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The searchlight problem in radiation transport: some analytical and computational results

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I. Introduction

In a recent paper [1] concerning radiation transport in finite plane-parallel media with nonuniform surface illumination, we developed a solution based on a two-dimensional Fourier transformation, a pseudo problem [2], an integral transformation technique [3] and the F_N method [4] that is applicable to the classic searchlight problem, as considered, for example, by Rybicki [5]. Here we establish some additional analytical results relevant to the considered three-dimensional radiation transport problem, we discuss in detail the basic computational aspects of the developed solution and we report some numerical results for the searchlight problem.

As our initial analysis of the considered problem was reported in detail in Ref. 1, we assume that work to be available and thus give here only a sketch of the material introductory to our current development.

For the searchlight problem we seek a solution of

$$\mu \frac{\partial}{\partial z} I(z, \varrho, \boldsymbol{\Omega}) + \boldsymbol{\omega} \cdot \frac{\partial}{\partial \boldsymbol{\varrho}} I(z, \varrho, \boldsymbol{\Omega}) + I(z, \varrho, \boldsymbol{\Omega}) = \frac{c}{4\pi} \iint I(z, \varrho, \boldsymbol{\Omega}') \, d\boldsymbol{\Omega}' \tag{1}$$

subject to the boundary conditions

$$I[0, \varrho, \boldsymbol{\Omega}(\mu, \phi)] = \frac{1}{2\pi\varrho} \delta(\varrho) \, \delta(\mu - \mu_0) \, \delta(\phi - \phi_0) \tag{2a}$$

and

$$I[a, \varrho, \boldsymbol{\Omega}(-\mu, \phi)] = 0 \tag{2b}$$

for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$. Following Refs. 1 and 5, we note that $z \in [0, a]$ and $\boldsymbol{\varrho}$, which lies in the $x - y$ plane, locate in optical units the position in the homogeneous medium, and $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mu, \phi)$, with $\mu = \cos\theta$, is a unit vector that defines the direction of propagation (see the accompanying figure). In

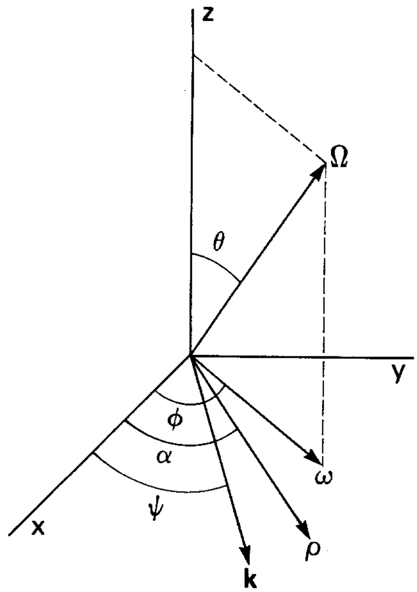

Figure 1
The Geometry for $\boldsymbol{\Omega}$, $\boldsymbol{\omega}$, $\boldsymbol{\varrho}$ and $\boldsymbol{k}$.

addition $\boldsymbol{\omega}$ is the projection of $\boldsymbol{\Omega}$ in the $x-y$ plane, $\boldsymbol{\Omega}_0 = \boldsymbol{\Omega}(\mu_0, \phi_0)$ defines the direction of the incident beam and $c < 1$ is the mean number of secondary particles per collision.

As shown in Ref. 1, the Fourier transform

$$\Psi(z, \mu, \phi) = \iint I(z, \boldsymbol{\varrho}, \boldsymbol{\Omega})\, e^{i\boldsymbol{k}\cdot\boldsymbol{\varrho}}\, \mathrm{d}\boldsymbol{\varrho} \tag{3}$$

satisfies

$$\mu \frac{\partial}{\partial z} \Psi(z, \mu, \phi) + u(\mu, \phi)\, \Psi(z, \mu, \phi) = \frac{c}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \Psi(z, \mu', \phi')\, \mathrm{d}\phi'\, \mathrm{d}\mu' \tag{4}$$

and the boundary conditions

$$\Psi(0, \mu, \phi) = \delta(\mu - \mu_0)\, \delta(\phi - \phi_0) \tag{5a}$$

and

$$\Psi(a, -\mu, \phi) = 0 \tag{5b}$$

for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$. Here we use $k = |\boldsymbol{k}|$ and

$$u(\mu, \phi) = 1 - ik(1 - \mu^2)^{1/2} \cos(\phi - \psi). \tag{6}$$

Wishing first to find the distribution of radiation exiting the medium, $I[0, \boldsymbol{\varrho}, \boldsymbol{\Omega}(-\mu, \phi)]$ and $I[a, \boldsymbol{\varrho}, \boldsymbol{\Omega}(\mu, \phi)]$ for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, we note from Ref. 1 that, for the considered searchlight problem, the transforms of the desired quantities can be expressed, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, as

$$\Psi(0, -\mu, \phi) = \frac{c}{4\pi\mu} S^*(U) \tag{7a}$$

and

$$\Psi(a, \mu, \phi) = \delta(\mu - \mu_0)\,\delta(\phi - \phi_0)\,e^{-a/U} + \frac{c}{4\pi\mu}\,e^{-a/U}\,S^*(-U) \tag{7b}$$

where $U = \mu/u(\mu, \phi)$ and, for $s \notin [-\gamma, \gamma]$,

$$S^*(s) = \Lambda^{-1}(s)\left[F^*(s) + cs\int_0^{\gamma} \tau\,\phi^3(\tau)\,\Phi[0, -p(\tau)]\,\frac{d\tau}{\tau - s} - cse^{-a/s}\int_0^{\gamma} \tau\,\phi^3(\tau)\,\Phi[a, p(\tau)]\,\frac{d\tau}{\tau + s}\right]. \tag{8}$$

