TRANSPORT THEORY AND STATISTICAL PHYSICS, 14(4), 437-483 (1985)

BENCHMARK RESULTS IN RADIATIVE TRANSFER

R. D. M. Garcia* and C. E. Siewert**

IBM Scientific Center
Palo Alto, CA 94304

ABSTRACT

Several new aspects of the F_N method are reported, and the method is used to solve accurately some benchmark problems in radiative transfer that have been posed in the literature. More specifically, modifications in the method are made so that cases of pure scattering can be solved, and an improved way to compute certain required integrals is given. In addition, a two-term recursion formula that provides a computationally attractive alternative to previously used three-term recursion formulas for generating matrix elements basic to the method is established. Also, a recently developed improvement in the way the method is used to compute the radiation intensity is generalized and thus made available for all component problems in a Fourier decomposition of the complete radiation field. The numerical results given are focused on a collection of test problems including five (for a Haze L phase function and a Cloud C_1 phase function) posed by the Radiation Commission of the International Association of Meteorology and Atmospheric Physics.

---

* Centro Técnico Aeroespacial
Instituto de Estudos Avançados
12200 São José dos Campos, SP
Brasil

** Mathematics Department
North Carolina State University
Raleigh, NC 27695-8205
USA

 0041-1450/85/1404-0437$3.50/0

I. INTRODUCTION

In 1977 the Radiation Commission of the International Association of Meteorology and Atmospheric Physics issued a report[1] that reviewed many of the standard procedures used to solve radiative-transfer problems relevant to the field of atmospheric physics. The report went on to pose five test problems, three based on a Haze L scattering model and two on a Cloud C_1 phase function, and to tabulate computational results obtained from several solution techniques by numerous researchers. In order to assess the merits of the various methods of solution the Commission investigated the computational effort (computer time) required to achieve a degree of accuracy commensurate with current experimental techniques. Here we investigate these same test problems from the point of view of computational mathematics and report reference quality results that can, we hope, be considered definitive.

In Section II we state, in general terms, the problem to be solved, and as the F_N method[2,3,4] is used here to establish our benchmark results, we review some aspects of the method in Section III. We also report in Section III a modification of the method required in order to accommodate the case of pure scattering, and we develop an improved procedure for computing the radiation intensity. In Section IV we summarize our algorithms for computing the various quantities that are used in the F_N method. In particular, we report in Section IV an improved scheme for evaluating certain integrals relevant to the method, and we give a two-term recursion relation that has proved useful for the numerical evaluation of matrix elements basic to the method. In Section V we discuss the methods used to solve the encountered linear algebraic equations, and Section VI is devoted to numerical results and our conclusions concerning this work.

II. STATEMENT OF THE PROBLEM

Following Chandrasekhar[5], we consider a general radiative-transfer problem defined by the equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau,\mu,\phi) + I(\tau,\mu,\phi) = \frac{\omega}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos\Theta) I(\tau,\mu',\phi') d\phi' d\mu' \qquad (1)$$

with a phase function of the form

$$p(\cos\Theta) = \sum_{\ell=0}^{L} \beta_\ell P_\ell(\cos\Theta), \qquad (2)$$

where $\beta_0=1$ and $|\beta_\ell|<2\ell+1$. We consider the boundary conditions, for $\mu>0$ and $\phi\epsilon[0,2\pi]$,

$$I(0,\mu,\phi) = F_1(\mu,\phi) \qquad (3a)$$

and

$$I(\tau_0,-\mu,\phi) = F_2(\mu,\phi) \qquad (3b)$$

where $F_1(\mu,\phi)$ and $F_2(\mu,\phi)$ are considered given. Here $\tau\epsilon[0,\tau_0]$ is the optical variable, $\mu\epsilon[-1,1]$ and $\phi\epsilon[0,2\pi]$ specify the direction of propagation, $\omega\epsilon[0,1]$ is the single-scattering albedo and we seek for all τ, μ, and ϕ the radiation intensity $I(\tau,\mu,\phi)$. As did Chandrasekhar[5], we can use the addition theorem for the Legendre polynomials and express the desired solution as

$$I(\tau,\mu,\phi) = F_1(\mu,\phi)e^{-\tau/\mu} + \sum_{m=0}^{L} (2-\delta_{0,m})[C^m(\tau,\mu)\cos m(\phi-\phi_r) + S^m(\tau,\mu)\sin m(\phi-\phi_r)] \qquad (4a)$$

and

$$I(\tau,-\mu,\phi) = F_2(\mu,\phi)e^{-(\tau_0-\tau)/\mu} + \sum_{m=0}^{L} (2-\delta_{0,m})[C^m(\tau,-\mu)\cos m(\phi-\phi_r) + S^m(\tau,-\mu)\sin m(\phi-\phi_r)] \qquad (4b)$$

for $\mu>0$ and all ϕ. Here ϕ_r is a reference angle, and

$$C^m(\tau,\mu) = I_c^m(\tau,\mu) - I_c^m(0,\mu)e^{-\tau/\mu}, \tag{5a}$$

$$C^m(\tau,-\mu) = I_c^m(\tau,-\mu) - I_c^m(\tau_0,-\mu)e^{-(\tau_0-\tau)/\mu}, \tag{5b}$$

$$S^m(\tau,\mu) = I_s^m(\tau,\mu) - I_s^m(0,\mu)e^{-\tau/\mu} \tag{6a}$$

and

$$S^m(\tau,-\mu) = I_s^m(\tau,-\mu) - I_s^m(\tau_0,-\mu)e^{-(\tau_0-\tau)/\mu} \tag{6b}$$

where $I_c^m(\tau,\mu)$ and $I_s^m(\tau,\mu)$ satisfy

$$\mu \frac{\partial}{\partial\tau} I(\tau,\mu) + I(\tau,\mu)$$

$$= \frac{\omega}{2} \sum_{\ell=m}^{L} \beta_\ell^m P_\ell^m(\mu) \int_{-1}^{1} P_\ell^m(\mu')I(\tau,\mu')d\mu' \tag{7}$$

and the boundary conditions, for $\mu>0$,

$$I(0,\mu) = F_1(\mu) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\mu,\phi) \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m(\phi-\phi_r)d\phi \tag{8a}$$

and

$$I(\tau_0,-\mu) = F_2(\mu) = \frac{1}{2\pi}\int_0^{2\pi} F_2(\mu,\phi) \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} m(\phi-\phi_r)d\phi. \tag{8b}$$

Here we use $P_\ell^m(\mu)$ to denote the associated Legendre functions and

$$\beta_\ell^m = \frac{(\ell-m)!}{(\ell+m)!}\,\beta_\ell. \tag{9}$$

III. THE F_N METHOD

Following the work of Devaux, Siewert and Yuan[4], we express our F_N solution to the radiative-transfer problem defined by Eqs.

(7) and (8) as

$$I(0,-\mu) = F_2(\mu)e^{-\tau_0/\mu} + \frac{\omega}{2}(1-\mu^2)^{m/2} \sum_{\alpha=0}^{N} a_\alpha P_\alpha(2\mu-1) \quad (10a)$$

and

$$I(\tau_0,\mu) = F_1(\mu)e^{-\tau_0/\mu} + \frac{\omega}{2}(1-\mu^2)^{m/2} \sum_{\alpha=0}^{N} b_\alpha P_\alpha(2\mu-1) \quad (10b)$$

for $\mu\varepsilon[0,1]$ and

$$I(\tau,-\mu) = F_2(\mu)e^{-(\tau_0-\tau)/\mu} + \frac{\omega}{2}(1-\mu^2)^{m/2} \sum_{\alpha=0}^{N} c_\alpha(\tau) P_\alpha(2\mu-1) \quad (11a)$$

and

$$I(\tau,\mu) = F_1(\mu)e^{-\tau/\mu} + \frac{\omega}{2}(1-\mu^2)^{m/2} \sum_{\alpha=0}^{N} d_\alpha(\tau) P_\alpha(2\mu-1) \quad (11b)$$

for $\mu\varepsilon[0,1]$ and $\tau\varepsilon(0,\tau_0)$. Here (for $\omega\neq1$) the constants $\{a_\alpha,b_\alpha\}$ are the solutions to the linear algebraic equations

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi_\beta) + b_\alpha A_\alpha(\xi_\beta)e^{-\tau_0/\xi_\beta}] = 2E(0,-\xi_\beta) \quad (12a)$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi_\beta) + a_\alpha A_\alpha(\xi_\beta)e^{-\tau_0/\xi_\beta}] = 2E(\tau_0,\xi_\beta), \quad (12b)$$

and similarly the coefficients $\{c_\alpha(\tau),d_\alpha(\tau)\}$ are the solutions to the linear system

$$\sum_{\alpha=0}^{N} [c_\alpha(\tau)B_\alpha(\xi_\beta) - d_\alpha(\tau)A_\alpha(\xi_\beta)] = 2E(\tau,-\xi_\beta) - e^{-(\tau_0-\tau)/\xi_\beta} \sum_{\alpha=0}^{N} b_\alpha A_\alpha(\xi_\beta) \quad (13a)$$

and

$$\sum_{\alpha=0}^{N} [d_\alpha(\tau)B_\alpha(\xi_\beta) - c_\alpha(\tau)A_\alpha(\xi_\beta)] = 2E(\tau,\xi_\beta)$$

$$- e^{-\tau/\xi_\beta} \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi_\beta). \tag{13b}$$

Here

$$E(\tau,-\xi) = \int_0^1 [G(\xi,\mu)F_2(\mu)C(\tau_0-\tau:\mu,\xi)$$

$$+ G(-\xi,\mu)F_1(\mu)e^{-\tau/\mu} S(\tau_0-\tau:\mu,\xi)]\mu d\mu \tag{14a}$$

and

$$E(\tau,\xi) = \int_0^1 [G(\xi,\mu)F_1(\mu)C(\tau:\mu,\xi)$$

$$+ G(-\xi,\mu)F_2(\mu)e^{-(\tau_0-\tau)/\mu} S(\tau:\mu,\xi)]\mu d\mu \tag{14b}$$

with

$$C(\tau:\mu,\xi) = \frac{e^{-\tau/\mu} - e^{-\tau/\xi}}{\mu - \xi} \tag{15a}$$

and

$$S(\tau:\mu,\xi) = \frac{1 - e^{-\tau/\mu} e^{-\tau/\xi}}{\mu + \xi} . \tag{15b}$$

Further

$$G(\xi,\mu) = \sum_{\ell=m}^{L} \beta_\ell^m P_\ell^m(\mu) g_\ell^m(\xi) \tag{16}$$

where the polynomials $g_\ell^m(\xi)$ are those of Chandrasekhar[5], *i.e.* $g_m^m(\xi)=(2m-1)!!$ and, for $\ell\geq m$,

$$(\ell-m+1)g_{\ell+1}^m(\xi) = h_\ell \xi g_\ell^m(\xi) - (\ell+m)(1-\delta_{m,\ell})g_{\ell-1}^m(\xi) \tag{17}$$

with

$$h_\ell = 2\ell+1 - \omega\beta_\ell. \tag{18}$$

We use here a variation on the collocation strategy introduced by Garcia and Siewert[6], *i.e.* $\xi_\beta=\nu_\beta$, $\beta=0,1,2,\cdots,\kappa-1$, where the ν_β are the positive discrete eigenvalues[7], and

$$\xi_\beta = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\beta+1-\kappa}{N+2-\kappa}\pi\right), \quad \beta = \kappa,\kappa+1,\cdots,N. \tag{19}$$

Here the ξ_β, $\beta=\kappa,\kappa+1,\cdots,N$, are the zeros of the Chebyshev polynomial of the second kind $U_{N+1-\kappa}(2\xi-1)$. The functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ appearing in Eqs. (12) and (13) are defined[4] by

$$A_\alpha(\xi) = \frac{2}{\xi}\int_0^1 \mu(1-\mu^2)^{m/2}\, P_\alpha(2\mu-1)\phi(-\xi,\mu)d\mu \tag{20a}$$

and

$$B_\alpha(\xi) = \frac{2}{\xi}\int_0^1 \mu(1-\mu^2)^{m/2}\, P_\alpha(2\mu-1)\phi(\xi,\mu)d\mu \tag{20b}$$

where the (generalized) functions $\phi(\xi,\mu)$ are those established by McCormick and Kuščer[7].

