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RADIATIVE TRANSFER CALCULATIONS IN SPHERES AND CYLINDERS

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Abstract—Integral transformation techniques and the F_N method are used to solve, for the case of isotropic scattering, radiative transfer problems in spherical and cylindrical geometries. Numerical results accurate to five or six significant figures are given for selected cases basic to problems with internal heat generation and emitting and diffusely reflecting surfaces.

1. INTRODUCTION

In this paper we use the results of two previously reported works[1,2] to solve a class of basic radiative transfer problems in spheres and cylinders.

2. SPHERICAL GEOMETRY

We consider, for $r \in [0, R]$ and $\mu \in [-1, 1]$, the equation of transfer

$$(BI)(r, \mu) = (1/2)(1 - \omega)Q, \tag{1}$$

where

$$(Bf)(r, \mu) = \left[\mu \frac{\partial}{\partial r} + \frac{1}{r}(1 - \mu^2)\frac{\partial}{\partial \mu} + 1\right] f(r, \mu) - \frac{\omega}{2}\int_{-1}^{1} f(r, \mu')\, \mathrm{d}\mu', \tag{2}$$

and the boundary condition

$$I(R, -\mu) = \frac{1}{2}F + 2\beta \int_0^1 I(R, \mu')\mu'\, \mathrm{d}\mu' \tag{3}$$

for $\mu \in [0, 1]$. Here Q represents a uniform internal source, the constant F is proportional to the fourth power of the temperature of the emitting surface and β is the coefficient for diffuse reflection.

We can express the intensity $I(r, \mu)$ as

$$I(r, \mu) = (1/2)\{Q + (1 - \beta A^*)^{-1}[F - (1 - \beta)Q]\psi(r, \mu)\} \tag{4}$$

where $\psi(r, \mu)$ is the solution, discussed in Ref. 1, of

$$(B\psi)(r, \mu) = 0, \tag{5a}$$

subject to the boundary condition

$$\psi(R, -\mu) = 1, \qquad \mu \in [0, 1], \tag{5b}$$

and

$$A^* = 2\int_0^1 \psi(R, \mu)\mu\, \mathrm{d}\mu \tag{6}$$

is the albedo. As we are interested here in the incident radiation and the heat flux,

$$E(r) = \int_{-1}^{1} I(r, \mu)\, d\mu \tag{7a}$$

and

$$q(r) = \int_{-1}^{1} I(r, \mu)\mu\, d\mu, \tag{7b}$$

we integrate Eq. (4) to find

$$E(r) = Q + (1/2)(1 - \beta A^*)^{-1}[F - (1 - \beta)Q]\rho(r) \tag{8a}$$

and

$$q(r) = (1/2)(1 - \beta A^*)^{-1}[F - (1 - \beta)Q]j(r), \tag{8b}$$

where

$$\rho(r) = \int_{-1}^{1} \psi(r, \mu)\, d\mu \tag{9a}$$

and

$$j(r) = \int_{-1}^{1} \psi(r, \mu)\mu\, d\mu \tag{9b}$$

can be found from Ref. 1, in the manner used in Ref. 2, to be

$$\rho(r) = \frac{1}{r}\left\{W(\nu_0)[e^{-(R-r)/\nu_0} - e^{-(R+r)/\nu_0}] + \int_0^1 W(\nu)[e^{-(R-r)/\nu} - e^{-(R+r)/\nu}]\, d\nu\right\} \tag{10a}$$

and

$$j(r) = \frac{1}{r^2}(1 - \omega)\left\{\nu_0 W(\nu_0)[(\nu_0 - r)\, e^{-(R-r)/\nu_0} - (\nu_0 + r)\, e^{-(R+r)/\nu_0}] + \int_0^1 \nu W(\nu)[(\nu - r)\, e^{-(R-r)/\nu} - (\nu + r)\, e^{-(R+r)/\nu}]\, d\nu\right\} \tag{10b}$$

where

$$W(\xi) = \frac{\omega\xi}{2N(\xi)}[RB_0(\xi) + B_1(\xi) + \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi)]. \tag{11}$$

Here ν_0 is the positive zero of

$$\Lambda(z) = 1 + \frac{\omega}{2} z \int_{-1}^{1} \frac{d\mu}{\mu - z}, \tag{12}$$

$$N(\nu_0) = \frac{\omega}{2}\nu_0^3\left(\frac{\omega}{\nu_0^2 - 1} - \frac{1}{\nu_0^2}\right), \tag{13a}$$

$$N(\nu) = \nu[(1 - \omega\nu \tanh^{-1} \nu)^2 + (1/4)(\omega\nu\pi)^2] \tag{13b}$$

and the F_N functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ and the constants a_α are as defined in Ref. 1.