Here

$$\Lambda(s) = 1 + \frac{c}{2}\,s\int_{-\gamma}^{\gamma} \phi(\tau)\,\frac{d\tau}{\tau - s}, \tag{9}$$

$$\phi(\tau) = (1 - k^2\tau^2)^{-1/2}, \tag{10}$$

$$\gamma = (1 + k^2)^{-1/2} \tag{11}$$

and

$$p(\tau) = \tau(1 - k^2\tau^2)^{-1/2}. \tag{12}$$

In addition

$$F^*(s) = \frac{s\,U_0}{s + U_0}\,[1 - e^{-a/s}\,e^{-a/U_0}], \tag{13}$$

where $U_0 = \mu_0/u(\mu_0, \phi_0)$, and $\Phi(z, \mu)$ satisfies Williams' [2] pseudo problem, *viz.*

$$\mu(1 + k^2\mu^2)^{1/2}\,\frac{\partial}{\partial z}\,\Phi(z, \mu) + (1 + k^2\mu^2)\,\Phi(z, \mu) = \frac{c}{2}\int_{-1}^{1} \Phi(z, \mu')\,d\mu' + \frac{1}{2}\,F(z) \tag{14}$$

with

$$F(z) = e^{-z/U_0}, \tag{15}$$

$$\Phi(0, \mu) = 0, \quad \mu \in [0, 1], \tag{16a}$$

and

$$\Phi(a, -\mu) = 0, \quad \mu \in [0, 1]. \tag{16b}$$

To evaluate $\Psi(0, -\mu, \phi)$ and $\Psi(a, \mu, \phi)$ as given by Eqs. (7) we clearly must solve the pseudo problem to obtain $\Phi[0, -p(\tau)]$ and $\Phi[a, p(\tau)]$ for $\tau \in [0, \gamma]$. In Ref. 1 a pair of coupled singular integral equations and a pair of integral constraints that together define $\Phi[0, -p(\tau)]$ and $\Phi[a, p(\tau)]$, for $\tau \in [0, \gamma]$, were derived by applying an integral transformation technique to Eq. (14). As the singular integral equations and constraints are basic to what follows in this work, we list them here:

$$2\nu\lambda(\nu)\phi^2(\nu)\Phi[0,-p(\nu)] + c\nu P\int_0^\gamma \tau\phi^3(\tau)\Phi[0,-p(\tau)]\frac{d\tau}{\nu-\tau} + c\nu e^{-a/\nu}\int_0^\gamma \tau\phi^3(\tau)\Phi[a,p(\tau)]\frac{d\tau}{\nu+\tau} = F^*(\nu) \quad (17\text{a})$$

and

$$2\nu\lambda(\nu)\phi^2(\nu)\Phi[a,p(\nu)] + c\nu P\int_0^\gamma \tau\phi^3(\tau)\Phi[a,p(\tau)]\frac{d\tau}{\nu-\tau} + c\nu e^{-a/\nu}\int_0^\gamma \tau\phi^3(\tau)\Phi[0,-p(\tau)]\frac{d\tau}{\nu+\tau} = F^*(-\nu)e^{-a/\nu}, \quad (17\text{b})$$

for $\nu \in [0,\gamma]$, and

$$cs_0\int_0^\gamma \tau\phi^3(\tau)\Phi[0,-p(\tau)]\frac{d\tau}{s_0-\tau} + cs_0 e^{-a/s_0}\int_0^\gamma \tau\phi^3(\tau)\Phi[a,p(\tau)]\frac{d\tau}{s_0+\tau} = F^*(s_0) \quad (17\text{c})$$

and

$$cs_0\int_0^\gamma \tau\phi^3(\tau)\Phi[a,p(\tau)]\frac{d\tau}{s_0-\tau} + cs_0 e^{-a/s_0}\int_0^\gamma \tau\phi^3(\tau)\Phi[0,-p(\tau)]\frac{d\tau}{s_0+\tau} = F^*(-s_0)e^{-a/s_0}, \quad (17\text{d})$$

where s_0 is the positive zero of $\Lambda(s)$ and

$$\lambda(\nu) = 1 + \frac{c\nu}{2}P\int_{-\gamma}^{\gamma}\phi(\tau)\frac{d\tau}{\tau-\nu}. \quad (18)$$

We now proceed to develop approximate (but accurate) solutions of Eqs. (17).