In the following section we summarize our methods for computing the discrete eigenvalues $\{\nu_\beta\}$ and the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ for $\xi\epsilon[0,1]\cup\{\nu_\beta\}$. Here we consider that these quantities are available, and we can therefore solve Eqs. (12) and (13) to find the constants required in Eqs. (10) and (11) to yield the desired components of the intensity for all μ and all τ. However we would like to show how a technique used previously[8,9] can be generalized and employed here to improve the intensity calculation. From Eq. (20b) we can deduce that

$$B_\alpha(\xi) = 2P_\alpha(2\xi-1) - [2\omega f_m - A_0(\xi)]P_\alpha(2\xi-1) - G_\alpha(\xi) \tag{21}$$

where the polynomials $G_\alpha(\xi)$ can be generated, for $\xi\epsilon[0,1]$, in the forward direction from

$$(\alpha+1)G_{\alpha+1}(\xi) = (2\alpha+1)(2\xi-1)G_\alpha(\xi) - \alpha G_{\alpha-1}(\xi) + $$

$$+ 2\omega(2\alpha+1)\sum_{\ell=m}^{L} \beta_\ell^m g_\ell^m(\xi)T_{\alpha,\ell}^m \tag{22}$$

with $G_0(\xi)=0$. We note that $f_m=\beta_m/(2m+1)$ and that the constants

$$T^m_{\alpha,\ell} = \int_0^1 \mu(1-\mu^2)^{m/2} P_\alpha(2\mu-1)P^m_\ell(\mu)d\mu \tag{23}$$

can be computed accurately in the way described in the following section. We can now use Eq. (21) in the versions of Eqs. (12) valid for $\xi\epsilon[0,1]$ and note Eqs. (10) to obtain, for $\mu\epsilon[0,1]$,

$$I(0,-\mu) = F_2(\mu)e^{-\tau_0/\mu} + \frac{\omega}{2}(1-\mu^2)^{m/2} E(0,-\mu) + \frac{\omega}{4}(1-\mu^2)^{m/2} F(0,-\mu) \tag{24a}$$

and

$$I(\tau_0,\mu) = F_1(\mu)e^{-\tau_0/\mu} + \frac{\omega}{2}(1-\mu^2)^{m/2} E(\tau_0,\mu) + \frac{\omega}{4}(1-\mu^2)^{m/2} F(\tau_0,\mu) \tag{24b}$$

where

$$F(0,-\mu) = [2\omega f_m - A_0(\mu)] \sum_{\alpha=0}^{N} a_\alpha P_\alpha(2\mu-1) + \sum_{\alpha=0}^{N} [a_\alpha G_\alpha(\mu) - b_\alpha A_\alpha(\mu)e^{-\tau_0/\mu}] \tag{25a}$$

and

$$F(\tau_0,\mu) = [2\omega f_m - A_0(\mu)] \sum_{\alpha=0}^{N} b_\alpha P_\alpha(2\mu-1) + \sum_{\alpha=0}^{N} [b_\alpha G_\alpha(\mu) - a_\alpha A_\alpha(\mu)e^{-\tau_0/\mu}]. \tag{25b}$$

In a similar way we can deduce, for $\tau\epsilon(0,\tau_0)$ and $\mu\epsilon[0,1]$, that

$$I(\tau,-\mu) = F_2(\mu)e^{-(\tau_0-\tau)/\mu} + \frac{\omega}{2}(1-\mu^2)^{m/2}\, E(\tau,-\mu) + \frac{\omega}{4}(1-\mu^2)^{m/2}\, F(\tau,-\mu) \quad (26a)$$

and

$$I(\tau,\mu) = F_1(\mu)e^{-\tau/\mu} + \frac{\omega}{2}(1-\mu^2)^{m/2}\, E(\tau,\mu) + \frac{\omega}{4}(1-\mu^2)^{m/2}\, F(\tau,\mu) \quad (26b)$$

where, for $\tau\varepsilon(0,\tau_0)$ and $\mu\varepsilon[0,1]$,

$$F(\tau,-\mu) = [2\omega f_m - A_0(\mu)] \sum_{\alpha=0}^{N} c_\alpha(\tau)P_\alpha(2\mu-1) + \sum_{\alpha=0}^{N} \{c_\alpha(\tau)G_\alpha(\mu) + [d_\alpha(\tau) - b_\alpha\, e^{-(\tau_0-\tau)/\mu}]A_\alpha(\mu)\} \quad (27a)$$

and

$$F(\tau,\mu) = [2\omega f_m - A_0(\mu)] \sum_{\alpha=0}^{N} d_\alpha(\tau)P_\alpha(2\mu-1) + \sum_{\alpha=0}^{N} \{d_\alpha(\tau)G_\alpha(\mu) + [c_\alpha(\tau) - a_\alpha\, e^{-\tau/\mu}]A_\alpha(\mu)\}. \quad (27b)$$

We observe that our final results given by Eqs. (24) and (26) are exact to order ω as $\omega\to 0$.

As mentioned, the collocation strategy we use here is based on using points from the continuum $\nu\varepsilon[0,1]$ and the discrete spectrum, *i.e.* the (positive) zeros $\{\nu_\beta\}$ of the dispersion function[7]

$$\Lambda(z) = 1 + z\int_{-1}^{1} \psi(\mu)\,\frac{d\mu}{\mu-z} \quad (28)$$

where

$$\psi(\mu) = \frac{\omega}{2}(1-\mu^2)^{m/2}\, G(\mu,\mu). \quad (29)$$

We note[3,10] that

$$\Lambda(\infty) = \prod_{\ell=m}^{L} (1-\omega f_\ell) \tag{30}$$

from which we conclude, since $0<\omega\le 1$, $f_0=1$ and $|f_\ell|<1$ for $\ell>0$, that there is a discrete eigenvalue at infinity only if $\omega=1$ and only if $m=0$. From the expressions given by Devaux and Siewert[3] we deduce for $\omega=1$ and $m=0$ that

$$\Lambda(z) \to \Lambda(\infty) - \frac{1}{3z^2} \prod_{\ell=2}^{L} (1-f_\ell) + \frac{a_4}{z^4} + \cdots \tag{31}$$

as $|z|\to\infty$. It follows that $\Lambda(z)$ cannot have more than one zero at infinity and that this zero has multiplicity two. Continuing with the case $m=0$ and $\omega=1$, we note that for $\xi_\beta\to\infty$ each of Eqs. (12) reduces to

$$\sum_{\alpha=0}^{N} (a_\alpha+b_\alpha)T^0_{\alpha,0} = 2\int_0^1 [F_1(\mu) + F_2(\mu)][1 - e^{-\tau_0/\mu}]\mu d\mu \tag{32}$$

and each of Eqs. (13) reduces to

$$\sum_{\alpha=0}^{N} [c_\alpha(\tau) - d_\alpha(\tau)]T^0_{\alpha,0} = 2\int_0^1 [F_1(\mu)e^{-\tau/\mu} + F_2(\mu)][1 - e^{-(\tau_0-\tau)/\mu}]\mu d\mu - \sum_{\alpha=0}^{N} b_\alpha T^0_{\alpha,0}. \tag{33}$$

It is thus apparent that for this case an additional equation is required to supplement each of Eqs. (32) and (33). If we multiply Eq. (7) by μ and integrate over μ from -1 to 1 we find, for $\omega=1$ and $m=0$, that

$$\frac{d}{d\tau}\int_{-1}^{1} \mu^2 I(\tau,\mu)d\mu + \frac{1}{3} h_1 \left(\frac{F_\star}{2\pi}\right) = 0 \tag{34}$$

where the diffuse component of the net flux,

$$F_* = 2\pi \int_{-1}^{1} \mu I(\tau,\mu)d\mu, \tag{35}$$

is a constant[5]. We can now integrate Eq. (34) to obtain

$$\int_{-1}^{1} \mu^2[I(\tau_0,\mu) - I(\tau,\mu)]d\mu + \frac{1}{3} h_1(\tau_0-\tau) \int_{-1}^{1} \mu I(\tau_0,\mu)d\mu = 0. \tag{36}$$

Substituting Eqs. (8) and (10) into Eq. (36) evaluated at τ=0 and substituting Eqs. (8b), (10b) and (11) into Eq. (36), we find equations supplemental to Eqs. (32) and (33), *viz.*

$$\sum_{\alpha=0}^{N} (a_\alpha - b_\alpha)T^0_{\alpha,1} - \frac{1}{3} h_1\tau_0 \sum_{\alpha=0}^{N} b_\alpha T^0_{\alpha,0} = 2R(0) \tag{37}$$

and

$$\sum_{\alpha=0}^{N} [c_\alpha(\tau)+d_\alpha(\tau)]T^0_{\alpha,1} = 2R(\tau) + \sum_{\alpha=0}^{N} b_\alpha T^0_{\alpha,1} + \frac{1}{3} h_1(\tau_0-\tau) \sum_{\alpha=0}^{N} b_\alpha T^0_{\alpha,0} \tag{38}$$

where

$$R(\tau) = \int_0^1 \mu^2[F_2(\mu) - F_1(\mu)e^{-\tau/\mu}][1 - e^{-(\tau_0 -\tau)/\mu}]d\mu + \frac{1}{3} h_1(\tau_0-\tau) \int_0^1 \mu[F_1(\mu)e^{-\tau_0/\mu} - F_2(\mu)]d\mu. \tag{39}$$

As our basic analysis is now complete, we are ready to consider the computational aspects of the developed solution.

