | 2.8374(−3) |
| 0.3 | | 1.7488(−1) | 2.1274(−1) | 1.7775(−1) | 4.7663(−2) | 1.3324(−2) | 3.4865(−3) |
| 0.4 | | 1.9107(−1) | 2.4339(−1) | 2.1027(−1) | 5.7080(−2) | 1.5973(−2) | 4.2332(−3) |
| 0.5 | | 2.0926(−1) | 2.7549(−1) | 2.4610(−1) | 6.8132(−2) | 1.9100(−2) | 5.1047(−3) |
| 0.6 | | 2.3025(−1) | 3.1022(−1) | 2.8549(−1) | 8.1121(−2) | 2.2814(−2) | 6.1341(−3) |
| 0.7 | | 2.5461(−1) | 3.4869(−1) | 3.2900(−1) | 9.6374(−2) | 2.7253(−2) | 7.3633(−3) |
| 0.8 | | 2.8288(−1) | 3.9191(−1) | 3.7743(−1) | 1.1425(−1) | 3.2583(−2) | 8.8467(−3) |
| 0.9 | | 3.1561(−1) | 4.4094(−1) | 4.3176(−1) | 1.3515(−1) | 3.9003(−2) | 1.0655(−2) |
| 1 | | 3.5345(−1) | 4.9685(−1) | 4.9312(−1) | 1.5955(−1) | 4.6750(−2) | 1.2876(−2) |

Table 12
The Stokes parameter $U(\tau, \mu, \varphi)$ for phase matrix $I$ with $F_V = 1$, $F_U = 0$, $\omega = 0.99$ and $\tau_0 = 10$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | |
| −0.9 | 1.1006(−3) | 4.6787(−4) | 1.8152(−4) | −9.2836(−6) | −2.2141(−5) | −6.6548(−6) | |
| −0.8 | 1.8555(−3) | 7.0621(−4) | 2.1190(−4) | −8.5106(−5) | −5.1754(−5) | −1.4931(−5) | |
| −0.7 | 2.2584(−3) | 7.1009(−4) | 8.8123(−5) | −2.2826(−4) | −8.8661(−5) | −2.4773(−5) | |
| −0.6 | 2.3018(−3) | 4.7351(−4) | −1.9352(−4) | −4.3971(−4) | −1.3263(−4) | −3.6093(−5) | |
| −0.5 | 1.9782(−3) | −1.0354(−5) | −6.3726(−4) | −7.2052(−4) | −1.8338(−4) | −4.8767(−5) | |
| −0.4 | 1.2819(−3) | −7.4853(−4) | −1.2477(−3) | −1.0717(−3) | −2.4050(−4) | −6.2631(−5) | |
| −0.3 | 2.1293(−4) | −1.7473(−3) | −2.0291(−3) | −1.4943(−3) | −3.0351(−4) | −7.7488(−5) | |
| −0.2 | −1.2140(−3) | −3.0099(−3) | −2.9850(−3) | −1.9888(−3) | −3.7175(−4) | −9.3118(−5) | |
| −0.1 | −2.9453(−3) | −4.5326(−3) | −4.1162(−3) | −2.5551(−3) | −4.4435(−4) | −1.0924(−4) | |
| −0.0 | −4.73 (−3) | −6.2941(−3) | −5.4175(−3) | −3.1913(−3) | −5.2021(−4) | −1.2549(−4) | |
| 0.0 | | −6.2941(−3) | −5.4175(−3) | −3.1913(−3) | −5.2021(−4) | −1.2549(−4) | −2.98 (−5) |
| 0.1 | | −8.1695(−3) | −6.8685(−3) | −3.8928(−3) | −5.9782(−4) | −1.4138(−4) | −3.3802(−5) |
| 0.2 | | −9.3811(−3) | −8.3421(−3) | −4.6478(−3) | −6.7518(−4) | −1.5630(−4) | −3.7285(−5) |
| 0.3 | | −9.7940(−3) | −9.4906(−3) | −5.4130(−3) | −7.4957(−4) | −1.6943(−4) | −4.0241(−5) |
| 0.4 | | −9.6791(−3) | −1.0104(−2) | −6.0752(−3) | −8.1696(−4) | −1.7973(−4) | −4.2412(−5) |
| 0.5 | | −9.1484(−3) | −1.0124(−2) | −6.4822(−3) | −8.7036(−4) | −1.8577(−4) | −4.3424(−5) |
| 0.6 | | −8.2259(−3) | −9.5267(−3) | −6.4888(−3) | −8.9589(−4) | −1.8544(−4) | −4.2762(−5) |
| 0.7 | | −6.8954(−3) | −8.2774(−3) | −5.9656(−3) | −8.6832(−4) | −1.7526(−4) | −3.9692(−5) |
| 0.8 | | −5.1187(−3) | −6.3241(−3) | −4.7918(−3) | −7.4836(−4) | −1.4904(−4) | −3.3070(−5) |
| 0.9 | | −2.8424(−3) | −3.5952(−3) | −2.8462(−3) | −4.8186(−4) | −9.6201(−5) | −2.0959(−5) |
| 1 | | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