II. The F_N method

As we must solve our pseudo problem for enough values of k that an accurate inverse Fourier transformation can be achieved, we clearly seek a solution that can be computed with modest effort. Thus, even though an F_N solution of Eqs. (17) was reported previously [1], we prefer to use here a variant of the F_N method that has some analytical and computational strengths in low order.

Letting $\Phi_0[0, -p(v)]$ and $\Phi_0[a, p(v)]$ denote the desired solutions for the case $c = 0$, we deduce from Eqs. (17a and b) that

$$\Phi_0[0, -p(v)] = \tfrac{1}{2} U_0 \phi^{-2}(v) S(a{:}\, v, U_0) \tag{19a}$$

and

$$\Phi_0[a, p(v)] = \tfrac{1}{2} U_0 \phi^{-2}(v) C(a{:}\, v, U_0), \tag{19b}$$

where

$$S(a{:}\, x, y) = \frac{1 - e^{-a/x} e^{-a/y}}{x + y} \tag{20a}$$

and

$$C(a{:}\, x, y) = \frac{e^{-a/x} - e^{-a/y}}{x - y}. \tag{20b}$$

Considering now the general case, $c \neq 0$, we write our approximate solutions to Eqs. (17) as

$$\Phi[0, -p(v)] = \Phi_0[0, -p(v)] + \frac{c}{2} \gamma \phi^{-2}(v) \sum_{\alpha=0}^{N} a_\alpha H_\alpha(v/\gamma) \tag{21a}$$

and

$$\Phi[a, p(v)] = \Phi_0[a, p(v)] + \frac{c}{2} \gamma \phi^{-2}(v) \sum_{\alpha=0}^{N} b_\alpha H_\alpha(v/\gamma) \tag{21b}$$

for $v \in [0, \gamma]$. The basis functions $H_\alpha(\mu)$ are to be chosen and the constants $\{a_\alpha, b_\alpha\}$ are to be found so that $\Phi[0, -p(v)]$ and $\Phi[a, p(v)]$ will satisfy Eqs. (17) at $N + 1$ values of $\zeta \in [0, \gamma] \cup s_0$.

Substituting Eqs. (21) into Eqs. (17), we find, after letting $\eta = v/\gamma$ and $\eta_0 = s_0/\gamma$,

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi) + b_\alpha A_\alpha(\xi) e^{-a/(\gamma\xi)}] = R_1(\xi) \tag{22a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi) + a_\alpha A_\alpha(\xi) e^{-a/(\gamma\xi)}] = R_2(\xi) \tag{22b}$$

for $\xi = \eta \in [0, 1]$ or $\xi = \eta_0 = s_0/\gamma$, which we hereafter abbreviate as $\xi \in P$. We find that the known right-hand sides of Eqs. (22) can be expressed as

$$R_1(\xi) = U_0 \Big\{ W(\xi) S(a{:}\, \gamma\xi, U_0) + \int_0^1 x \phi(\gamma x) [S(a{:}\, \gamma x, U_0) - S(a{:}\, \gamma\xi, U_0)] \frac{dx}{x - \xi} - e^{-a/(\gamma\xi)} \int_0^1 x \phi(\gamma x) C(a{:}\, \gamma x, U_0) \frac{dx}{x + \xi} \Big\} \tag{23a}$$

and

$$R_2(\xi) = U_0 \Bigg\{ W(\xi)\, C(a\colon \gamma\xi, U_0) + \int_0^1 x\,\phi(\gamma x)\,[C(a\colon \gamma x, U_0) - C(a\colon \gamma\xi, U_0)]\,\frac{dx}{x-\xi} - e^{-a/(\gamma\xi)} \int_0^1 x\,\phi(\gamma x)\, S(a\colon \gamma x, U_0)\,\frac{dx}{x+\xi} \Bigg\} \tag{23 b}$$

where

$$W(\xi) = 2\int_0^1 \phi(\gamma x)\,dx - \int_0^1 x\,\phi(\gamma x)\,\frac{dx}{x+\xi}. \tag{24}$$

We note also that the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ here are defined as

$$A_\alpha(\xi) = c\gamma \int_0^1 x\,\phi(\gamma x)\, H_\alpha(x)\,\frac{dx}{x+\xi}, \quad \xi \in P, \tag{25}$$

$$B_\alpha(\eta) = 2\lambda(\gamma\eta)\, H_\alpha(\eta) + c\gamma P \int_0^1 x\,\phi(\gamma x)\, H_\alpha(x)\,\frac{dx}{\eta - x}, \tag{26 a}$$

for $\eta \in [0, 1]$, and

$$B_\alpha(\eta_0) = c\gamma \int_0^1 x\,\phi(\gamma x)\, H_\alpha(x)\,\frac{dx}{\eta_0 - x}. \tag{26 b}$$

Now to complete our F_N solution, we consider Eqs. (22) at $N + 1$ selected values of $\xi \in P$, say ξ_β, $\beta = 0, 1, \cdots, N$, and solve the linear algebraic equations

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi_\beta) + b_\alpha A_\alpha(\xi_\beta)\, e^{-a/(\gamma\xi_\beta)}] = R_1(\xi_\beta) \tag{27 a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi_\beta) + a_\alpha A_\alpha(\xi_\beta)\, e^{-a/(\gamma\xi_\beta)}] = R_2(\xi_\beta) \tag{27 b}$$

to find the required constants $\{a_\alpha, b_\alpha\}$. We proceed therefore to a discussion of the numerical methods we use to compute the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ for a particular choice of basis functions $H_\alpha(x)$.