IV. COMPUTATIONAL METHODS

The first task we face in evaluating our developed F_N solution is to calculate $\{\nu_\beta\}$, the κ positive zeros of the dispersion function $\Lambda(z)$ given by Eq. (28). As all of our previous F_N work concerned problems for which κ was less than four, we have generally evaluated the exact expressions[11] for ν_β to obtain results accurate to, say, six significant figures; and those results were then refined by the Newton iteration scheme. However, in attempting to use the "exact" technique[11] to compute the discrete eigenvalues for the cloud problem discussed in Section VI we encountered computational difficulties that derived from the facts that κ was large (*e.g.* κ=25) and that many of the zeros of $\Lambda(z)$ were not well separated (from a computational point of view). Thus for computing the $\{\nu_\beta\}$ we use here a particularly simple and effective approximate technique that is common to the discrete ordinates method[5] and the spherical harmonics method[12,13,14]. In a recent work[15] we used some ideas reported by İnönü[16] for the m=0 case to derive the expression

$$(1-z^2)^{m/2}\, P^m_{L+1}(z)\Lambda(z) = (1-z^2)^m\, g^m_{L+1}(z) - 2z\psi(z)Q^m_{L+1}(z) \quad (40)$$

where

$$Q^m_\ell(z) = \frac{1}{2}\int_{-1}^{1} (1-\mu^2)^{m/2}\, P^m_\ell(\mu)\, \frac{d\mu}{z-\mu}. \quad (41)$$

Considering that $z\notin[-1,1]$, we can rewrite Eq. (40) as

$$\Lambda(z) = (1-z^2)^{m/2}\,\frac{g^m_{L+1}(z)}{P^m_{L+1}(z)} - 2z(1-z^2)^{-m/2}\,\psi(z)\,\frac{Q^m_{L+1}(z)}{P^m_{L+1}(z)}. \quad (42)$$

Since $Q^m_{L+1}(z)$ is bounded for $|z|>1$ and since $P^m_{L+1}(z)$ has a polynomial factor (of degree L+1-m), it is clear that for $|z|>1$

$$\lim_{L\to\infty} \frac{Q^m_{L+1}(z)}{P^m_{L+1}(z)} = 0 \quad (43)$$

so that we can deduce from Eq. (42) that

$$\Lambda(z) = \lim_{L\to\infty} (1-z^2)^{m/2}\,\frac{g^m_{L+1}(z)}{P^m_{L+1}(z)}\ , \ |z| > 1. \quad (44)$$

Thus for some sufficiently large L the zeros of $g^m_{L+1}(z)$, all of which are real, outside the real interval [-1,1] provide a good approximation to the zeros of $\Lambda(z)$. We note that in contrast to the discrete ordinates method and the spherical harmonics method where all the zeros of $g^m_{L+1}(z)$ are of interest, we require only those zeros outside the real interval [-1,1].

A spherical harmonics approximation of order N (odd) yields (N+1)/2 pairs of eigenvalues $\{\eta_j\}$ that are the zeros of the polynomial $g^m_{m+N+1}(\xi)$ as defined by Eq. (17). A recursive relation involving only even $g^m_\ell(\xi)$ can be easily obtained from Eq. (17). We have for ℓ=m,m+2,$\cdots$

$$(1-\delta_{m,\ell})\left[\frac{(\ell+m-1)(\ell+m)}{h_{\ell-1}}\right] g^m_{\ell-2}(\xi) + \left\{\left[\frac{(\ell+1)^2 - m^2}{h_{\ell+1}}\right] + \left[\frac{\ell^2 - m^2}{h_{\ell-1}}\right] - h_\ell \xi^2\right\} g^m_\ell(\xi) + \left[\frac{(\ell-m+1)(\ell-m+2)}{h_{\ell+1}}\right] g^m_{\ell+2}(\xi) = 0. \quad (45)$$

By using Eq. (45) for ℓ = m,m+2,$\cdots$,m+N-1, we see that the problem of finding the positive zeros of $g^m_{m+N+1}(\xi)$, say $\{\eta_j\}$, can be restated as one of finding the eigenvalues $\{\eta_j^2\}$ of the tridiagonal matrix $\underset{\sim}{E}$ of order (N+1)/2 with elements

$$E_{\alpha,\alpha+1} = \frac{(2\alpha-1)2\alpha}{h_{m+2\alpha-2}\, h_{m+2\alpha-1}}, \quad (46a)$$

$$E_{\alpha,\alpha} = \frac{1}{h_{m+2\alpha-2}}\left[\frac{4(\alpha-1)(m+\alpha-1)}{h_{m+2\alpha-3}} + \frac{(2\alpha-1)(2m+2\alpha-1)}{h_{m+2\alpha-1}}\right] \quad (46b)$$

and

$$E_{\alpha+1,\alpha} = \frac{2(2m+2\alpha-1)(m+\alpha)}{h_{m+2\alpha}\, h_{m+2\alpha-1}}, \quad (46c)$$

where $\alpha\geq 1$. As discussed in Section III, in the event that $\omega=1$ and $m=0$ there is one pair of unbounded eigenvalues, and thus the above scheme requires a minor modification. Since $h_0=0$ in this case, we note from Eq. (45) that both $g_0(\xi)$ and $g_2(\xi)$ are constants. After using Eq. (45) for $\ell=m+2,m+4,\cdots,m+N-1$ and expressing $g_0(\xi)$ in terms of $g_2(\xi)$ we conclude that the squares of the bounded zeros of $g_{N+1}(\xi)$ for $\omega=1$ are the eigenvalues of the tridiagonal matrix $\underset{\sim}{F}$ of order $(N-1)/2$ given, for $\alpha\geq 1$, by

$$F_{\alpha,\alpha+1} = E_{\alpha+1,\alpha+2}, \tag{47a}$$

$$F_{\alpha,\alpha} = E_{\alpha+1,\alpha+1} - \left(\frac{4}{h_1 h_2}\right) \delta_{\alpha,1} \tag{47b}$$

and

$$F_{\alpha+1,\alpha} = E_{\alpha+2,\alpha+1}. \tag{47c}$$

In this work we have used a FORTRAN program in the EISPACK package[17] to compute the required eigenvalues of the appropriate tridiagonal matrix.

Clearly we can increase N in the foregoing spherical harmonics version of our eigenvalue problem until all of the eigenvalues $\eta_j>1$ converge to a desired degree of accuracy (we seek 15 correct figures); however, we have found the convergence of this approach to be slow in the event any of the $\eta_j>1$ are very close to unity (say $\eta_j\approx 1.000001$). For this reason we deduce those eigenvalues that are close to unity by using the spherical harmonics results (obtained for a computationally reasonable value of N, say $N\leq 1999$) to initiate a bisection calculation of the zeros of $\Lambda(z)$, which we express here as

$$\Lambda(z) = 1 + z\psi(z)\,log\left(\frac{z-1}{z+1}\right) + \omega z \sum_{\ell=m}^{L} \beta_\ell^m\, g_\ell^m(z)\Gamma_\ell^m(z) \tag{48}$$

where, as discussed previously[3], the polynomials

$$\Gamma_\ell^m(z) = \frac{1}{2}\int_{-1}^{1} [(1-\mu^2)^{m/2}\, P_\ell^m(\mu) - (1-z^2)^{m/2}\, P_\ell^m(z)]\,\frac{d\mu}{\mu-z} \tag{49}$$

can be evaluated recursively.

We now discuss a scheme for computing in a convenient and accurate way the constants $T^m_{\alpha,\ell}$ defined by Eq. (23). To start, for m=0, the first two rows of the $\underset{\sim}{T}$ matrix can be computed from

$$T^0_{0,\ell+1} = \left(\frac{2-\ell}{3+\ell}\right) T^0_{0,\ell-1}, \tag{50a}$$

$$T^0_{1,\ell+1} = \left(\frac{4}{4+\ell}\right) T^0_{0,\ell} - T^0_{0,\ell+1} \tag{50b}$$

and the initial values $T^0_{0,0}=1/2$, $T^0_{0,1}=1/3$ and $T^0_{1,0}=1/6$. The remaining non-zero $T^0_{\alpha,\ell}$ (in general $T^m_{\alpha,\ell}=0$ for $\alpha>\ell+m+1$) can be computed from[9]

$$T^0_{\alpha,\ell+1} = \frac{1}{2}\left(\frac{2\ell+1}{\ell+1}\right)\left[\left(\frac{\alpha}{2\alpha+1}\right) T^0_{\alpha-1,\ell} + T^0_{\alpha,\ell} + \left(\frac{\alpha+1}{2\alpha+1}\right) T^0_{\alpha+1,\ell}\right] - \left(\frac{\ell}{\ell+1}\right) T^0_{\alpha,\ell-1}. \tag{51}$$

We have found that the recursion formula previously derived[4] to generate the initial values for m>0 does not yield accurate results when m is large, say m>70, and thus here for m>0 we use

$$(2\ell+1)T^{m+1}_{\alpha,\ell} = (\ell+m)(\ell+m+1)T^m_{\alpha,\ell-1} - (\ell-m)(\ell-m+1)T^m_{\alpha,\ell+1} \tag{52}$$

which can be readily deduced from basic properties of the associated Legendre functions $P^m_\ell(\mu)$.

For $\xi<1$ the Chandrasekhar polynomials $g^m_\ell(\xi)$ can be evaluated accurately by forward recursion of Eq. (17) with $g^m_m(\xi)=(2m-1)!!$. For $\xi=\nu_\beta>1$ the Chandrasekhar polynomials can be evaluated accurately by backward recursion. Following Gautschi[18] and considering that we require $g^m_\ell(\xi)$ for $\ell=m,m+1,\cdots,Q$ we take, for some $M\geq Q$, $R^m_M(\xi)=0$ and compute

$$R^m_{\ell-1}(\xi) = \left(\frac{\ell+m}{\ell-m+1}\right)\left[\left(\frac{\xi}{\ell-m+1}\right) h_\ell - R^m_\ell(\xi)\right]^{-1} \tag{53}$$

for ℓ=M,M-1,$\cdots$,m+3. We then take

$$g_m^m(\xi) = (2m-1)!! \tag{54}$$

$$g_{m+1}^m(\xi) = (2m-1)!!h_m\xi \tag{55}$$

$$g_{m+2}^m(\xi) = \frac{1}{2}\,(2m-1)!!(h_{m+1}h_m\xi^2 - 2m-1) \tag{56}$$

and

$$g_\ell^m(\xi) = R_{\ell-1}^m(\xi)g_{\ell-1}^m(\xi) \quad , \ \ell = m+3,m+4,\cdots,Q. \tag{57}$$

Finally we increase M and repeat the calculation until the required $g_\ell^m(\xi)$, ℓ=m,m+1,$\cdots$,Q, have converged to the desired degree of accuracy.

The functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$, for $\xi\epsilon[0,1]\cup\{\nu_\beta\}$, can also be evaluated conveniently by recursion formulas. First of all for $\nu\epsilon[0,1]$ we compute the starting value $B_0(\nu)$ from

$$B_0(\nu) = A_0(\nu) + \left(\frac{2}{2m+1}\right) h_m \tag{58}$$

where

$$A_0(\nu) = \omega \sum_{\ell=m}^{L} \beta_\ell^m\, g_\ell^m(\nu)\pi_\ell^m(\nu) - 2\nu\psi(\nu)\, ln\left(1 + \frac{1}{\nu}\right) \tag{59}$$

and the polynomials $\pi_\ell^m(\nu)$ can be computed efficiently by previously reported[3] recursion formulas. Then we use forward recursion[4] to find the remaining $B_\alpha(\nu)$, *i.e.*

$$B_{\alpha+1}(\nu) = \left(\frac{2\alpha+1}{\alpha+1}\right) [(2\nu-1)B_\alpha(\nu) - 2\omega Z_\alpha(-\nu)] - \left(\frac{\alpha}{\alpha+1}\right) B_{\alpha-1}(\nu) \tag{60}$$

for α=0,1,$\cdots$,N-1, where for $\pm\xi\epsilon[0,1]\cup\{\nu_\beta\}$ we use the definition

$$Z_\alpha(\xi) = \sum_{\ell=m}^{L} (-1)^{\ell-m} \beta_\ell^m g_\ell^m(\xi) T_{\alpha,\ell}^m. \tag{61}$$