Table 13
The Stokes parameter $V_*(\tau, \mu, \varphi)$ for phase matrix $II$ with $F_V = 1$, $F_U = 0$, $\omega = 0.99$ and $\tau_0 = 1$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | −4.5155(−2) | −4.2041(−2) | −3.9085(−2) | −3.3585(−2) | −1.9596(−2) | −9.6479(−3) | |
| −0.9 | 2.6007(−2) | 2.4817(−2) | 2.3586(−2) | 2.1037(−2) | 1.3003(−2) | 6.2599(−3) | |
| −0.8 | 2.0209(−2) | 1.9517(−2) | 1.8746(−2) | 1.7023(−2) | 1.0874(−2) | 5.2422(−3) | |
| −0.7 | 2.4678(−2) | 2.3990(−2) | 2.3185(−2) | 2.1302(−2) | 1.4050(−2) | 6.9397(−3) | |
| −0.6 | 3.0455(−2) | 2.9850(−2) | 2.9072(−2) | 2.7100(−2) | 1.8607(−2) | 9.4940(−3) | |
| −0.5 | 3.8002(−2) | 3.7605(−2) | 3.6954(−2) | 3.5035(−2) | 2.5238(−2) | 1.3409(−2) | |
| −0.4 | 4.7569(−2) | 4.7621(−2) | 4.7309(−2) | 4.5794(−2) | 3.5042(−2) | 1.9621(−2) | |
| −0.3 | 5.8569(−2) | 5.9531(−2) | 5.9986(−2) | 5.9668(−2) | 4.9632(−2) | 3.0049(−2) | |
| −0.2 | 6.8303(−2) | 7.0939(−2) | 7.2861(−2) | 7.5146(−2) | 7.0410(−2) | 4.8717(−2) | |
| −0.1 | 7.1140(−2) | 7.6526(−2) | 8.0672(−2) | 8.6910(−2) | 9.3942(−2) | 8.1448(−2) | |
| −0.0 | 5.80 (−2) | 7.0752(−2) | 7.8645(−2) | 9.0171(−2) | 1.0913(−1) | 1.1239(−1) | |
| 0.0 | | 7.0752(−2) | 7.8645(−2) | 9.0171(−2) | 1.0913(−1) | 1.1239(−1) | 9.00 (−2) |
| 0.1 | | 2.7118(−2) | 4.7870(−2) | 7.6204(−2) | 1.1585(−1) | 1.2883(−1) | 1.2730(−1) |
| 0.2 | | 1.6988(−2) | 3.2675(−2) | 5.9762(−2) | 1.1371(−1) | 1.3780(−1) | 1.4767(−1) |
| 0.3 | | 1.4166(−2) | 2.7775(−2) | 5.2911(−2) | 1.1148(−1) | 1.4348(−1) | 1.6217(−1) |
| 0.4 | | 1.3714(−2) | 2.7003(−2) | 5.2049(−2) | 1.1419(−1) | 1.5172(−1) | 1.7704(−1) |
| 0.5 | | 1.4619(−2) | 2.8783(−2) | 5.5594(−2) | 1.2352(−1) | 1.6634(−1) | 1.9733(−1) |
| 0.6 | | 1.6843(−2) | 3.3100(−2) | 6.3788(−2) | 1.4154(−1) | 1.9109(−1) | 2.2799(−1) |
| 0.7 | | 2.0873(−2) | 4.0908(−2) | 7.8486(−2) | 1.7258(−1) | 2.3194(−1) | 2.7626(−1) |
| 0.8 | | 2.7757(−2) | 5.4237(−2) | 1.0351(−1) | 2.2465(−1) | 2.9936(−1) | 3.5428(−1) |
| 0.9 | | 3.8178(−2) | 7.4581(−2) | 1.4227(−1) | 3.0832(−1) | 4.1019(−1) | 4.8464(−1) |
| 1 | | 5.6190(−1) | 1.0754 | 1.9699 | 3.7876 | 4.5713 | 4.9107 |