III. Computational aspects of the F_N solution

The ease with which we can evaluate the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ defined by Eqs. (25) and (26), the numerical stability of the linear system, Eqs. (27), and the accuracy of the final solution all clearly depend on the choice of basis functions $H_\alpha(x)$. As we ultimately must carry out a Fourier inversion by numerical integration, we use here only small values of N, say $N \leqq 15$, in order to keep the computer-time requirement within a modest limit. With such values of N in mind, we choose to use the basis functions $H_\alpha(x) = x^\alpha$ which lead to especially easily evaluated functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$.

We now let $r = \gamma k$ and write Eq. (25) as

$$A_\alpha(\xi) = c\gamma \int_0^1 x^{\alpha+1}(1 - r^2 x^2)^{-1/2} \frac{dx}{x+\xi}, \quad \xi \in P, \tag{28}$$

from which we can readily deduce the recursion formula, for $\alpha \geqq 0$,

$$A_{\alpha+1}(\xi) = -\xi A_\alpha(\xi) + c\gamma M_\alpha \tag{29}$$

where

$$M_\alpha = \int_0^1 x^{\alpha+1}(1 - r^2 x^2)^{-1/2}\,dx. \tag{30}$$

We can integrate Eq. (30) to find

$$M_0 = \frac{1}{r^2}[1 - (1 - r^2)^{1/2}], \tag{31 a}$$

$$M_1 = \frac{1}{2r^3}[\mathrm{Sin}^{-1} r - r(1 - r^2)^{1/2}] \tag{31 b}$$

and the recursion formula

$$M_\alpha = \frac{1}{(\alpha+1)r^2}[\alpha M_{\alpha-2} - (1 - r^2)^{1/2}], \quad \alpha \geqq 2. \tag{32}$$

We have found that Eqs. (31) and (32) provide a fast and accurate way to compute the constants M_k for k greater than, say, 2. For $k \leqq 2$ we use the series

$$M_\alpha = \frac{1}{\alpha+2} + \frac{1}{2}\left(\frac{1}{\alpha+4}\right) r^2 + \frac{3}{8}\left(\frac{1}{\alpha+6}\right) r^4 + \frac{5}{16}\left(\frac{1}{\alpha+8}\right) r^6 + \cdots, \tag{33}$$

that is readily available from Eq. (30), to compute the required $\{M_\alpha\}$.

We can now use Eqs. (26) to deduce the recursion formula, for $\alpha \geqq 0$,

$$B_{\alpha+1}(\xi) = \xi B_\alpha(\xi) - c\gamma M_\alpha, \quad \xi \in P, \tag{34}$$

that provides a convenient way to compute the functions $B_\alpha(\xi)$. Of course a computation of the $A_\alpha(\xi)$ or the $B_\alpha(\xi)$ by forward recursion requires the starting values $A_0(\xi)$ and $B_0(\xi)$, and thus we integrate Eq. (28) to obtain

$$A_0(\xi) = c\gamma \left\{\frac{1}{r}\mathrm{Sin}^{-1} r - \xi(1 - r^2\xi^2)^{-1/2} \ln\left[\left(1 + \frac{1}{\xi}\right) Z(\xi)\right]\right\}, \tag{35}$$

for $r^2 \xi^2 < 1$, where

$$Z(\xi) = [1 + (1 - r^2 \xi^2)^{1/2}] [1 + r^2 \xi + (1 - r^2)^{1/2} (1 - r^2 \xi^2)^{1/2}]^{-1}. \tag{36}$$

Now, as $\xi \in P$ yields $r^2 \xi^2 < 1$, we can use Eq. (35) to evaluate $A_0(\eta)$, $\eta \in [0, 1]$, and $A_0(\eta_0)$; and from Eq. (9) we find we can express $B_0(\xi)$ in terms of $A_0(\xi)$, *viz.*

$$B_0(\xi) = 2\Lambda(\infty) + A_0(\xi), \quad \xi \in P, \tag{37}$$

where

$$\Lambda(\infty) = 1 - \frac{c}{k} \operatorname{Tan}^{-1} k. \tag{38}$$

We have found that the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ can be evaluated by forward recursion without significant loss of accuracy for all $\xi = \eta \in [0, 1]$ and $\xi = \eta_0$ for $\eta_0 \leqq 1.2$, say; for $\xi = \eta_0 > 1.2$ we use the series

$$A_M(\xi) = c \frac{\gamma}{\xi} \left(M_M - \frac{1}{\xi} M_{M+1} + \frac{1}{\xi^2} M_{M+2} - \frac{1}{\xi^3} M_{M+3} + \cdots \right) \tag{39a}$$

and

$$B_M(\xi) = c \frac{\gamma}{\xi} \left(M_M + \frac{1}{\xi} M_{M+1} + \frac{1}{\xi^2} M_{M+2} + \frac{1}{\xi^3} M_{M+3} + \cdots \right), \tag{39b}$$

for some $M > N$, and use Eqs. (29) and (34) recursively in the backward direction to find $A_\alpha(\xi)$ and $B_\alpha(\xi)$ for $\alpha = 0, 1, 2, \cdots, N$.