If we consider the definition

$$A_\alpha(\xi) = \frac{2}{\xi} \int_0^1 \mu(1-\mu^2)^{m/2} P_\alpha(2\mu-1)\phi(-\xi,\mu)d\mu \tag{62}$$

to be valid for $\xi\varepsilon\{-\nu_\beta\}$ as well as $\xi\varepsilon[0,1]\cup\{\nu_\beta\}$, then clearly we have $B_\alpha(\nu_\beta)=-A_\alpha(-\nu_\beta)$, and thus we summarize our methods for computing $A_\alpha(\xi)$ for $\xi\varepsilon[0,1]\cup\{\pm\nu_\beta\}$. Our general procedure for computing $A_\alpha(\xi)$ is to use backward recursion; however, for ξ near 0 or -1, say for example $\xi=\nu\varepsilon[0,0.001]$ and $\xi=-\nu_\beta\varepsilon[-1.001,-1)$, we find it expedient to use forward recursion. Thus, for $\nu_\beta\varepsilon(1,1.001]$ we use

$$B_0(\nu_\beta) = A_0(\nu_\beta) + \left(\frac{2}{2m+1}\right) h_m, \tag{63}$$

with

$$A_0(\nu_\beta) = \omega \sum_{\ell=m}^{L} \beta_\ell^m g_\ell^m(\nu_\beta)\pi_\ell^m(\nu_\beta) - 2\nu_\beta\psi(\nu_\beta)\, ln\, (1 + 1/\nu_\beta), \tag{64}$$

and

$$B_{\alpha+1}(\nu_\beta) = \left(\frac{2\alpha+1}{\alpha+1}\right) [(2\nu_\beta-1)B_\alpha(\nu_\beta) - 2\omega Z_\alpha(-\nu_\beta)] - \left(\frac{\alpha}{\alpha+1}\right) B_{\alpha-1}(\nu_\beta) \tag{65}$$

for $\alpha=0,1,\cdots,N-1$; and for $\nu\varepsilon[0,0.001]$ and $\alpha=0,1,\cdots,N-1$ we use

$$A_{\alpha+1}(\nu) = \left(\frac{2\alpha+1}{\alpha+1}\right) [2\omega Z_\alpha(\nu) - (2\nu+1)A_\alpha(\nu)] - \left(\frac{\alpha}{\alpha+1}\right) A_{\alpha-1}(\nu), \tag{66}$$

where the starting value $A_0(\nu)$ is given by Eq. (59).

Considering now that $\xi\epsilon[0,1]\cup\{\pm\nu_\beta\}$ and $\xi\notin[-1.001,0.001]$, we describe the method of backward recursion[9] we use to compute $A_\alpha(\xi)$ for $\alpha=0,1,\cdots,N$. First of all we use the starting value

$$U_0(\xi) = \frac{1}{2\xi+1} \tag{67}$$

and forward recursion, *i.e.*

$$U_\alpha(\xi) = \left[\left(\frac{2\alpha+1}{\alpha+1}\right)(2\xi+1) - \left(\frac{\alpha}{\alpha+1}\right)U_{\alpha-1}(\xi)\right]^{-1} \tag{68}$$

for $\alpha=1,2,\cdots$ to compute the ratios $U_\alpha(\xi)=P_\alpha(2\xi+1)/P_{\alpha+1}(2\xi+1)$. We then use forward recursion, *i.e.*

$$V_0(\xi) = U_0(\xi)Z_0(\xi) \tag{69}$$

and

$$V_\alpha(\xi) = -U_\alpha(\xi)[V_{\alpha-1}(\xi) - (2\alpha+1)Z_\alpha(\xi)] \tag{70}$$

for $\alpha=1,2,\cdots$ to compute the functions $V_\alpha(\xi)$, and we express our two-term recursion formula[9] for $A_\alpha(\xi)$ as

$$A_\alpha(\xi) = -U_\alpha(\xi)A_{\alpha+1}(\xi) + \left(\frac{2\omega}{\alpha+1}\right)V_\alpha(\xi). \tag{71}$$

Now given a starting value $A_{M+1}(\xi)$ for some $M\geq N$ we can compute $A_\alpha(\xi)$ for $\alpha=M,M-1,\cdots,0$ from Eq. (71). For $M+1>L+m$ we can deduce from Eq. (62) that

$$A_{M+1}(\xi) = 2\psi(\xi)F_{M+1}(\xi) \tag{72}$$

where

$$F_{M+1}(\xi) = \int_0^1 \mu P_{M+1}(2\mu-1)\,\frac{d\mu}{\mu+\xi}. \tag{73}$$

We therefore use two different procedures for computing $A_\alpha(\xi)$ from Eq. (71). First of all for $\xi=-\nu_\beta<-1.001$ or $\xi=\nu_\beta>1$ we take $A_{M+1}(\xi)=0$, for some $M>\max\{N,L+m-1\}$ and use Eq. (71) to compute

$A_\alpha(\xi)$ for $\alpha=M,M-1,\cdots,0$; we then increase M and repeat the calculation until $A_\alpha(\xi)$, for $\alpha=0,1,\cdots,N$, has converged to the desired degree of accuracy. On the other hand, for $\xi\epsilon(0.001,1]$ we compute (in a manner to be discussed) $F_{M+1}(\xi)$ for some $M>\max\{N,L+m-1\}$ and use Eq. (72) to establish the starting value $A_{M+1}(\xi)$ so that Eq. (71) can be used to compute $A_\alpha(\xi)$ for $\alpha=M,M-1,\cdots,0$.

To complete this section we summarize our way to use backward recursion to compute $F_{M+1}(\xi)$ for $\xi\epsilon(0.001,1]$. We again follow Gautschi[18] and write, for some $L>M$,

$$S_L = 0 \tag{74}$$

and, for $\alpha=L,L-1,\cdots,2$,

$$S_{\alpha-1} = -\left(\frac{\alpha}{\alpha+1}\right)\left[\left(\frac{2\alpha+1}{\alpha+1}\right)(2\xi+1) + S_\alpha\right]^{-1}. \tag{75}$$

We then take

$$F_1(\xi) = \xi\left[(2\xi+1)\,ln\left(1+\frac{1}{\xi}\right) - 2\right] \tag{76}$$

and

$$F_{\alpha+1}(\xi) = S_\alpha(\xi)F_\alpha(\xi) \tag{77}$$

for $\alpha=1,2,\cdots,M$. We then increase L and repeat the calculation until $F_{M+1}(\xi)$ has converged to the desired degree of accuracy.

V. LINEAR ALGEBRAIC EQUATIONS

We consider, first of all, $\omega\epsilon(0,1)$ and $m\geq 0$, and we write Eqs. (12) in the form

$$\begin{vmatrix} \underset{\sim}{B} & \underset{\sim}{D}(\tau_0)\underset{\sim}{A} \\ \underset{\sim}{D}(\tau_0)\underset{\sim}{A} & \underset{\sim}{B} \end{vmatrix} \begin{vmatrix} \underset{\sim}{a} \\ \underset{\sim}{b} \end{vmatrix} = 2 \begin{vmatrix} \underset{\sim}{E}_1(0) \\ \underset{\sim}{E}_2(\tau_0) \end{vmatrix} \tag{78}$$

where

$$\underset{\sim}{B} = \begin{vmatrix} B_0(\xi_0) & \cdots & B_N(\xi_0) \\ \vdots & & \vdots \\ B_0(\xi_N) & \cdots & B_N(\xi_N) \end{vmatrix}, \tag{79}$$

$$\underset{\sim}{A} = \begin{vmatrix} A_0(\xi_0) & \cdots & A_N(\xi_0) \\ \vdots & & \vdots \\ A_0(\xi_N) & \cdots & A_N(\xi_N) \end{vmatrix}, \tag{80}$$

$$\underset{\sim}{D}(x) = \text{diag}\left\{e^{-x/\xi_0}, e^{-x/\xi_1}, \cdots, e^{-x/\xi_N}\right\} \tag{81}$$

and the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{E}_1(x)$ and $\underset{\sim}{E}_2(x)$ respectively have elements $\{a_\alpha\}$, $\{b_\alpha\}$, $\{E(x,-\xi_\beta)\}$ and $\{E(x,\xi_\beta)\}$. Now following a suggestion of Dubrulle[19], we write the solution to Eq. (78) as

$$\underset{\sim}{a} = \underset{\sim}{X} + \underset{\sim}{Y} \quad \text{and} \quad \underset{\sim}{b} = \underset{\sim}{X} - \underset{\sim}{Y} \tag{82a,b}$$

where $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are solutions of

$$[\underset{\sim}{B} + \underset{\sim}{D}(\tau_0)\underset{\sim}{A}]\underset{\sim}{X} = \underset{\sim}{E}_1(0) + \underset{\sim}{E}_2(\tau_0) \tag{83a}$$

and

$$[\underset{\sim}{B} - \underset{\sim}{D}(\tau_0)\underset{\sim}{A}]\underset{\sim}{Y} = \underset{\sim}{E}_1(0) - \underset{\sim}{E}_2(\tau_0) \quad . \tag{83b}$$

In this way we have the advantage of solving two linear systems [Eqs. (83)] of order N+1 rather than one linear system [Eq. (78)] of order 2(N+1).

Proceeding in a manner analogous to the foregoing development, we write Eqs. (13) as

$$\begin{vmatrix} \underset{\sim}{B} & -\underset{\sim}{A} \\ -\underset{\sim}{A} & \underset{\sim}{B} \end{vmatrix} \begin{vmatrix} \underset{\sim}{c}(\tau) \\ \underset{\sim}{d}(\tau) \end{vmatrix} = 2 \begin{vmatrix} \underset{\sim}{F}_1(\tau) \\ \underset{\sim}{F}_2(\tau) \end{vmatrix} \tag{84}$$

where

$$\underset{\sim}{F}_1(\tau) = \underset{\sim}{E}_1(\tau) - \frac{1}{2}\underset{\sim}{D}(\tau_0-\tau)\underset{\sim}{A}\underset{\sim}{b} \tag{85a}$$

and

$$\underset{\sim}{F}_2(\tau) = \underset{\sim}{E}_2(\tau) - \frac{1}{2}\underset{\sim}{D}(\tau)\underset{\sim}{A}\underset{\sim}{a}, \tag{85b}$$

and express the desired solution as

$$\underset{\sim}{c}(\tau) = \underset{\sim}{X}(\tau) + \underset{\sim}{Y}(\tau) \quad \text{and} \quad \underset{\sim}{d}(\tau) = \underset{\sim}{X}(\tau) - \underset{\sim}{Y}(\tau) \tag{86a,b}$$

where now

$$(\underset{\sim}{B}-\underset{\sim}{A})\underset{\sim}{X}(\tau) = \underset{\sim}{F}_1(\tau) + \underset{\sim}{F}_2(\tau) \tag{87a}$$

and

$$(\underset{\sim}{B}+\underset{\sim}{A})\underset{\sim}{Y}(\tau) = \underset{\sim}{F}_1(\tau) - \underset{\sim}{F}_2(\tau). \tag{87b}$$

Upon considering the case $m=0$ and $\omega=1$ and noting that Eqs. (32) and (37) are not symmetric in $\{a_\alpha\}$ and $\{b_\alpha\}$, we deduce that a minor modification is required before we can write our linear system in a form analogous to Eq. (78). We therefore replace Eq. (32) by

$$\sum_{\alpha=0}^{N} (b_\alpha - a_\alpha)T^0_{\alpha,1} - \frac{1}{3}h_1\tau_0 \sum_{\alpha=0}^{N} a_\alpha T^0_{\alpha,0} = -2\left[\frac{1}{3}h_1\tau_0 R_1 + R(0)\right], \tag{88}$$

a linear combination of Eqs. (32) and (37). Here

$$R_1 = \int_0^1 [F_1(\mu) + F_2(\mu)][1-e^{-\tau_0/\mu}]\mu d\mu. \tag{89}$$

In a similar manner, we replace Eqs. (33) and (38) by

$$\sum_{\alpha=0}^{N} \left\{c_\alpha(\tau)[T^0_{\alpha,0}+T^0_{\alpha,1}] - d_\alpha(\tau)[T^0_{\alpha,0}-T^0_{\alpha,1}]\right\} = W_1(\tau) + W_2(\tau) \tag{90a}$$

and

$$\sum_{\alpha=0}^{N} \left\{c_\alpha(\tau)[T^0_{\alpha,0}-T^0_{\alpha,1}] - d_\alpha(\tau)[T^0_{\alpha,0}+T^0_{\alpha,1}]\right\} = W_1(\tau) - W_2(\tau), \tag{90b}$$

where $W_1(\tau)$ and $W_2(\tau)$ represent respectively the right-hand sides of Eqs. (33) and (38). We now have available a set of equations analogous to Eq. (84).

