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# ON THE SINGULAR COMPONENTS OF THE SOLUTION TO THE SEARCHLIGHT PROBLEM IN RADIATIVE TRANSFER

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**Abstract**—Explicit expressions for the singular components of the solution to the searchlight problem are reported.

## 1. INTRODUCTION

In a recent study[1] of the classical searchlight problem in radiative transfer,[2–5] Dunn and Siewert found, for the case of a normally incident beam, explicit results for the singular components of the complete radiation field on the surfaces of a finite plane-parallel layer. Here we develop the equivalent results for the case of a beam incident in an arbitrary direction. We follow previous works basic to the searchlight problem[1,4,5] and seek, in general, a solution of

$$\mu \frac{\partial}{\partial z} I(z, \rho, \Omega) + \omega \cdot \frac{\partial}{\partial \rho} I(z, \rho, \Omega) + I(z, \rho, \Omega) = \frac{c}{4\pi} \int\int I(z, \rho, \Omega')\, d\Omega' \tag{1}$$

subject to the boundary conditions

$$I[0, \rho, \Omega(\mu, \phi)] = \frac{1}{2\pi\rho}\, \delta(\rho)\delta(\mu - \mu_0)\delta(\phi - \phi_0) \tag{2a}$$

and

$$I[a, \rho, \Omega(-\mu, \phi)] = 0 \tag{2b}$$

for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$. Following Refs. 1, 4 and 5, we note that $z \in [0, a]$ and $\rho$, which lies in the $x$–$y$ plane, locate in optical units the position in the homogeneous medium and $\Omega = \Omega\,(\mu, \phi)$, with $\mu = \cos\theta$, is a unit vector that defines the direction of propagation (see Fig. 1). In addition $\omega$ is the projection of $\Omega$ in the $x$–$y$ plane, $\Omega_0 = \Omega_0\,(\mu_0, \phi_0)$ defines the direction of the incident beam and $c < 1$ is the mean number of secondary particles per collision.

As the boundary condition given by Eq. (2a) is expressed in terms of generalized functions, we find that the boundary solutions we seek, $I[0, \rho, \Omega(-\mu, \phi)]$ and $I[a, \rho, \Omega(\mu, \phi)]$ for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, also have components that are generalized functions. Here we report explicit expressions for these singular components.

## 2. SINGULAR COMPONENTS

In their analysis of the searchlight problem as defined by Eqs. (1) and (2), Dunn and Siewert[1] expressed, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, the two-dimensional Fourier transformations

$$\Psi(0, -\mu, \phi) = \int\int I[0, \rho, \Omega(-\mu, \phi)]\, e^{i\mathbf{k}\cdot\rho}\, d\rho \tag{3a}$$

and

$$\Psi(a, \mu, \phi) = \int\int I[a, \rho, \Omega(\mu, \phi)]\, e^{i\mathbf{k}\cdot\rho}\, d\rho \tag{3b}$$

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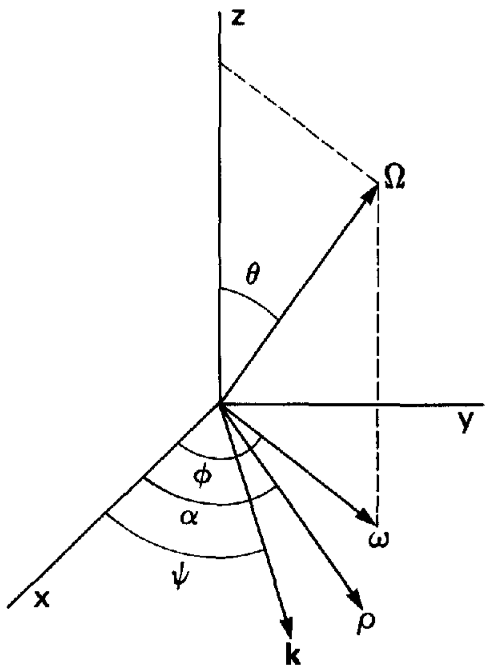


Fig. 1. The geometry for $\Omega$, $\omega$, $\rho$ and $\mathbf{k}$.