Table 14
The Stokes parameter $U(\tau, \mu, \varphi)$ for phase matrix $II$ with $F_V = 1$, $F_U = 0$, $\omega = 0.99$ and $\tau_0 = 1$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | |
| −0.9 | 5.7415(−2) | 5.4144(−2) | 5.0936(−2) | 4.4698(−2) | 2.7143(−2) | 1.3402(−2) | |
| −0.8 | 3.3526(−2) | 3.1735(−2) | 2.9960(−2) | 2.6459(−2) | 1.6301(−2) | 8.0994(−3) | |
| −0.7 | 2.5853(−2) | 2.4549(−2) | 2.3241(−2) | 2.0629(−2) | 1.2851(−2) | 6.4120(−3) | |
| −0.6 | 2.2649(−2) | 2.1599(−2) | 2.0528(−2) | 1.8348(−2) | 1.1622(−2) | 5.8495(−3) | |
| −0.5 | 2.1977(−2) | 2.1071(−2) | 2.0125(−2) | 1.8154(−2) | 1.1789(−2) | 6.0415(−3) | |
| −0.4 | 2.2792(−2) | 2.2003(−2) | 2.1151(−2) | 1.9320(−2) | 1.3025(−2) | 6.8931(−3) | |
| −0.3 | 2.4437(−2) | 2.3806(−2) | 2.3087(−2) | 2.1467(−2) | 1.5348(−2) | 8.5945(−3) | |
| −0.2 | 2.6096(−2) | 2.5763(−2) | 2.5296(−2) | 2.4113(−2) | 1.8887(−2) | 1.1805(−2) | |
| −0.1 | 2.6291(−2) | 2.6582(−2) | 2.6560(−2) | 2.6097(−2) | 2.2895(−2) | 1.7522(−2) | |
| −0.0 | 2.17 (−2) | 2.4349(−2) | 2.5358(−2) | 2.6149(−2) | 2.5154(−2) | 2.2532(−2) | |
| 0.0 | | 2.4349(−2) | 2.5358(−2) | 2.6149(−2) | 2.5154(−2) | 2.2532(−2) | 1.64 (−2) |
| 0.1 | | 8.0974(−3) | 1.3795(−2) | 2.0404(−2) | 2.5247(−2) | 2.4415(−2) | 2.1767(−2) |
| 0.2 | | 4.1236(−3) | 7.7590(−3) | 1.3533(−2) | 2.2203(−2) | 2.3953(−2) | 2.3278(−2) |
| 0.3 | | 2.7679(−3) | 5.3471(−3) | 9.8516(−3) | 1.8603(−2) | 2.1849(−2) | 2.2679(−2) |
| 0.4 | | 2.1307(−3) | 4.1535(−3) | 7.8146(−3) | 1.5770(−2) | 1.9477(−2) | 2.1125(−2) |
| 0.5 | | 1.7689(−3) | 3.4608(−3) | 6.5710(−3) | 1.3692(−2) | 1.7381(−2) | 1.9368(−2) |
| 0.6 | | 1.5541(−3) | 3.0408(−3) | 5.7858(−3) | 1.2206(−2) | 1.5696(−2) | 1.7753(−2) |
| 0.7 | | 1.4468(−3) | 2.8235(−3) | 5.3562(−3) | 1.1274(−2) | 1.4525(−2) | 1.6505(−2) |
| 0.8 | | 1.4889(−3) | 2.8879(−3) | 5.4221(−3) | 1.1142(−2) | 1.4134(−2) | 1.5870(−2) |
| 0.9 | | 9.8381(−4) | 1.8685(−3) | 3.3674(−3) | 6.1499(−3) | 7.0982(−3) | 7.2734(−3) |
| 1 | | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