To complete this section we note that we have used two subroutine packages from the LINPACK collection[20] to solve our linear systems. The first of these packages, essentially the FORTRAN subroutines DGEFA and DGESL, is based on the L-U factorization variant of the Gaussian elimination method and was found adequate for most of our work. However, as discussed in the next section of this paper, we also required a more powerful package, essentially the FORTRAN subroutine DSVDC, that is based on the singular-value decomposition method[21] to solve some of our problems.

VI. NUMERICAL RESULTS

As a first numerical example we consider a problem based on the Mie scattering theory for spherical particles (with size parameter $\alpha=2$ and index of refraction $m=1.33$) and defined by a $L=8$ phase function with $\beta_0=1$, $\beta_1=2.00916$, $\beta_2=1.56339$, $\beta_3=0.67407$, $\beta_4=0.22215$, $\beta_5=0.04725$, $\beta_6=0.00671$, $\beta_7=0.00068$ and $\beta_8=0.00005$. We take $\omega=0.95$, $\tau_0=1$ and $\mu_0=1/2$, and to define the boundary conditions we use

$$F_1(\mu,\phi) = \pi\delta(\mu-\mu_0)\delta(\phi-\phi_0) \tag{91a}$$

and

$$F_2(\mu,\phi) = 0 \tag{91b}$$

for $\mu\epsilon[0,1]$ and $\phi\epsilon[0,2\pi]$. Here μ_0 and ϕ_0 are used to define the direction of the incident beam. Devaux and Siewert used the F_N method to solve this problem and reported[3] numerical results for all of the Fourier components of the intensity - but only at the boundaries. Here we list in Tables 1-9 numerical results which we believe accurate to within ±1 in the sixth significant figure for the Fourier components defined, with $\phi_r=\phi_0$, by

$$I_*(\tau,-\mu) = (2-\delta_{0,m})I(\tau,-\mu) \tag{92a}$$

and

Table 1. The Fourier Component $I_*(\tau,\mu)$ for m = 0.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	4.76807(-2)	4.41912(-2)	4.06467(-2)	3.37099(-2)	1.58572(-2)	5.45297(-3)	
-0.9	6.45644(-2)	6.03743(-2)	5.60139(-2)	4.72899(-2)	2.38121(-2)	9.01083(-3)	
-0.8	8.45877(-2)	7.96763(-2)	7.44425(-2)	6.37540(-2)	3.38287(-2)	1.36919(-2)	
-0.7	1.08350(-1)	1.02719(-1)	9.65678(-2)	8.37488(-2)	4.64918(-2)	1.98845(-2)	
-0.6	1.36504(-1)	1.30204(-1)	1.23128(-1)	1.08063(-1)	6.26048(-2)	2.81721(-2)	
-0.5	1.69677(-1)	1.62852(-1)	1.54919(-1)	1.37618(-1)	8.32921(-2)	3.94792(-2)	
-0.4	2.08230(-1)	2.01201(-1)	1.92626(-1)	1.73358(-1)	1.10124(-1)	5.53631(-2)	
-0.3	2.51677(-1)	2.45065(-1)	2.36310(-1)	2.15806(-1)	1.45144(-1)	7.86298(-2)	
-0.2	2.97523(-1)	2.92401(-1)	2.84241(-1)	2.63800(-1)	1.90035(-1)	1.14557(-1)	
-0.1	3.40126(-1)	3.38476(-1)	3.31984(-1)	3.12888(-1)	2.41217(-1)	1.70697(-1)	
-0.0	3.59379(-1)	3.74485(-1)	3.73800(-1)	3.59347(-1)	2.88258(-1)	2.25623(-1)	
0.0		3.74485(-1)	3.73800(-1)	3.59347(-1)	2.88258(-1)	2.25623(-1)	1.51520(-1)
0.1		1.55620(-1)	2.51427(-1)	3.38864(-1)	3.30703(-1)	2.69855(-1)	2.03073(-1)
0.2		9.24370(-2)	1.65080(-1)	2.61887(-1)	3.35517(-1)	3.01553(-1)	2.44242(-1)
0.3		6.70726(-2)	1.24144(-1)	2.10383(-1)	3.14805(-1)	3.09713(-1)	2.70302(-1)
0.4		5.30170(-2)	9.98939(-2)	1.75304(-1)	2.88183(-1)	3.02939(-1)	2.81066(-1)
0.5		4.37182(-2)	8.32383(-2)	1.49251(-1)	2.61304(-1)	2.88464(-1)	2.80822(-1)
0.6		3.67864(-2)	7.05194(-2)	1.28323(-1)	2.35259(-1)	2.69793(-1)	2.73029(-1)
0.7		3.11411(-2)	5.99908(-2)	1.10387(-1)	2.09939(-1)	2.48476(-1)	2.59822(-1)
0.8		2.62225(-2)	5.07166(-2)	9.42001(-2)	1.84982(-1)	2.25179(-1)	2.42453(-1)
0.9		2.17133(-2)	4.21538(-2)	7.90039(-2)	1.60034(-1)	2.00187(-1)	2.21667(-1)
1.0		1.74231(-2)	3.39716(-2)	6.43193(-2)	1.34819(-1)	1.73627(-1)	1.97932(-1)