In regard to the right-hand side of the linear system given by Eqs. (27), we note that the integrals in Eqs. (23) must be evaluated numerically; however, we can integrate Eq. (24) to obtain, for $r^2 \xi^2 < 1$,

$$W(\xi) = \frac{1}{r} \operatorname{Sin}^{-1} r + \xi (1 - r^2 \xi^2)^{-1/2} \ln \left[\left(1 + \frac{1}{\xi} \right) z(\xi) \right]. \tag{40}$$

We consider now that we can compute accurately the functions $A_\alpha(\xi)$, $B_\alpha(\xi)$, $R_1(\xi)$ and $R_2(\xi)$ for $\xi \in P$, and so to find the constants $\{a_\alpha, b_\alpha\}$ we must simply specify a collocation strategy to define the points $\{\xi_\beta\}$ and solve the linear system given by Eqs. (27). We follow a previous work [6] and use $\xi_0 = \eta_0$ and the zeros of the Chebyshev polynomial of the first kind $T_N(2x - 1)$, *i.e.*,

$$\xi_\beta = \tfrac{1}{2} + \tfrac{1}{2} \cos [(2\beta - 1) \pi/(2N)], \quad \beta = 1, 2, \cdots, N. \tag{41}$$

As the constants $\{a_\alpha, b_\alpha\}$ are now available, we go on to formulate our Fourier inversion to obtain the desired final results.

IV. The Fourier inversion

Now that the boundary fluxes for the pseudo problem are available we can evaluate the Fourier transforms $\Psi(0, -\mu, \phi)$ and $\Psi(a, \mu, \phi)$ as given by Eqs. (7).

Thus we substitute Eqs. (21) into Eq. (8) and use the resulting expression in Eqs. (7) to obtain our F_N approximations to $\Psi(0, -\mu, \phi)$ and $\Psi(a, \mu, \phi)$. We write these results as

$$\Psi(0, -\mu, \phi) = \Psi_1(0, -\mu, \phi) + \Psi_2(0, -\mu, \phi) \tag{42a}$$

and

$$\Psi(a, \mu, \phi) = \Psi_0(a, \mu, \phi) + \Psi_1(a, \mu, \phi) + \Psi_2(a, \mu, \phi) \tag{42b}$$

for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$. Here

$$\Psi_0(a, \mu, \phi) = \delta(\mu - \mu_0)\,\delta(\phi - \phi_0)\,e^{-a/U}, \tag{43a}$$

$$\Psi_1(0, -\mu, \phi) = c\,U_0\left[\frac{1}{4\pi u(\mu, \phi)}\right] S(a\colon U, U_0), \tag{43b}$$

$$\Psi_1(a, \mu, \phi) = c\,U_0\left[\frac{1}{4\pi u(\mu, \phi)}\right] C(a\colon U, U_0), \tag{43c}$$

$$\Psi_2(0, -\mu, \phi) = \left[\frac{c^2}{4\pi u(\mu, \phi)\,\Lambda(U)}\right]\left[U_0\,\Xi(\mu, \phi) + \frac{1}{2}\gamma X(\mu, \phi)\right] \tag{43d}$$

and

$$\Psi_2(a, \mu, \phi) = \left[\frac{c^2}{4\pi u(\mu, \phi)\,\Lambda(U)}\right]\left[U_0\,T(\mu, \phi) + \frac{1}{2}\gamma Y(\mu, \phi)\right], \tag{43e}$$

where

$$\Xi(\mu, \phi) = \Gamma(U)\,S(a\colon U, U_0) + \tfrac{1}{2}\gamma\,[J(U/\gamma) - K(-U/\gamma)\,e^{-a/U}], \tag{44a}$$

$$T(\mu, \phi) = \Gamma(U)\,C(a\colon U, U_0) + \tfrac{1}{2}\gamma\,[K(U/\gamma) - J(-U/\gamma)\,e^{-a/U}], \tag{44b}$$

$$X(\mu, \phi) = \sum_{\alpha=0}^{N} [a_\alpha E_\alpha(U/\gamma) - b_\alpha E_\alpha(-U/\gamma)\,e^{-a/U}] \tag{44c}$$

and

$$Y(\mu, \phi) = \sum_{\alpha=0}^{N} [b_\alpha E_\alpha(U/\gamma) - a_\alpha E_\alpha(-U/\gamma)\,e^{-a/U}]. \tag{44d}$$

In addition we have introduced

$$\Gamma(s) = \frac{1}{c}[1 - \Lambda(s)], \tag{45}$$

$$E_\alpha(z) = c\gamma \int_0^1 x\,\phi(\gamma x)\,H_\alpha(x)\,\frac{dx}{x - z}, \quad z \notin (0, 1], \tag{46}$$

$$J(z) = \int_0^1 x\,\phi(\gamma x)\,S(a\colon \gamma x, U_0)\,\frac{dx}{x - z}, \quad z \notin (0, 1], \tag{47a}$$

and

$$K(z) = \int_0^1 x\,\phi(\gamma x)\,C(a{:}\gamma x, U_0)\frac{dx}{x - z}, \quad z \notin (0, 1]. \tag{47b}$$