3. CYLINDRICAL GEOMETRY

We consider, for $r \in [0, R]$, $\mu \in [-1, 1]$ and $\phi \in [0, 2\pi]$, the equation of transfer

$$(BI)(r, \mu, \phi) = \frac{1}{4\pi}(1 - \omega)Q, \tag{14}$$

where

$$(Bf)(r, \mu, \phi) = \left[(1 - \mu^2)^{1/2}\left(\cos\phi \frac{\partial}{\partial r} - \frac{1}{r}\sin\phi \frac{\partial}{\partial \phi}\right) + 1\right] f(r, \mu, \phi) - \frac{\omega}{4\pi}\int_{-1}^{1}\int_0^{2\pi} f(r, \mu', \phi')\, d\phi'\, d\mu', \quad (15)$$

and the boundary condition

$$I(R, \mu, \phi) = \frac{F}{4\pi} + \frac{4\beta}{\pi}\int_0^1\int_0^{\pi/2} I(R, \mu', \phi')(1 - \mu'^2)^{1/2}\cos\phi'\, d\phi'\, d\mu' \quad (16)$$

for $\mu \in [-1, 1]$ and $\phi \in [\pi/2, 3\pi/2]$. Here we write

$$I(r, \mu, \phi) = \frac{1}{4\pi}\{Q + (1 - \beta A^*)^{-1}[F - (1 - \beta)Q]\psi(r, \mu, \phi)\} \quad (17)$$

where $\psi(r, \mu, \phi)$, which satisfies

$$(B\psi)(r, \mu, \phi) = 0 \quad (18)$$

subject to

$$\psi(R, \mu, \phi) = 1, \quad (19)$$

for $\mu \in [-1, 1]$ and $\phi \in [\pi/2, 3\pi/2]$, is available from Ref. 2. Here

$$A^* = \frac{4}{\pi}\int_0^1\int_0^{\pi/2} \psi(R, \mu, \phi)(1 - \mu^2)^{1/2}\cos\phi\, d\phi\, d\mu \quad (20)$$

is the albedo for the bare cylinder that is tabulated in Ref. 2.

To find the incident radiation and the heat flux,

$$E(r) = \int_{-1}^1\int_0^{2\pi} I(r, \mu, \phi)\, d\phi\, d\mu \quad (21a)$$

and

$$q(r) = \int_{-1}^1\int_0^{2\pi} I(r, \mu, \phi)(1 - \mu^2)^{1/2}\cos\phi\, d\phi\, d\mu, \quad (21b)$$

we integrate Eq. (17) to obtain

$$E(r) = Q + \frac{1}{4\pi}(1 - \beta A^*)^{-1}[F - (1 - \beta)Q]\rho(r) \quad (22a)$$

and

$$q(r) = \frac{1}{4\pi}(1 - \beta A^*)^{-1}[F - (1 - \beta)Q]j(r) \quad (22b)$$

where

$$\rho(r) = \int_{-1}^1\int_0^{2\pi} \psi(r, \mu, \phi)\, d\phi\, d\mu \quad (23a)$$

and

$$j(r) = \int_{-1}^1\int_0^{2\pi} \psi(r, \mu, \phi)(1 - \mu^2)^{1/2}\cos\phi\, d\phi\, d\mu \quad (23b)$$

are available from the tabulations given in Ref. 2.

4. NUMERICAL RESULTS

In Tables 1–8 we give values for $\rho(r)$ and $j(r)$ for spheres and cylinders that we believe to be accurate to within ±1 in the last significant figure shown. We also list in Table 9

Table 1. Function $\rho(r)$ for a sphere with radius $R = 1$.

r/R	$\omega = 0.7$	$\omega = 0.8$	$\omega = 0.9$	$\omega = 0.99$
0.0	1.37928	1.54655	1.74960	1.97240
0.1	1.38182	1.54851	1.75074	1.97253
0.2	1.38949	1.55441	1.75418	1.97293
0.3	1.40249	1.56439	1.75998	1.97360
0.4	1.42114	1.57867	1.76825	1.97455
0.5	1.44600	1.59761	1.77917	1.97581
0.6	1.47793	1.62178	1.79301	1.97738
0.7	1.51833	1.65215	1.81027	1.97934
0.8	1.56975	1.69043	1.83181	1.98175
0.9	1.63792	1.74054	1.85964	1.98483
1.0	1.75266	1.82312	1.90452	1.98970

Table 2. Function $\rho(r)$ for a sphere with radius $R = 10$.