Table 2. The Fourier Component $I_*(\tau,\mu)$ for m = 1.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	3.60181(-2)	3.36746(-2)	3.12064(-2)	2.62787(-2)	1.32903(-2)	5.19802(-3)	
-0.8	6.16596(-2)	5.78589(-2)	5.38160(-2)	4.56764(-2)	2.38338(-2)	9.72915(-3)	
-0.7	9.10359(-2)	8.57207(-2)	8.00113(-2)	6.84223(-2)	3.67820(-2)	1.56203(-2)	
-0.6	1.26254(-1)	1.19295(-1)	1.11742(-1)	9.62823(-2)	5.33398(-2)	2.35586(-2)	
-0.5	1.68911(-1)	1.60192(-1)	1.50609(-1)	1.30818(-1)	7.48561(-2)	3.44878(-2)	
-0.4	2.20425(-1)	2.09925(-1)	1.98188(-1)	1.73691(-1)	1.03142(-1)	4.99460(-2)	
-0.3	2.81715(-1)	2.69635(-1)	2.55778(-1)	2.26467(-1)	1.40633(-1)	7.26822(-2)	
-0.2	3.52226(-1)	3.39192(-1)	3.23503(-1)	2.89666(-1)	1.89868(-1)	1.07900(-1)	
-0.1	4.28566(-1)	4.16266(-1)	3.99406(-1)	3.61374(-1)	2.49443(-1)	1.63513(-1)	
-0.0	4.90592(-1)	4.93468(-1)	4.79629(-1)	4.40175(-1)	3.12405(-1)	2.22786(-1)	
0.0		4.93468(-1)	4.79629(-1)	4.40175(-1)	3.12405(-1)	2.22786(-1)	1.40086(-1)
0.1		2.12101(-1)	3.36358(-1)	4.36674(-1)	3.78485(-1)	2.79507(-1)	1.92369(-1)
0.2		1.27652(-1)	2.24094(-1)	3.44034(-1)	3.98038(-1)	3.26312(-1)	2.41187(-1)
0.3		9.31423(-2)	1.69547(-1)	2.78455(-1)	3.79153(-1)	3.42883(-1)	2.74467(-1)
0.4		7.34351(-2)	1.36109(-1)	2.31599(-1)	3.47468(-1)	3.37079(-1)	2.87903(-1)
0.5		5.97940(-2)	1.11999(-1)	1.94715(-1)	3.11216(-1)	3.17429(-1)	2.84863(-1)
0.6		4.89801(-2)	9.23641(-2)	1.62887(-1)	2.72157(-1)	2.88095(-1)	2.68557(-1)
0.7		3.94407(-2)	7.47181(-2)	1.33107(-1)	2.29842(-1)	2.50372(-1)	2.40503(-1)
0.8		3.01812(-2)	5.73658(-2)	1.02969(-1)	1.82413(-1)	2.03320(-1)	2.00130(-1)
0.9		2.00568(-2)	3.82174(-2)	6.90058(-2)	1.24817(-1)	1.41801(-1)	1.42461(-1)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 3. The Fourier Component $I_*(\tau,\mu)$ for m = 2.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	6.13369(-3)	5.63258(-3)	5.14146(-3)	4.22728(-3)	2.08109(-3)	8.51138(-4)	
-0.8	1.40473(-2)	1.29199(-2)	1.18125(-2)	9.74639(-3)	4.86758(-3)	2.02908(-3)	
-0.7	2.42374(-2)	2.23272(-2)	2.04470(-2)	1.69330(-2)	8.59010(-3)	3.65704(-3)	
-0.6	3.72977(-2)	3.44146(-2)	3.15724(-2)	2.62520(-2)	1.35554(-2)	5.91266(-3)	
-0.5	5.39364(-2)	4.98590(-2)	4.58321(-2)	3.82838(-2)	2.01851(-2)	9.06805(-3)	
-0.4	7.49853(-2)	6.94667(-2)	6.40030(-2)	5.37486(-2)	2.90737(-2)	1.35709(-2)	
-0.3	1.01366(-1)	9.41479(-2)	8.69734(-2)	7.34927(-2)	4.10537(-2)	2.02184(-2)	
-0.2	1.33969(-1)	1.24806(-1)	1.15624(-1)	9.83434(-2)	5.71259(-2)	3.05232(-2)	
-0.1	1.73539(-1)	1.62321(-1)	1.50803(-1)	1.28934(-1)	7.75407(-2)	4.68764(-2)	
-0.0	2.17610(-1)	2.07223(-1)	1.93748(-1)	1.66780(-1)	1.01645(-1)	6.54823(-2)	
0.0		2.07223(-1)	1.93748(-1)	1.66780(-1)	1.01645(-1)	6.54823(-2)	3.94802(-2)
0.1		9.13850(-2)	1.41198(-1)	1.74338(-1)	1.30506(-1)	8.62088(-2)	5.45248(-2)
0.2		5.47834(-2)	9.40008(-2)	1.38386(-1)	1.41997(-1)	1.05409(-1)	7.12370(-2)
0.3		3.94265(-2)	7.02551(-2)	1.11034(-1)	1.35966(-1)	1.12775(-1)	8.32078(-2)
0.4		3.03255(-2)	5.50820(-2)	9.03843(-2)	1.22905(-1)	1.10240(-1)	8.74206(-2)
0.5		2.37574(-2)	4.36468(-2)	7.32857(-2)	1.06676(-1)	1.01104(-1)	8.46547(-2)
0.6		1.83638(-2)	3.39907(-2)	5.79582(-2)	8.84652(-2)	8.72842(-2)	7.61525(-2)
0.7		1.35348(-2)	2.51835(-2)	4.34139(-2)	6.86137(-2)	6.97897(-2)	6.28579(-2)
0.8		8.96183(-3)	1.67383(-2)	2.90913(-2)	4.72161(-2)	4.91804(-2)	4.54269(-2)
0.9		4.47932(-3)	8.39018(-3)	1.46739(-2)	2.43199(-2)	2.58191(-2)	2.43409(-2)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 4. The Fourier Component $I_*(\tau,\mu)$ for m = 3.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	4.71882(-4)	4.27922(-4)	3.86504(-4)	3.12323(-4)	1.49303(-4)	6.12974(-5)	
-0.8	1.63700(-3)	1.48492(-3)	1.34202(-3)	1.08650(-3)	5.24875(-4)	2.18940(-4)	
-0.7	3.62794(-3)	3.29256(-3)	2.97807(-3)	2.41643(-3)	1.18117(-3)	5.01304(-4)	
-0.6	6.65125(-3)	6.04087(-3)	5.46943(-3)	4.44992(-3)	2.20550(-3)	9.55263(-4)	
-0.5	1.09558(-2)	9.96045(-3)	9.02968(-3)	7.37068(-3)	3.71563(-3)	1.65092(-3)	
-0.4	1.68361(-2)	1.53260(-2)	1.39154(-2)	1.14039(-2)	5.87321(-3)	2.70031(-3)	
-0.3	2.46361(-2)	2.24603(-2)	2.04291(-2)	1.68191(-2)	8.90196(-3)	4.29905(-3)	
-0.2	3.47536(-2)	3.17326(-2)	2.89122(-2)	2.39119(-2)	1.30799(-2)	6.81521(-3)	
-0.1	4.77051(-2)	4.36203(-2)	3.97892(-2)	3.29968(-2)	1.85468(-2)	1.08405(-2)	
-0.0	6.40589(-2)	5.89247(-2)	5.38601(-2)	4.47620(-2)	2.52999(-2)	1.55900(-2)	
0.0		5.89247(-2)	5.38601(-2)	4.47620(-2)	2.52999(-2)	1.55900(-2)	9.40491(-3)
0.1		2.65947(-2)	4.04696(-2)	4.85984(-2)	3.37870(-2)	2.12142(-2)	1.30156(-2)
0.2		1.59125(-2)	2.69468(-2)	3.87779(-2)	3.75052(-2)	2.66263(-2)	1.73623(-2)
0.3		1.12859(-2)	1.98696(-2)	3.07668(-2)	3.58014(-2)	2.85953(-2)	2.04272(-2)
0.4		8.43804(-3)	1.51551(-2)	2.44012(-2)	3.16829(-2)	2.74886(-2)	2.11807(-2)
0.5		6.31959(-3)	1.14883(-2)	1.89493(-2)	2.64242(-2)	2.42961(-2)	1.98181(-2)
0.6		4.56717(-3)	8.36998(-3)	1.40335(-2)	2.05701(-2)	1.97297(-2)	1.67997(-2)
0.7		3.04358(-3)	5.61002(-3)	9.51751(-3)	1.44725(-2)	1.43314(-2)	1.26141(-2)
0.8		1.71669(-3)	3.17792(-3)	5.43950(-3)	8.50729(-3)	8.63650(-3)	7.80304(-3)
0.9		6.32858(-4)	1.17545(-3)	2.02593(-3)	3.23965(-3)	3.35492(-3)	3.09582(-3)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 5. The Fourier Component $I_*(\tau,\mu)$ for m = 4.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	6.69258(-5)	6.02239(-5)	5.40838(-5)	4.33436(-5)	2.05304(-5)	8.51494(-6)	
-0.8	3.02494(-4)	2.72377(-4)	2.44794(-4)	1.96548(-4)	9.39014(-5)	3.93973(-5)	
-0.7	7.61720(-4)	6.86418(-4)	6.17472(-4)	4.96877(-4)	2.39851(-4)	1.02062(-4)	
-0.6	1.50358(-3)	1.35622(-3)	1.22134(-3)	9.85431(-4)	4.81766(-4)	2.08696(-4)	
-0.5	2.59147(-3)	2.34013(-3)	2.11014(-3)	1.70803(-3)	8.48455(-4)	3.76292(-4)	
-0.4	4.09363(-3)	3.70143(-3)	3.34275(-3)	2.71612(-3)	1.37699(-3)	6.30869(-4)	
-0.3	6.08324(-3)	5.50815(-3)	4.98269(-3)	4.06626(-3)	2.11599(-3)	1.01667(-3)	
-0.2	8.63901(-3)	7.83189(-3)	7.09526(-3)	5.81430(-3)	3.12207(-3)	1.61559(-3)	
-0.1	1.18659(-2)	1.07645(-2)	9.75898(-3)	8.01357(-3)	4.41270(-3)	2.55497(-3)	
-0.0	1.59510(-2)	1.44948(-2)	1.31484(-2)	1.08036(-2)	5.96834(-3)	3.63154(-3)	
0.0		1.44948(-2)	1.31484(-2)	1.08036(-2)	5.96834(-3)	3.63154(-3)	2.19670(-3)
0.1		6.42599(-3)	9.72886(-3)	1.15783(-2)	7.86933(-3)	4.86919(-3)	2.96365(-3)
0.2		3.72830(-3)	6.28643(-3)	8.98115(-3)	8.53292(-3)	5.98094(-3)	3.86184(-3)
0.3		2.53962(-3)	4.45371(-3)	6.85213(-3)	7.85349(-3)	6.20648(-3)	4.39482(-3)
0.4		1.80267(-3)	3.22602(-3)	5.16374(-3)	6.61455(-3)	5.68628(-3)	4.34786(-3)
0.5		1.26267(-3)	2.28772(-3)	3.75288(-3)	5.16865(-3)	4.71326(-3)	3.81819(-3)
0.6		8.35644(-4)	1.52664(-3)	2.54654(-3)	3.68968(-3)	3.51216(-3)	2.97182(-3)
0.7		4.93531(-4)	9.07021(-4)	1.53136(-3)	2.30332(-3)	2.26473(-3)	1.98170(-3)
0.8		2.32528(-4)	4.29264(-4)	7.31400(-4)	1.13210(-3)	1.14161(-3)	1.02575(-3)
0.9		6.20029(-5)	1.14862(-4)	1.97113(-4)	3.12097(-4)	3.21141(-4)	2.94782(-4)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 6. The Fourier Component $I_*(\tau,\mu)$ for m = 5.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	6.28063(-6)	5.64038(-6)	5.05780(-6)	4.04469(-6)	1.91071(-6)	7.93886(-7)	
-0.8	3.73124(-5)	3.35331(-5)	3.00940(-5)	2.41116(-5)	1.14857(-5)	4.82442(-6)	
-0.7	1.07458(-4)	9.66557(-5)	8.68253(-5)	6.97210(-5)	3.35504(-5)	1.42849(-5)	
-0.6	2.29470(-4)	2.06608(-4)	1.85802(-4)	1.49599(-4)	7.28951(-5)	3.15824(-5)	
-0.5	4.15167(-4)	3.74235(-4)	3.36993(-4)	2.72200(-4)	1.34741(-4)	5.97452(-5)	
-0.4	6.75372(-4)	6.09580(-4)	5.49751(-4)	4.45739(-4)	2.25137(-4)	1.03090(-4)	
-0.3	1.01957(-3)	9.21504(-4)	8.32420(-4)	6.77822(-4)	3.51320(-4)	1.68651(-4)	
-0.2	1.45558(-3)	1.31706(-3)	1.19142(-3)	9.74059(-4)	5.20758(-4)	2.69137(-4)	
-0.1	1.99273(-3)	1.80388(-3)	1.63272(-3)	1.33734(-3)	7.32823(-4)	4.23514(-4)	
-0.0	2.65403(-3)	2.40328(-3)	2.17551(-3)	1.78221(-3)	9.79104(-4)	5.94123(-4)	
0.0		2.40328(-3)	2.17551(-3)	1.78221(-3)	9.79104(-4)	5.94123(-4)	3.60119(-4)
0.1		1.04360(-3)	1.57774(-3)	1.87309(-3)	1.26582(-3)	7.80603(-4)	4.74525(-4)
0.2		5.87087(-4)	9.88700(-4)	1.40973(-3)	1.33332(-3)	9.31756(-4)	6.00443(-4)
0.3		3.83899(-4)	6.72490(-4)	1.03283(-3)	1.17920(-3)	9.29569(-4)	6.57017(-4)
0.4		2.58469(-4)	4.62075(-4)	7.38429(-4)	9.42637(-4)	8.08592(-4)	6.17257(-4)
0.5		1.69091(-4)	3.06064(-4)	5.01326(-4)	6.88265(-4)	6.26405(-4)	5.06701(-4)
0.6		1.02291(-4)	1.86706(-4)	3.10999(-4)	4.49276(-4)	4.26898(-4)	3.60733(-4)
0.7		5.34205(-5)	9.80933(-5)	1.65394(-4)	2.48078(-4)	2.43517(-4)	2.12817(-4)
0.8		2.09675(-5)	3.86763(-5)	6.58150(-5)	1.01604(-4)	1.02297(-4)	9.18069(-5)
0.9		4.03113(-6)	7.46208(-6)	1.27901(-5)	2.02003(-5)	2.07549(-5)	1.90300(-5)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 7. The Fourier Component $I_*(\tau,\mu)$ for m = 6.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	3.64612(-7)	3.27338(-7)	2.93459(-7)	2.34597(-7)	1.10774(-7)	4.60360(-8)	
-0.8	2.90480(-6)	2.60976(-6)	2.34156(-6)	1.87544(-6)	8.92958(-7)	3.75132(-7)	
-0.7	9.72738(-6)	8.74687(-6)	7.85546(-6)	6.30582(-6)	3.03290(-6)	1.29146(-6)	
-0.6	2.27862(-5)	2.05098(-5)	1.84403(-5)	1.48421(-5)	7.22835(-6)	3.13189(-6)	
-0.5	4.37887(-5)	3.94595(-5)	3.55246(-5)	2.86845(-5)	1.41912(-5)	6.29250(-6)	
-0.4	7.40943(-5)	6.68560(-5)	6.02802(-5)	4.88581(-5)	2.46633(-5)	1.12928(-5)	
-0.3	1.14597(-4)	1.03542(-4)	9.35097(-5)	7.61152(-5)	3.94263(-5)	1.89249(-5)	
-0.2	1.65614(-4)	1.49803(-4)	1.35478(-4)	1.10719(-4)	5.91524(-5)	3.05665(-5)	
-0.1	2.27191(-4)	2.05580(-4)	1.86022(-4)	1.52303(-4)	8.33921(-5)	4.81822(-5)	
-0.0	3.00490(-4)	2.71920(-4)	2.46056(-4)	2.01467(-4)	1.10580(-4)	6.70733(-5)	
0.0		2.71920(-4)	2.46056(-4)	2.01467(-4)	1.10580(-4)	6.70733(-5)	4.06782(-5)
0.1		1.16112(-4)	1.75494(-4)	2.08255(-4)	1.40600(-4)	8.66594(-5)	5.26746(-5)
0.2		6.35970(-5)	1.07078(-4)	1.52621(-4)	1.44236(-4)	1.00747(-4)	6.49064(-5)
0.3		4.00550(-5)	7.01511(-5)	1.07705(-4)	1.22887(-4)	9.68330(-5)	6.84233(-5)
0.4		2.56452(-5)	4.58379(-5)	7.32305(-5)	9.34258(-5)	8.01121(-5)	6.11407(-5)
0.5		1.56977(-5)	2.84085(-5)	4.65197(-5)	6.38310(-5)	5.80755(-5)	4.69672(-5)
0.6		8.68979(-6)	1.58582(-5)	2.64083(-5)	3.81303(-5)	3.62204(-5)	3.06005(-5)
0.7		4.01435(-6)	7.37013(-6)	1.24236(-5)	1.86253(-5)	1.82779(-5)	1.59706(-5)
0.8		1.31217(-6)	2.42003(-6)	4.11715(-6)	6.35302(-6)	6.39473(-6)	5.73797(-6)
0.9		1.81710(-7)	3.36316(-7)	5.76320(-7)	9.09821(-7)	9.34571(-7)	8.56760(-7)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 8. The Fourier Component $I_*(\tau,\mu)$ for m = 7.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	1.27031(-8)	1.14041(-8)	1.02235(-8)	8.17255(-9)	3.85880(-9)	1.60370(-9)	
-0.8	1.42973(-7)	1.28447(-7)	1.15244(-7)	9.22992(-8)	4.39445(-8)	1.84614(-8)	
-0.7	5.79668(-7)	5.21220(-7)	4.68089(-7)	3.75735(-7)	1.80707(-7)	7.69491(-8)	
-0.6	1.53649(-6)	1.38294(-6)	1.24336(-6)	1.00071(-6)	4.87335(-7)	2.11154(-7)	
-0.5	3.21082(-6)	2.89328(-6)	2.60469(-6)	2.10309(-6)	1.04041(-6)	4.61327(-7)	
-0.4	5.75018(-6)	5.18825(-6)	4.67782(-6)	3.79130(-6)	1.91370(-6)	8.76244(-7)	
-0.3	9.22443(-6)	8.33427(-6)	7.52656(-6)	6.12622(-6)	3.17306(-6)	1.52308(-6)	
-0.2	1.36060(-5)	1.23065(-5)	1.11294(-5)	9.09502(-6)	4.85872(-6)	2.51067(-6)	
-0.1	1.87914(-5)	1.70031(-5)	1.53850(-5)	1.25956(-5)	6.89601(-6)	3.98428(-6)	
-0.0	2.47201(-5)	2.23679(-5)	2.02394(-5)	1.65707(-5)	9.09428(-6)	5.51598(-6)	
0.0		2.23679(-5)	2.02394(-5)	1.65707(-5)	9.09428(-6)	5.51598(-6)	3.34558(-6)
0.1		9.38991(-6)	1.41917(-5)	1.68401(-5)	1.13680(-5)	7.00631(-6)	4.25867(-6)
0.2		4.99787(-6)	8.41462(-6)	1.19931(-5)	1.13331(-5)	7.91561(-6)	5.09952(-6)
0.3		3.02141(-6)	5.29147(-6)	8.12382(-6)	9.26819(-6)	7.30285(-6)	5.16014(-6)
0.4		1.83088(-6)	3.27241(-6)	5.22780(-6)	6.66903(-6)	5.71842(-6)	4.36413(-6)
0.5		1.04255(-6)	1.88668(-6)	3.08937(-6)	4.23874(-6)	3.85640(-6)	3.11870(-6)
0.6		5.24606(-7)	9.57346(-7)	1.59419(-6)	2.30167(-6)	2.18631(-6)	1.84705(-6)
0.7		2.12804(-7)	3.90688(-7)	6.58548(-7)	9.87231(-7)	9.68785(-7)	8.46476(-7)
0.8		5.74705(-8)	1.05990(-7)	1.80314(-7)	2.78220(-7)	2.80038(-7)	2.51273(-7)
0.9		5.68457(-9)	1.05210(-8)	1.80285(-8)	2.84596(-8)	2.92330(-8)	2.67986(-8)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 9. The Fourier Component $I_*(\tau,\mu)$ for m = 8.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	6.56254(-10)	5.89143(-10)	5.28152(-10)	4.22197(-10)	1.99346(-10)	8.28473(-11)	
-0.8	9.16369(-9)	8.23263(-9)	7.38637(-9)	5.91575(-9)	2.81653(-9)	1.18325(-9)	
-0.7	4.02493(-8)	3.61908(-8)	3.25016(-8)	2.60890(-8)	1.25472(-8)	5.34291(-9)	
-0.6	1.09663(-7)	9.87034(-8)	8.87413(-8)	7.14229(-8)	3.47818(-8)	1.50704(-8)	
-0.5	2.29190(-7)	2.06523(-7)	1.85923(-7)	1.50118(-7)	7.42637(-8)	3.29292(-8)	
-0.4	4.03647(-7)	3.64199(-7)	3.28368(-7)	2.66136(-7)	1.34335(-7)	6.15090(-8)	
-0.3	6.29436(-7)	5.68693(-7)	5.13578(-7)	4.18023(-7)	2.16513(-7)	1.03927(-7)	
-0.2	8.94476(-7)	8.09046(-7)	7.31658(-7)	5.97914(-7)	3.19414(-7)	1.65053(-7)	
-0.1	1.18131(-6)	1.06889(-6)	9.67164(-7)	7.91809(-7)	4.33507(-7)	2.50465(-7)	
-0.0	1.47573(-6)	1.33530(-6)	1.20823(-6)	9.89214(-7)	5.42893(-7)	3.29281(-7)	
0.0		1.33530(-6)	1.20823(-6)	9.89214(-7)	5.42893(-7)	3.29281(-7)	1.99719(-7)
0.1		5.28593(-7)	7.98899(-7)	9.47982(-7)	6.39936(-7)	3.94403(-7)	2.39731(-7)
0.2		2.63292(-7)	4.43289(-7)	6.31803(-7)	5.97029(-7)	4.16994(-7)	2.68642(-7)
0.3		1.47638(-7)	2.58560(-7)	3.96958(-7)	4.52873(-7)	3.56839(-7)	2.52139(-7)
0.4		8.20707(-8)	1.46688(-7)	2.34338(-7)	2.98940(-7)	2.56328(-7)	1.95622(-7)
0.5		4.22497(-8)	7.64582(-8)	1.25197(-7)	1.71775(-7)	1.56280(-7)	1.26385(-7)
0.6		1.88253(-8)	3.43539(-8)	5.72066(-8)	8.25936(-8)	7.84537(-8)	6.62795(-8)
0.7		6.54561(-9)	1.20171(-8)	2.02561(-8)	3.03658(-8)	2.97984(-8)	2.60363(-8)
0.8		1.42836(-9)	2.63425(-9)	4.48146(-9)	6.91474(-9)	6.95991(-9)	6.24498(-9)
0.9		9.88569(-11)	1.82964(-10)	3.13522(-10)	4.94919(-10)	5.08367(-10)	4.66032(-10)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 10. The Legendre Coefficients for the Haze L Phase Function.