as

$$\Psi(0, -\mu, \phi) = \Psi_1(0, -\mu, \phi) + \Psi_2(0, -\mu, \phi) \tag{4a}$$

and

$$\Psi(a, \mu, \phi) = \Psi_0(a, \mu, \phi) + \Psi_1(a, \mu, \phi) + \Psi_2(a, \mu, \phi) \tag{4b}$$

where

$$\Psi_0(a, \mu, \phi) = \delta(\mu - \mu_0)\delta(\phi - \phi_0)\, e^{-a/U} \tag{5}$$

is the transform of the uncollided beam, $\Psi_1(0, -\mu, \phi)$ and $\Psi_1(a, \mu, \phi)$ are the transforms of the once-collided exiting intensities and $\Psi_2(0, -\mu, \phi)$ and $\Psi_2(a, \mu, \phi)$ are the transforms of exiting intensities resulting from two or more scattering events in the layer. Here $U = \mu/u(\mu, \phi)$,

$$u(\mu, \phi) = 1 - ik(1 - \mu^2)^{1/2} \cos(\phi - \psi), \tag{6}$$

$$\Psi_1(0, -\mu, \phi) = cU_0\left(\frac{1}{4\pi u(\mu, \phi)}\right)S(a\colon U, U_0) \tag{7a}$$

and

$$\Psi_1(a, \mu, \phi) = cU_0\left(\frac{1}{4\pi u(\mu, \phi)}\right)C(a\colon U, U_0), \tag{7b}$$

for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, where $U_0 = \mu_0/u(\mu_0, \phi_0)$,

$$S(a\colon x, y) = \frac{1 - e^{-a/x}\, e^{-a/y}}{x + y} \tag{8a}$$

and

$$C(a\colon x, y) = \frac{e^{-a/x} - e^{-a/y}}{x - y}. \tag{8b}$$

The complete solutions are given by the inverse Fourier transformations

$$I[0, \rho, \Omega(-\mu, \phi)] = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{\infty} \Psi(0, -\mu, \phi)\, e^{-ik\rho\cos(\alpha-\psi)} k\, dk\, d\psi \tag{9a}$$

and

$$I[a, \rho, \Omega(\mu, \phi)] = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{\infty} \Psi(a, \mu, \phi)\, e^{-ik\rho\cos(\alpha-\psi)} k\, dk\, d\psi, \tag{9b}$$

for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$.

Now substituting Eqs. (4) into Eqs. (9), we write, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$,

$$I[0, \rho, \Omega(-\mu, \phi)] = \sum_{\alpha=1}^{2} I_\alpha[0, \rho, \Omega(-\mu, \phi)] \tag{10a}$$

and

$$I[a, \rho, \Omega(\mu, \phi)] = \sum_{\alpha=0}^{2} I_\alpha[a, \rho, \Omega(\mu, \phi)], \tag{10b}$$

where the singular

$$I_0[a, \rho, \Omega(\mu, \phi)] = \frac{1}{\rho}\, \delta[\rho - a(1 - \mu_0^2)^{1/2}/\mu_0]\delta(\alpha - \phi)\delta(\mu - \mu_0)\delta(\phi - \phi_0)\, e^{-a/\mu_0} \tag{11}$$

is the uncollided component of the solution. The components $I_2[0, \rho, \Omega(-\mu, \phi)]$ and $I_2[a, \rho, \Omega(\mu, \phi)]$ in Eqs. (10) represent transforms of $\Psi_2(0, -\mu, \phi)$ and $\Psi_2(a, \mu, \phi)$ and thus, as previously discussed,[1] are to be evaluated by numerical methods. However, we have deduced singular components $I_1[0, \rho, \Omega(-\mu, \phi)]$ and $I_1[a, \rho, \Omega(\mu, \phi)]$, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, whose Fourier transforms yield $\Psi_1(0, -\mu, \phi)$ and $\Psi_1(a, \mu, \phi)$.