Formally our results for the distribution of radiation exiting the medium are now available, and so we write for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$

$$I[0, \varrho, \boldsymbol{\Omega}(-\mu, \phi)] = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} \Psi(0, -\mu, \phi)\,e^{-ik\varrho\cos(\alpha-\psi)}\,k\,dk\,d\psi \tag{48a}$$

and

$$I[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)] = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} \Psi(a, \mu, \phi)\,e^{-ik\varrho\cos(\alpha-\psi)}\,k\,dk\,d\psi \tag{48b}$$

where $\Psi(0, -\mu, \phi)$ and $\Psi(a, \mu, \phi)$ are given by Eqs. (42). In the following section of this paper we focus our attention on the case of normal incidence, $\mu_0 = 1$, and proceed to simplify the Fourier inversion indicated by Eqs. (48).

V. The case of normal incidence

For the case $\mu_0 = 1$ we note that Eqs. (43) can be written as

$$\Psi_0(a, \mu, \phi) = \delta(\mu - 1)\,\delta(\phi - \phi_0)\,e^{-a}, \tag{49a}$$

$$\Psi_1(0, -\mu, \phi) = \left[\frac{c}{4\pi u(\mu, \phi)}\right] S(a{:}\,U, 1), \tag{49b}$$

$$\Psi_1(a, \mu, \phi) = \left[\frac{c}{4\pi u(\mu, \phi)}\right] C(a{:}\,U, 1), \tag{49c}$$

$$\Psi_2(0, -\mu, \phi) = \left[\frac{c^2}{4\pi u(\mu, \phi)\,\Lambda(U)}\right]\left[\Xi(\mu, \phi) + \frac{1}{2}\gamma X(\mu, \phi)\right] \tag{49d}$$

and

$$\Psi_2(a, \mu, \phi) = \left[\frac{c^2}{4\pi u(\mu, \phi)\,\Lambda(U)}\right]\left[T(\mu, \phi) + \frac{1}{2}\gamma Y(\mu, \phi)\right]. \tag{49e}$$

Now generalizing the result given in Ref. 1 for the case $\mu_0 = 1$ and $a \to \infty$, we note that we can find components $I_0[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)]$, $I_1[0, \varrho, \boldsymbol{\Omega}(-\mu, \phi)]$ and $I_1[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)]$ of the complete solution whose two-dimensional Fourier transforms $\Psi_0(a, \mu, \phi)$, $\Psi_1(0, -\mu, \phi)$ and $\Psi_1(a, \mu, \phi)$ are given, respectively, by Eqs. (49a, b and c). We write these results, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, as

$$I_0[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)] = \frac{1}{2\pi\varrho}\,\delta(\varrho)\,\delta(\mu - 1)\,\delta(\phi - \phi_0)\,e^{-a}, \tag{50a}$$

$$I_1[0, \varrho, \boldsymbol{\Omega}(-\mu, \phi)] = \left[\frac{c}{4\pi\varrho(1 - \mu^2)^{1/2}}\right]\delta(\alpha - \phi)\,e^{-\varrho(1+\mu)/(1-\mu^2)^{1/2}}, \tag{50b}$$

for $0 < \varrho \leqq a(1-\mu^2)^{1/2}/\mu$, and $I_1[0, \varrho, \boldsymbol{\Omega}(-\mu, \phi)] = 0$, otherwise, and

$$I_1[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)] = \left[\frac{c}{4\pi\varrho(1-\mu^2)^{1/2}}\right]\delta(\alpha-\phi)\, e^{-a}\, e^{-\varrho(1-\mu)/(1-\mu^2)^{1/2}}, \quad (50\text{c})$$

for $0 < \varrho \leqq a(1-\mu^2)^{1/2}/\mu$, and $I_1[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)] = 0$, otherwise.

With the components $I_0[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)]$, $I_1[0, \varrho, \boldsymbol{\Omega}(-\mu, \phi)]$ and $I_1[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)]$ given by Eqs. (50), we can now express our desired solutions as

$$I[0, \varrho, \boldsymbol{\Omega}(-\mu, \phi)] = I_1[0, \varrho, \boldsymbol{\Omega}(-\mu, \phi)] + I_2[0, \varrho, \boldsymbol{\Omega}(-\mu, \phi)] \quad (51\text{a})$$

and

$$I[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)] = I_0[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)] + I_1[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)] + I_2[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)] \quad (51\text{b})$$

for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$. Here the Fourier inversion integrals

$$I_2[0, \varrho, \boldsymbol{\Omega}(-\mu, \phi)] = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} \Psi_2(0, -\mu, \phi)\, e^{-ik\varrho\cos(\alpha-\psi)}\, k\, dk\, d\psi \quad (52\text{a})$$

and

$$I_2[a, \varrho, \boldsymbol{\Omega}(\mu, \phi)] = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{\infty} \Psi_2(a, \mu, \phi)\, e^{-ik\varrho\cos(\alpha-\psi)}\, k\, dk\, d\psi \quad (52\text{b})$$

are to be evaluated by numerical methods.