ℓ	β_ℓ	$\beta_{\ell+16}$	$\beta_{\ell+32}$	$\beta_{\ell+48}$	$\beta_{\ell+64}$	$\beta_{\ell+80}$
0	1	0.34688	0.01711	0.00107	0.00008	0.00001
1	2.41260	0.28351	0.01298	0.00082	0.00006	0.00001
2	3.23047	0.23317	0.01198	0.00077	0.00006	0.00001
3	3.37296	0.18963	0.00904	0.00059	0.00005	
4	3.23150	0.15788	0.00841	0.00055	0.00004	
5	2.89350	0.12739	0.00634	0.00043	0.00004	
6	2.49594	0.10762	0.00592	0.00040	0.00003	
7	2.11361	0.08597	0.00446	0.00031	0.00003	
8	1.74812	0.07381	0.00418	0.00029	0.00002	
9	1.44692	0.05828	0.00316	0.00023	0.00002	
10	1.17714	0.05089	0.00296	0.00021	0.00002	
11	0.96643	0.03971	0.00225	0.00017	0.00001	
12	0.78237	0.03524	0.00210	0.00015	0.00001	
13	0.64114	0.02720	0.00160	0.00012	0.00001	
14	0.51966	0.02451	0.00150	0.00011	0.00001	
15	0.42563	0.01874	0.00115	0.00009	0.00001	

$$I_*(\tau,\mu) = (2-\delta_{0,m})[I(\tau,\mu) - \frac{1}{2}\,\delta(\mu-\mu_0)e^{-\tau/\mu}]. \tag{92b}$$

The foregoing problem, with a modest value of L, was chosen so that we could, in a reasonable amount of space, tabulate each of the Fourier components of the intensity. However, in order to demonstrate the applicability of the F_N method to more realistic problems, based on many-term phase functions, we have solved the five test problems posed[1] by the Radiation Commission of the International Association of Meteorology and Atmospheric Physics. The test problems are based on a Haze L scattering model and a Cloud C_1 phase function[1]. In Tables 10 and 17 we list the relevant Legendre coefficients computed by de Haan[22] and Karp[23] and discussed in a recent work[14] that used the classical spherical harmonics method to solve the mentioned five test problems.

Table 11. The Discrete Eigenvalues for the Haze L Phase Function.

	$\omega = 0.9$	$\omega = 1$
m = 0	1.000030758268	1.001460826123
	1.050018045879	1.081719072113
	1.380760539074	1.567448253230
	3.817062777373	∞
m = 1	1.004008389389	1.011931753504
	1.119278151573	1.175989383903
	1.737436136571	2.101952588198
m = 2	1.012730514626	1.029026630920
	1.211873478485	1.303106894309
m = 3	1.020111932171	1.044881108960
m = 4-82	none	none

In Tables 11 and 18 we report the discrete eigenvalues computed, for the haze and cloud problems, in the manner discussed in Section IV and thought to be correct to within ±1 in the last digit given. We write

$$I(\tau,\mu,\phi) = I_*(\tau,\mu,\phi) + \pi\delta(\mu-\mu_0)\delta(\phi-\phi_0)e^{-\tau/\mu} \tag{93}$$

and in Tables 12-16, 19 and 20 we report, relevant to the boundary conditions given by Eqs. (92), our results, thought to be correct to within ±1 in the sixth significant figure, for $I_*(\tau,\mu,\phi)$ for the five test problems summarized in Table 21. Table 22 is devoted to the fluxes

$$q_\pm(\tau) = \int_0^1 \int_0^{2\pi} \mu I(\tau,\pm\mu,\phi)d\phi d\mu \tag{94}$$