r/R	$\omega = 0.7$	$\omega = 0.8$	$\omega = 0.9$	$\omega = 0.99$
0.0	3.88362 (−3)	1.34692 (−2)	7.84066 (−2)	1.19428
0.1	4.34648 (−3)	1.46332 (−2)	8.20657 (−2)	1.20021
0.2	5.93762 (−3)	1.84922 (−2)	9.36619 (−2)	1.21812
0.3	9.37234 (−3)	2.62865 (−2)	1.15178 (−1)	1.24831
0.4	1.62664 (−2)	4.06086 (−2)	1.50367 (−1)	1.29135
0.5	3.00441 (−2)	6.64327 (−2)	2.05541 (−1)	1.34800
0.6	5.79624 (−2)	1.13176 (−1)	2.90939 (−1)	1.41932
0.7	1.15725 (−1)	1.98869 (−1)	4.23121 (−1)	1.50668
0.8	2.38783 (−1)	3.59215 (−1)	6.29502 (−1)	1.61195
0.9	5.15768 (−1)	6.71835 (−1)	9.59917 (−1)	1.73850
1.0	1.36009	1.45555	1.60201	1.90990

Table 3. Function $\rho(r)$ for a cylinder with radius $R = 1$.

r/R	$\omega = 0.7$	$\omega = 0.8$	$\omega = 0.9$	$\omega = 0.99$
0.0	7.49719	8.73803	1.03632 (1)	1.23119 (1)
0.1	7.51643	8.75348	1.03727 (1)	1.23131 (1)
0.2	7.57458	8.80012	1.04012 (1)	1.23166 (1)
0.3	7.67306	8.87891	1.04493 (1)	1.23225 (1)
0.4	7.81442	8.99157	1.05179 (1)	1.23308 (1)
0.5	8.00276	9.14088	1.06082 (1)	1.23417 (1)
0.6	8.24451	9.33123	1.07225 (1)	1.23555 (1)
0.7	8.55013	9.56980	1.08645 (1)	1.23724 (1)
0.8	8.93834	9.86954	1.10409 (1)	1.23931 (1)
0.9	9.45052	1.02594 (1)	1.12669 (1)	1.24194 (1)
1.0	1.02978 (1)	1.08891 (1)	1.16230 (1)	1.24598 (1)

Table 4. Function $\rho(r)$ for a cylinder with radius $R = 10$.

r/R	$\omega = 0.7$	$\omega = 0.8$	$\omega = 0.9$	$\omega = 0.99$
0.0	1.00761 (−2)	3.77053 (−2)	2.53713 (−1)	6.11781
0.1	1.18915 (−2)	4.26224 (−2)	2.71533 (−1)	6.16341
0.2	1.83340 (−2)	5.93159 (−2)	3.28766 (−1)	6.30124
0.3	3.30099 (−2)	9.44753 (−2)	4.37627 (−1)	6.53438
0.4	6.44020 (−2)	1.62535 (−1)	6.21684 (−1)	6.86808
0.5	1.31344 (−1)	2.92275 (−1)	9.21565 (−1)	7.30988
0.6	2.75656 (−1)	5.40418 (−1)	1.40506	7.86988
0.7	5.91887 (−1)	1.01986	2.18494	8.56125
0.8	1.30208	1.96217	3.45276	9.40178
0.9	2.97996	3.88514	5.56268	1.04214 (1)
1.0	8.33566	8.91947	9.81739	1.18115 (1)

Table 5. Function $-j(r)$ for a sphere with radius $R = 1$.

r/R	$\omega = 0.7$	$\omega = 0.8$	$\omega = 0.9$	$\omega = 0.99$
0.0	0.0	0.0	0.0	0.0
0.1	1.38080 (−2)	1.03182 (−2)	5.83429 (−3)	6.57494 (−4)
0.2	2.77079 (−2)	2.06834 (−2)	1.16823 (−2)	1.31515 (−3)
0.3	4.17940 (−2)	3.11441 (−2)	1.75581 (−2)	1.97312 (−3)
0.4	5.61665 (−2)	4.17510 (−2)	2.34762 (−2)	2.63158 (−3)
0.5	7.09348 (−2)	5.25591 (−2)	2.94523 (−2)	3.29071 (−3)
0.6	8.62238 (−2)	6.36301 (−2)	3.55038 (−2)	3.95069 (−3)
0.7	1.02184 (−1)	7.50372 (−2)	4.16508 (−2)	4.61176 (−3)
0.8	1.19008 (−1)	8.68734 (−2)	4.79184 (−2)	5.27418 (−3)
0.9	1.36985 (−1)	9.92740 (−2)	5.43427 (−2)	5.93833 (−3)
1.0	1.56715 (−1)	1.12518 (−1)	6.09969 (−2)	6.60498 (−3)