Table 15
The Stokes parameter $V_*(\tau, \mu, \varphi)$ for phase matrix *II* with $F_V = 1$, $F_U = 0$, $\omega = 0.99$ and $\tau_0 = 10$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 1.4032(−2) | 3.7606(−2) | 4.8856(−2) | 5.2261(−2) | 2.6567(−2) | 9.3615(−3) | |
| −0.9 | 9.9886(−2) | 9.4752(−2) | 8.7631(−2) | 7.1530(−2) | 3.1186(−2) | 1.1336(−2) | |
| −0.8 | 9.3095(−2) | 9.3254(−2) | 8.9393(−2) | 7.6280(−2) | 3.5310(−2) | 1.3576(−2) | |
| −0.7 | 9.7680(−2) | 1.0004(−1) | 9.7212(−2) | 8.4387(−2) | 4.0264(−2) | 1.6270(−2) | |
| −0.6 | 1.0209(−1) | 1.0744(−1) | 1.0603(−1) | 9.3683(−2) | 4.5925(−2) | 1.9438(−2) | |
| −0.5 | 1.0597(−1) | 1.1518(−1) | 1.1560(−1) | 1.0407(−1) | 5.2337(−2) | 2.3101(−2) | |
| −0.4 | 1.0894(−1) | 1.2305(−1) | 1.2587(−1) | 1.1560(−1) | 5.9592(−2) | 2.7261(−2) | |
| −0.3 | 1.1025(−1) | 1.3073(−1) | 1.3668(−1) | 1.2833(−1) | 6.7805(−2) | 3.1922(−2) | |
| −0.2 | 1.0860(−1) | 1.3773(−1) | 1.4782(−1) | 1.4228(−1) | 7.7116(−2) | 3.7121(−2) | |
| −0.1 | 1.0104(−1) | 1.4333(−1) | 1.5896(−1) | 1.5748(−1) | 8.7687(−2) | 4.2958(−2) | |
| −0.0 | 7.71 (−2) | 1.4634(−1) | 1.6962(−1) | 1.7390(−1) | 9.9703(−2) | 4.9566(−2) | |
| 0.0 | | 1.4634(−1) | 1.6962(−1) | 1.7390(−1) | 9.9703(−2) | 4.9566(−2) | 1.45 (−2) |
| 0.1 | | 1.4412(−1) | 1.7905(−1) | 1.9151(−1) | 1.1339(−1) | 5.7093(−2) | 2.1807(−2) |
| 0.2 | | 1.3383(−1) | 1.8597(−1) | 2.1024(−1) | 1.2900(−1) | 6.5711(−2) | 2.7524(−2) |
| 0.3 | | 1.2544(−1) | 1.9111(−1) | 2.3019(−1) | 1.4687(−1) | 7.5628(−2) | 3.3308(−2) |
| 0.4 | | 1.2398(−1) | 1.9888(−1) | 2.5237(−1) | 1.6747(−1) | 8.7102(−2) | 3.9632(−2) |
| 0.5 | | 1.3049(−1) | 2.1374(−1) | 2.7959(−1) | 1.9152(−1) | 1.0048(−1) | 4.6768(−2) |
| 0.6 | | 1.4655(−1) | 2.4029(−1) | 3.1676(−1) | 2.2030(−1) | 1.1625(−1) | 5.4969(−2) |
| 0.7 | | 1.7619(−1) | 2.8544(−1) | 3.7174(−1) | 2.5625(−1) | 1.3522(−1) | 6.4562(−2) |
| 0.8 | | 2.2724(−1) | 3.6105(−1) | 4.5779(−1) | 3.0428(−1) | 1.5894(−1) | 7.6067(−2) |
| 0.9 | | 3.1009(−1) | 4.8945(−1) | 6.0930(−1) | 3.8084(−1) | 1.9223(−1) | 9.0756(−2) |
| 1 | | 3.7885 | 4.9135 | 4.2115 | 1.0381 | 3.2368(−1) | 1.2198(−1) |

Table 16
The Stokes parameter $U(\tau, \mu, \varphi)$ for phase matrix *II* with $F_V = 1$, $F_U = 0$, $\omega = 0.99$ and $\tau_0 = 10$.