As a first numerical implementation of our developed solutions, we wish to evaluate Eqs. (52) for the special case $\mu = 1$; for this case we note that $\Psi_2(0, -1, \phi)$ and $\Psi_2(a, 1, \phi)$ are independent of ψ, and so we can integrate Eqs. (52) to obtain

$$I_2[0, \varrho, \boldsymbol{\Omega}(-1, \phi)] = \frac{1}{2\pi}\int_0^{\infty} \Psi_2(0, -1, \phi)\, J_0(k\varrho)\, k\, dk \quad (53\text{a})$$

and

$$I_2[a, \varrho, \boldsymbol{\Omega}(1, \phi)] = \frac{1}{2\pi}\int_0^{\infty} \Psi_2(a, 1, \phi)\, J_0(k\varrho)\, k\, dk, \quad (53\text{b})$$

where $J_0(x)$ is used to denote the zero-th-order Bessel function of the first kind [7]. Making use of Eqs. (49) and (44), we write Eqs. (53) as

$$I_2[0, \varrho, \boldsymbol{\Omega}(-1, \phi)] = \left(\frac{c^2}{16\pi^2\varrho}\right)\int_0^{\infty} M(x/\varrho)\, J_0(x)\, dx \quad (54\text{a})$$

and

$$I_2[a, \varrho, \boldsymbol{\Omega}(1, \phi)] = \left(\frac{c^2}{16\pi^2\varrho}\right)\int_0^{\infty} N(x/\varrho)\, J_0(x)\, dx \quad (54\text{b})$$

where

$$M(k) = \frac{k}{\Lambda(1)} \{\Gamma(1)(1 - e^{-2a}) + \gamma [J(1/\gamma) - K(-1/\gamma) e^{-a} + X(1, \phi)]\}, \tag{55a}$$

$$N(k) = \frac{k}{\Lambda(1)} \{2a\Gamma(1) e^{-a} + \gamma [K(1/\gamma) - J(-1/\gamma) e^{-a} + Y(1, \phi)]\}, \tag{55b}$$

$$X(1, \phi) = \sum_{\alpha=0}^{N} [a_\alpha E_\alpha(1/\gamma) - b_\alpha E_\alpha(-1/\gamma) e^{-a}] \tag{56a}$$

and

$$Y(1, \phi) = \sum_{\alpha=0}^{N} [b_\alpha E_\alpha(1/\gamma) - a_\alpha E_\alpha(-1/\gamma) e^{-a}]. \tag{56b}$$

VI. Numerical results

Proceeding with our solution for the case $\mu_0 = \mu = 1$, we note from Eqs. (55) that

$$M(\infty) = \frac{\pi}{2}(1 - e^{-2a}) \tag{57a}$$

and

$$N(\infty) = \pi a e^{-a}, \tag{57b}$$

and so we choose to write Eqs. (54) as

$$I_2[0, \varrho, \boldsymbol{\Omega}(-1, \phi)] = \frac{c^2}{32\pi\varrho}(1 - e^{-2a})[1 - D(\varrho)] \tag{58a}$$

and

$$I_2[a, \varrho, \boldsymbol{\Omega}(1, \phi)] = \frac{c^2}{16\pi\varrho} a [e^{-a} - E(\varrho)] \tag{58b}$$

where

$$D(\varrho) = \int_0^\infty \left[1 - \frac{2}{\pi}(1 - e^{-2a})^{-1} M(x/\varrho)\right] J_0(x)\,\mathrm{d}x \tag{59a}$$

and

$$E(\varrho) = \int_0^\infty \left[e^{-a} - \frac{1}{\pi a} N(x/\varrho)\right] J_0(x)\,\mathrm{d}x. \tag{59b}$$

The integrals $J(\pm 1/\gamma)$ and $K(\pm 1/\gamma)$ appearing in Eqs. (55) must be evaluated by numerical methods; however we have available, from Ref. 1,

$$\Gamma(1) = \frac{1}{2}(1-k^2)^{-1/2} \ln\left[\frac{1+(1-k^2)^{1/2}}{1-(1-k^2)^{1/2}}\right], \quad k<1, \tag{60a}$$

$$\Gamma(1) = 1, \quad k = 1, \tag{60b}$$

and

$$\Gamma(1) = (k^2-1)^{-1/2}\,\mathrm{Tan}^{-1}(k^2-1)^{1/2}, \quad k>1. \tag{60c}$$

Finally we note that, as the definition given by Eq. (28) for $A_\alpha(\xi)$ can be extended to include any $\xi > 1$, we write $E_\alpha(-1/\gamma) = A_\alpha(1/\gamma)$ and use the technique discussed in Sect. III to compute $E_\alpha(-1/\gamma)$; we also compute $E_\alpha(1/\gamma)$ from the recursion formula

$$E_{\alpha+1}(1/\gamma) = (1/\gamma)\,E_\alpha(1/\gamma) + c\gamma M_\alpha. \tag{61}$$