Table 6. Function $-j(r)$ for a sphere with radius $R = 10$.

r/R	$\omega = 0.7$	$\omega = 0.8$	$\omega = 0.9$	$\omega = 0.99$
0.0	0.0	0.0	0.0	0.0
0.1	4.15861 (−4)	9.44173 (−4)	2.68645 (−3)	3.99279 (−3)
0.2	1.01376 (−3)	2.18639 (−3)	5.82781 (−3)	8.05704 (−3)
0.3	2.07144 (−3)	4.13725 (−3)	9.97094 (−3)	1.22658 (−2)
0.4	4.11389 (−3)	7.47939 (−3)	1.58669 (−2)	1.66950 (−2)
0.5	8.21233 (−3)	1.34422 (−2)	2.46296 (−2)	2.14256 (−2)
0.6	1.66210 (−2)	2.43152 (−2)	3.79773 (−2)	2.65450 (−2)
0.7	3.41870 (−2)	4.44474 (−2)	5.86219 (−2)	3.21491 (−2)
0.8	7.16198 (−2)	8.22813 (−2)	9.09277 (−2)	3.83459 (−2)
0.9	1.53882 (−1)	1.54935 (−1)	1.42178 (−1)	4.52619 (−2)
1.0	3.52590 (−1)	3.04218 (−1)	2.26857 (−1)	5.30905 (−2)

Table 7. Function $-j(r)$ for a cylinder with radius $R = 1$.

r/R	$\omega = 0.7$	$\omega = 0.8$	$\omega = 0.9$	$\omega = 0.99$
0.0	0.0	0.0	0.0	0.0
0.1	1.12602 (−1)	8.74575 (−2)	5.18397 (−2)	6.15626 (−3)
0.2	2.26074 (−1)	1.75380 (−1)	1.03822 (−1)	1.23143 (−2)
0.3	3.41307 (−1)	2.64243 (−1)	1.56092 (−1)	1.84758 (−2)
0.4	4.59244 (−1)	3.54545 (−1)	2.08799 (−1)	2.46426 (−2)
0.5	5.80910 (−1)	4.46820 (−1)	2.62105 (−1)	3.08167 (−2)
0.6	7.07460 (−1)	5.41663 (−1)	3.16183 (−1)	3.70001 (−2)
0.7	8.40264 (−1)	6.39767 (−1)	3.71233 (−1)	4.31951 (−2)
0.8	9.81057 (−1)	7.41992 (−1)	4.27498 (−1)	4.94044 (−2)
0.9	1.13232	8.49544 (−1)	4.85311 (−1)	5.56318 (−2)
1.0	1.29894	9.64758 (−1)	5.45307 (−1)	6.18840 (−2)

Table 8. Function $-j(r)$ for a cylinder with radius $R = 10$.

r/R	$\omega = 0.7$	$\omega = 0.8$	$\omega = 0.9$	$\omega = 0.99$
0.0	0.0	0.0	0.0	0.0
0.1	1.64563 (−3)	4.01382 (−3)	1.31286 (−2)	3.07030 (−2)
0.2	4.19381 (−3)	9.61458 (−3)	2.90394 (−2)	6.20930 (−2)
0.3	9.10074 (−3)	1.90886 (−2)	5.11708 (−2)	9.48741 (−2)
0.4	1.92920 (−2)	3.64481 (−2)	8.44367 (−2)	1.29785 (−1)
0.5	4.10553 (−2)	6.92743 (−2)	1.36417 (−1)	1.67618 (−1)
0.6	8.82334 (−2)	1.32288 (−1)	2.19234 (−1)	2.09239 (−1)
0.7	1.91816 (−1)	2.54508 (−1)	3.52630 (−1)	2.55609 (−1)
0.8	4.22798 (−1)	4.94171 (−1)	5.69276 (−1)	3.07822 (−1)
0.9	9.51856 (−1)	9.72696 (−1)	9.24912 (−1)	3.67163 (−1)
1.0	2.27686	1.99013	1.53048	4.35512 (−1)

Table 9. Albedo A^* for spheres and cylinders.

R	ω	Sphere	Cylinder
1	0.7	0.686570	0.586536
	0.8	0.774963	0.692908
	0.9	0.878006	0.826423
	0.99	0.986790	0.980302
10	0.7	0.294821	0.275252
	0.8	0.391564	0.366522
	0.9	0.546286	0.512832
	0.99	0.893819	0.861372

some typical values of the albedo A^* for spheres and cylinders; these results are in agreement with previous works.[1-4]

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