We first consider the surface at $z = 0$ and express, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, the singular component of the exiting intensity as

$$I_1[0, \rho, \Omega(-\mu, \phi)] = \left(\frac{c\mu_0}{4\pi\rho D(\mu, \phi)}\right)\delta[\alpha - \chi(\mu, \phi)]\, e^{-\rho(\mu_0+\mu)/D(\mu,\phi)} \tag{12a}$$

for $0 < \rho \le aD(\mu, \phi)/(\mu\mu_0)$ and

$$I_1[0, \rho, \Omega(-\mu, \phi)] = 0 \tag{12b}$$

for $\rho > aD(\mu, \phi)/(\mu\mu_0)$. Here

$$D(\mu, \phi) = [\mu^2(1 - \mu_0^2) + \mu_0^2(1 - \mu^2) + 2\mu\mu_0(1 - \mu^2)^{1/2}(1 - \mu_0^2)^{1/2} \cos(\phi - \phi_0)]^{1/2} \tag{13}$$

and the angle $\chi(\mu, \phi)$ is such that

$$\cos \chi(\mu, \phi) = [\mu(1 - \mu_0^2)^{1/2} \cos \phi_0 + \mu_0(1 - \mu^2)^{1/2} \cos \phi]/D(\mu, \phi) \tag{14a}$$

and

$$\sin \chi(\mu, \phi) = [\mu(1 - \mu_0^2)^{1/2} \sin \phi_0 + \mu_0(1 - \mu^2)^{1/2} \sin \phi]/D(\mu, \phi). \tag{14b}$$

In order to express our result for the second singular component of the intensity exiting the surface $z = a$ [the first is given by Eq. (11)], we first define

$$\rho_*(\rho, \alpha) = \left[\rho^2 + \left(\frac{a}{\mu_0}\right)^2(1 - \mu_0^2) - 2\left(\frac{a}{\mu_0}\right)\rho(1 - \mu_0^2)^{1/2} \cos(\alpha - \phi_0)\right]^{1/2} \tag{15}$$

and the angle $\alpha_*(\rho, \alpha)$ such that

$$\cos \alpha_*(\rho, \alpha) = \left[\rho \cos \alpha - \left(\frac{a}{\mu_0}\right)(1 - \mu_0^2)^{1/2} \cos \phi_0\right] \Big/ \rho_*(\rho, \alpha) \tag{16a}$$

and

$$\sin \alpha_*(\rho, \alpha) = \left[\rho \sin \alpha - \left(\frac{a}{\mu_0}\right)(1 - \mu_0^2)^{1/2} \sin \phi_0\right] \Big/ \rho_*(\rho, \alpha). \tag{16b}$$

Now we can write, for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$,

$$I_1[a, \rho, \Omega(\mu, \phi)] = \left(\frac{c\mu_0}{4\pi\rho_*(\rho, \alpha)D(-\mu, \phi)}\right) e^{-a/\mu_0}\delta[\alpha_*(\rho, \alpha) - \chi(-\mu, \phi)]\, e^{-\rho_*(\rho,\alpha)(\mu_0-\mu)/D(-\mu,\phi)} \tag{17a}$$

for $0 < \rho_*(\rho, \alpha) \leq aD(-\mu, \phi)/(\mu\mu_0)$ and

$$I_1[a, \rho, \Omega(\mu, \phi)] = 0 \tag{17b}$$

for $\rho_*(\rho, \alpha) > aD(-\mu, \phi)/(\mu\mu_0)$.

We note that the expressions given by Eqs. (11) and (17) reduce for the special case $\mu_0 = 1$ to the results reported in Ref. 1.

## 3. COMMENTS

To conclude this note we record a few remarks concerning this work: (i) Though we have not been able to obtain Eqs. (12) and (17) directly from Eqs. (9), we can readily show that the Fourier transformations of Eqs. (12) and (17) yield Eqs. (7). (ii) While the presence of the singular component $I_0[a, \rho, \Omega(\mu, \phi)]$ can be immediately seen, it is perhaps only on second thought that it becomes apparent that there can be additional singular components on both surfaces. (iii) It is, of course, essential to remove all singular components of the desired solution before attempting any numerical inverse Fourier transformation calculations (this was the basic motivation for this work). (iv) Finally, it is clear that Eqs. (12) and (17) can be multiplied by an appropriate scattering probability to obtain analogous results for the case of anisotropic scattering.

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