For $1/\gamma \leqq 1.2$ we use Eq. (61) recursively forward, with the starting value

$$E_0(1/\gamma) = \frac{c}{k}\,\mathrm{Tan}^{-1} k + c(1-k^2)^{-1/2} \ln\left[\frac{(1+k^2)^{1/2}-(1-k^2)^{1/2}}{(1+k^2)^{1/2}+(1-k^2)^{1/2}}\right]. \tag{62}$$

For $1/\gamma > 1.2$ we use Eq. (61) recursively backward, using

$$E_M(1/\gamma) = -c\gamma^2\,[M_M + \gamma M_{M+1} + \gamma^2 M_{M+2} + \cdots] \tag{63}$$

for some $M > N$ to start.

We would now like to report a first demonstration that our formalism, developed in detail for the case $\mu_0 = \mu = 1$, can in fact be evaluated with modest effort to yield the desired intensities $I_2\,[0, \varrho, \boldsymbol{\Omega}(-1, \phi)]$ and $I_2\,[a, \varrho, \boldsymbol{\Omega}(1, \phi)]$. We consider the case $c = 0.8$ and $a = 1.0$ and note from various orders of the F_N approximation, $0 < N < 20$, that

$$R(k) = 1 - \frac{2}{\pi}(1-e^{-2a})^{-1}\,M(k) \tag{64a}$$

and

$$T(k) = e^{-a} - \frac{1}{\pi a}\,N(k) \tag{64b}$$

are both positive and monotonically decreasing functions of k. We therefore use the method reported by Longman [8] to evaluate, for selected values of ϱ, the integrals defined by Eqs. (59). Longman's method [8] is based on using the zeros of $J_0(x)$ as break points to subdivide the integration interval $[0, \infty)$; subsequently an Euler transformation [9] is used to sum the resulting slowly converging series in a more rapidly convergent manner.

Table 1
Components of the Exit Fluxes.

ϱ	$I_2[0, \varrho, \boldsymbol{\Omega}(-1, \phi)]$	$I_2[a, \varrho, \boldsymbol{\Omega}(1, \phi)]$
0.001	5.4902	4.6781
0.01	5.3874(−1)	4.6264(−1)
0.1	4.6331(−2)	4.1430(−2)
0.2	1.9915(−2)	1.8281(−2)
0.4	7.4794(−3)	7.0846(−3)
0.6	3.7910(−3)	3.6552(−3)
0.8	2.1804(−3)	2.1251(−3)
1.0	1.3470(−3)	1.3219(−3)
1.2	8.7174(−4)	8.5945(−4)
1.4	5.8301(−4)	5.7661(−4)
1.6	3.9959(−4)	3.9609(−4)
1.8	2.7913(−4)	2.7715(−4)
2.0	1.9797(−4)	1.9681(−4)
2.2	1.4217(−4)	1.4147(−4)
2.4	1.0316(−4)	1.0273(−4)
2.6	7.5513(−5)	7.5240(−5)
2.8	5.5692(−5)	5.5518(−5)
3.0	4.1343(−5)	4.1230(−5)
4.0	1.0031(−5)	1.0017(−5)
5.0	2.6403(−6)	2.6402(−6)

We list in the accompanying table our numerical results deduced from the formalism herein discussed. To establish our belief that the reported results are correct to within ± 1 in the last digits given, we have used several orders of the F_N approximation, $0 < N < 20$, to compute the functions $R(k)$ and $T(k)$ and we have used several variants [defined by the specific zero of $J_0(k)$ where we first employ the method and the total number of terms in the series we use] of Longman's method to evaluate the improper integrals given by Eqs. (59). Finally we have gained additional confidence in our reported results by finding agreement (to, say, two significant figures) with independent Monte Carlo calculations [10, 11].

The results we report in the table represent, of course, only a partial solution to the classic searchlight problem, and thus we hope to be able to report more general numerical results, $I_2[0, \varrho, \boldsymbol{\Omega}(-\mu, \phi)]$ and $I_2[\mathrm{a}, \varrho, \boldsymbol{\Omega}(\mu, \phi)]$ for all μ and μ_0, in future works.

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Abstract

Integral transformation techniques and the F_N method are used to formulate a general solution to the classic searchlight problem for a finite plane-parallel layer. The special case of a normally incident beam is then considered, and the resulting expressions for the intensities exiting the two surfaces in the normal directions are reduced to one-dimensional inversion integrals which are evaluated to yield accurate numerical results for a selected case.

Zusammenfassung

Integrale Transformationstechniken und die F_N Methode werden benützt um eine generelle Lösung für das klassische Durchdringungsproblem einer endlichen plan-parallelen Schicht zu formulieren. Der Spezialfall eines gewöhnlich zufälligen Strahles wird dann betrachtet und die sich ergebenden Ausdrücke für die aus den zwei Oberflächen in Normalrichtung heraustretenden Intensitäten werden auf eindimensionale Inversionsintegrale reduziert, welche ausgewertet werden um genaue numerische Resultate für einen bestimmten Fall zu erbringen.

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