| $\mu$ | $\tau = 0$ | $\tau = \tau_0/20$ | $\tau = \tau_0/10$ | $\tau = \tau_0/5$ | $\tau = \tau_0/2$ | $\tau = 3\tau_0/4$ | $\tau = \tau_0$ |
|---|---|---|---|---|---|---|---|
| −1 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | |
| −0.9 | 7.5099(−2) | 5.2771(−2) | 3.7133(−2) | 1.8543(−2) | 2.6382(−3) | 6.2799(−4) | |
| −0.8 | 4.7267(−2) | 3.5508(−2) | 2.6632(−2) | 1.5059(−2) | 3.0691(−3) | 9.3264(−4) | |
| −0.7 | 3.7981(−2) | 2.9748(−2) | 2.3174(−2) | 1.4090(−2) | 3.5074(−3) | 1.2107(−3) | |
| −0.6 | 3.3826(−2) | 2.7488(−2) | 2.2070(−2) | 1.4146(−2) | 3.9901(−3) | 1.4855(−3) | |
| −0.5 | 3.2258(−2) | 2.7014(−2) | 2.2186(−2) | 1.4760(−2) | 4.5007(−3) | 1.7576(−3) | |
| −0.4 | 3.1905(−2) | 2.7435(−2) | 2.2948(−2) | 1.5692(−2) | 5.0251(−3) | 2.0227(−3) | |
| −0.3 | 3.1984(−2) | 2.8271(−2) | 2.4052(−2) | 1.6814(−2) | 5.5531(−3) | 2.2747(−3) | |
| −0.2 | 3.1885(−2) | 2.9234(−2) | 2.5316(−2) | 1.8046(−2) | 6.0766(−3) | 2.5093(−3) | |
| −0.1 | 3.0604(−2) | 3.0067(−2) | 2.6596(−2) | 1.9329(−2) | 6.5875(−3) | 2.7251(−3) | |
| −0.0 | 2.45 (−2) | 3.0367(−2) | 2.7715(−2) | 2.0599(−2) | 7.0765(−3) | 2.9201(−3) | |
| 0.0 | | 3.0367(−2) | 2.7715(−2) | 2.0599(−2) | 7.0765(−3) | 2.9201(−3) | 9.93 (−4) |
| 0.1 | | 2.9124(−2) | 2.8358(−2) | 2.1761(−2) | 7.5314(−3) | 3.0903(−3) | 1.2754(−3) |
| 0.2 | | 2.4870(−2) | 2.7862(−2) | 2.2657(−2) | 7.9347(−3) | 3.2293(−3) | 1.3740(−3) |
| 0.3 | | 2.0351(−2) | 2.5953(−2) | 2.3033(−2) | 8.2593(−3) | 3.3276(−3) | 1.4168(−3) |
| 0.4 | | 1.6901(−2) | 2.3442(−2) | 2.2695(−2) | 8.4626(−3) | 3.3715(−3) | 1.4234(−3) |
| 0.5 | | 1.4419(−2) | 2.0985(−2) | 2.1707(−2) | 8.4824(−3) | 3.3408(−3) | 1.3912(−3) |
| 0.6 | | 1.2670(−2) | 1.8861(−2) | 2.0298(−2) | 8.2405(−3) | 3.2044(−3) | 1.3102(−3) |
| 0.7 | | 1.1568(−2) | 1.7247(−2) | 1.8729(−2) | 7.6365(−3) | 2.9107(−3) | 1.1629(−3) |
| 0.8 | | 1.1327(−2) | 1.6348(−2) | 1.6930(−2) | 6.3464(−3) | 2.3325(−3) | 9.1214(−4) |
| 0.9 | | 6.2615(−3) | 7.5558(−3) | 5.8308(−3) | 1.8587(−3) | 9.2308(−4) | 4.4892(−4) |
| 1 | | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

## Acknowledgements

The authors wish to express their appreciation to J. F. de Haan, J. W. Hovenier, A. H. Karp and C. V. M. van der Mee for their interest in and contributions to this work. The authors are pleased to acknowledge the use of the solution by the doubling method provided by J. F. de Haan that helped establish confidence in their final results. This work was supported in part by the Consiglio Nazionale delle Ricerche of Italy, the IBM Scientific Center at Palo Alto and the U. S. National Science Foundation.

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### Abstract

A generalized spherical harmonics solution for the azimuthally symmetric component of a Fourier representation of the vector of Stokes parameters basic to the scattering of polarized light in a plane-parallel layer is reported. The final result is continuous in both independent variables and is valid for a general class of scattering matrices. Computational aspects of the developed solution are discussed, and numerical results for the relevant Stokes vector are given for several realistic cases.

### Zusammenfassung

Es wird über eine verallgemeinerte Lösung in Kugelfunktionen berichtet für die azimutalsymmetrische Komponente der Fourier-Darstellung eines Vektors von Stokes-Parametern, die für die Theorie der Streuung von polarisiertem Licht in ebenen parallelen Schichten grundlegend ist. Das Endresultat ist kontinuierlich in beiden unabhängigen Veränderlichen und ist für eine allgemeine Klasse von Streumatrizen gültig. Rechnerische Aspekte der gewonnenen Lösung wurden diskutiert, und numerische Lösungen für den relevanten Stokes-Vektor werden für verschiedene realistische Fälle angegeben.

(Received: April 10, 1984)