and

Table 12. The Intensity $I_*(\tau,\mu,\phi)$ for the Haze L Phase Function with $\tau_0 = 1$, $\omega = 1$ and $\mu_0 = 1$.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	3.61452(-2)	3.43394(-2)	3.25109(-2)	2.88122(-2)	1.76286(-2)	8.52589(-3)	
-0.9	3.97819(-2)	3.78723(-2)	3.59207(-2)	3.19303(-2)	1.96202(-2)	9.45731(-3)	
-0.8	4.27313(-2)	4.08406(-2)	3.88734(-2)	3.47677(-2)	2.16019(-2)	1.03959(-2)	
-0.7	4.80051(-2)	4.61319(-2)	4.41307(-2)	3.98292(-2)	2.52479(-2)	1.22171(-2)	
-0.6	5.58214(-2)	5.40432(-2)	5.20594(-2)	4.75986(-2)	3.11837(-2)	1.53618(-2)	
-0.5	6.60942(-2)	6.46296(-2)	6.28449(-2)	5.84971(-2)	4.02740(-2)	2.05621(-2)	
-0.4	7.81481(-2)	7.74403(-2)	7.62508(-2)	7.27049(-2)	5.37300(-2)	2.91285(-2)	
-0.3	8.99682(-2)	9.07706(-2)	9.08784(-2)	8.94711(-2)	7.29643(-2)	4.34688(-2)	
-0.2	9.70815(-2)	1.00421(-1)	1.02789(-1)	1.05506(-1)	9.83777(-2)	6.79949(-2)	
-0.1	9.29328(-2)	9.98187(-2)	1.05195(-1)	1.13497(-1)	1.24037(-1)	1.08399(-1)	
-0.0	6.98774(-2)	8.46673(-2)	9.41663(-2)	1.08727(-1)	1.35762(-1)	1.42779(-1)	
0.0		8.46673(-2)	9.41663(-2)	1.08727(-1)	1.35762(-1)	1.42779(-1)	1.14808(-1)
0.1		2.95418(-2)	5.24346(-2)	8.45649(-2)	1.35096(-1)	1.56106(-1)	1.56976(-1)
0.2		1.64907(-2)	3.22817(-2)	6.07527(-2)	1.24350(-1)	1.58925(-1)	1.76818(-1)
0.3		1.23421(-2)	2.48488(-2)	4.93968(-2)	1.14811(-1)	1.57937(-1)	1.88301(-1)
0.4		1.11879(-2)	2.26450(-2)	4.57547(-2)	1.12269(-1)	1.60862(-1)	2.00019(-1)
0.5		1.17959(-2)	2.37910(-2)	4.80003(-2)	1.19079(-1)	1.73191(-1)	2.19633(-1)
0.6		1.42049(-2)	2.84584(-2)	5.68731(-2)	1.39051(-1)	2.01445(-1)	2.55983(-1)
0.7		1.95833(-2)	3.89248(-2)	7.67454(-2)	1.82004(-1)	2.58986(-1)	3.25125(-1)
0.8		3.19532(-2)	6.29430(-2)	1.22045(-1)	2.77182(-1)	3.82767(-1)	4.68658(-1)
0.9		6.87267(-2)	1.33917(-1)	2.54259(-1)	5.44601(-1)	7.19447(-1)	8.46084(-1)
1.0		3.64940(-1)	7.00266(-1)	1.28955	2.52255	3.09319	3.38091

Table 13. The Intensity $I_*(\tau,\mu,\phi)$ for the Haze L Phase Function with $\tau_0 = 1$, $\omega = 0.9$ and $\mu_0 = 1$.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	2.79717(-2)	2.65834(-2)	2.51795(-2)	2.23421(-2)	1.37519(-2)	6.70439(-3)	
-0.9	3.01802(-2)	2.87427(-2)	2.72763(-2)	2.42824(-2)	1.50370(-2)	7.32793(-3)	
-0.8	3.14478(-2)	3.00707(-2)	2.86411(-2)	2.56621(-2)	1.60962(-2)	7.85504(-3)	
-0.7	3.43839(-2)	3.30557(-2)	3.16407(-2)	2.86060(-2)	1.83142(-2)	9.00263(-3)	
-0.6	3.91308(-2)	3.78910(-2)	3.65135(-2)	3.34278(-2)	2.20977(-2)	1.10619(-2)	
-0.5	4.56379(-2)	4.46171(-2)	4.33840(-2)	4.04032(-2)	2.80086(-2)	1.45143(-2)	
-0.4	5.35113(-2)	5.29854(-2)	5.21410(-2)	4.96862(-2)	3.68540(-2)	2.02308(-2)	
-0.3	6.15420(-2)	6.19914(-2)	6.19786(-2)	6.08863(-2)	4.96165(-2)	2.98277(-2)	
-0.2	6.69562(-2)	6.90564(-2)	7.04996(-2)	7.20372(-2)	6.66970(-2)	4.63006(-2)	
-0.1	6.55296(-2)	7.00041(-2)	7.34324(-2)	7.85904(-2)	8.44628(-2)	7.36110(-2)	
-0.0	5.17485(-2)	6.17096(-2)	6.80163(-2)	7.74665(-2)	9.39979(-2)	9.74837(-2)	
0.0		6.17096(-2)	6.80163(-2)	7.74665(-2)	9.39979(-2)	9.74837(-2)	7.93126(-2)
0.1		2.24949(-2)	3.95181(-2)	6.26530(-2)	9.62543(-2)	1.08718(-1)	1.08189(-1)
0.2		1.31007(-2)	2.53401(-2)	4.67729(-2)	9.15916(-2)	1.13815(-1)	1.24212(-1)
0.3		1.01943(-2)	2.02703(-2)	3.94722(-2)	8.74675(-2)	1.16620(-1)	1.35712(-1)
0.4		9.52906(-3)	1.90677(-2)	3.77703(-2)	8.83323(-2)	1.22503(-1)	1.48268(-1)
0.5		1.02637(-2)	2.05023(-2)	4.06492(-2)	9.64183(-2)	1.35817(-1)	1.67516(-1)
0.6		1.25293(-2)	2.49095(-2)	4.90656(-2)	1.15336(-1)	1.62230(-1)	2.00701(-1)
0.7		1.74171(-2)	3.44152(-2)	6.70811(-2)	1.53982(-1)	2.13563(-1)	2.61672(-1)
0.8		2.85622(-2)	5.60204(-2)	1.07697(-1)	2.38486(-1)	3.22550(-1)	3.86921(-1)
0.9		6.16331(-2)	1.19763(-1)	2.26124(-1)	4.76103(-1)	6.19703(-1)	7.17745(-1)
1.0		3.28124(-1)	6.29065(-1)	1.15632	2.24839	2.74147	2.97766

Table 14. The Intensity $I_*(\tau,\mu,\phi)$ for the Haze L Phase Function with $\tau_0 = 1$, $\omega = 0.9$, $\mu_0 = 0.5$ and $\phi - \phi_0 = 0$.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	2.28190(-2)	2.14170(-2)	1.99920(-2)	1.71574(-2)	9.34719(-3)	4.02513(-3)	
-0.9	4.11125(-2)	3.86276(-2)	3.60559(-2)	3.08703(-2)	1.64542(-2)	6.81049(-3)	
-0.8	6.49983(-2)	6.12972(-2)	5.73958(-2)	4.94000(-2)	2.66078(-2)	1.10213(-2)	
-0.7	9.99446(-2)	9.47422(-2)	8.91210(-2)	7.73509(-2)	4.26132(-2)	1.79328(-2)	
-0.6	1.50993(-1)	1.44083(-1)	1.36343(-1)	1.19657(-1)	6.81169(-2)	2.95065(-2)	
-0.5	2.24768(-1)	2.16263(-1)	2.06169(-1)	1.83479(-1)	1.09068(-1)	4.93125(-2)	
-0.4	3.29336(-1)	3.20182(-1)	3.08045(-1)	2.78889(-1)	1.75210(-1)	8.41056(-2)	
-0.3	4.72536(-1)	4.65520(-1)	4.52954(-1)	4.18719(-1)	2.82082(-1)	1.47239(-1)	
-0.2	6.56834(-1)	6.58390(-1)	6.49495(-1)	6.15206(-1)	4.51466(-1)	2.66183(-1)	
-0.1	8.70325(-1)	8.94524(-1)	8.97446(-1)	8.72172(-1)	6.97514(-1)	4.91555(-1)	
-0.0	1.03177	1.14775	1.18797	1.19440	1.00873	7.95748(-1)	
0.0		1.14775	1.18797	1.19440	1.00873	7.95748(-1)	5.24167(-1)
0.1		6.07548(-1)	9.98592(-1)	1.38473	1.41181	1.15806	8.76470(-1)
0.2		5.08883(-1)	9.07223(-1)	1.43790	1.82630	1.61273	1.29144
0.3		5.49994(-1)	9.99898(-1)	1.64825	2.30787	2.15587	1.80549
0.4		6.38615(-1)	1.16925	1.95778	2.87301	2.77822	2.40104
0.5		6.34271(-1)	1.16869	1.98321	3.03333	3.03426	2.71055
0.6		4.18260(-1)	7.78087(-1)	1.34620	2.18395	2.29506	2.15369
0.7		2.04883(-1)	3.85810(-1)	6.83932(-1)	1.19334	1.33200	1.32682
0.8		8.64752(-2)	1.65150(-1)	3.01013(-1)	5.69583(-1)	6.78608(-1)	7.19934(-1)
0.9		3.13664(-2)	6.09010(-2)	1.14612(-1)	2.37335(-1)	3.03207(-1)	3.43478(-1)
1.0		5.07113(-3)	1.01191(-2)	2.00435(-2)	4.76083(-2)	6.73492(-2)	8.37579(-2)

Table 15. The Intensity $I_*(\tau,\mu,\phi)$ for the Haze L Phase Function with $\tau_0 = 1$, $\omega = 0.9$, $\mu_0 = 0.5$ and $\phi - \phi_0 = \pi/2$.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	2.28190(-2)	2.14170(-2)	1.99920(-2)	1.71574(-2)	9.34719(-3)	4.02513(-3)	
-0.9	2.69861(-2)	2.54001(-2)	2.37700(-2)	2.04885(-2)	1.12507(-2)	4.83998(-3)	
-0.8	3.23251(-2)	3.05433(-2)	2.86841(-2)	2.48816(-2)	1.38576(-2)	5.98687(-3)	
-0.7	3.90915(-2)	3.71288(-2)	3.50364(-2)	3.06624(-2)	1.74617(-2)	7.63435(-3)	
-0.6	4.75194(-2)	4.54446(-2)	4.31587(-2)	3.82274(-2)	2.24929(-2)	1.00585(-2)	
-0.5	5.76960(-2)	5.56800(-2)	5.33274(-2)	4.79966(-2)	2.95696(-2)	1.37243(-2)	
-0.4	6.92921(-2)	6.76843(-2)	6.55506(-2)	6.02592(-2)	3.95485(-2)	1.94423(-2)	
-0.3	8.09723(-2)	8.04082(-2)	7.90373(-2)	7.47154(-2)	5.34553(-2)	2.86762(-2)	
-0.2	8.94088(-2)	9.08993(-2)	9.11597(-2)	8.93864(-2)	7.18225(-2)	4.41114(-2)	
-0.1	8.86327(-2)	9.36078(-2)	9.65669(-2)	9.91642(-2)	9.15413(-2)	6.94491(-2)	
-0.0	6.76014(-2)	8.16018(-2)	8.92220(-2)	9.83762(-2)	1.03484(-1)	9.32369(-2)	
0.0		8.16018(-2)	8.92220(-2)	9.83762(-2)	1.03484(-1)	9.32369(-2)	6.29164(-2)
0.1		2.74475(-2)	4.83619(-2)	7.57571(-2)	1.04622(-1)	1.04387(-1)	8.95907(-2)
0.2		1.41330(-2)	2.75945(-2)	5.09061(-2)	9.28868(-2)	1.04678(-1)	1.01178(-1)
0.3		9.26887(-3)	1.87294(-2)	3.68737(-2)	7.85470(-2)	9.74004(-2)	1.02990(-1)
0.4		6.96644(-3)	1.42411(-2)	2.88244(-2)	6.68034(-2)	8.82947(-2)	9.95180(-2)
0.5		5.75720(-3)	1.17847(-2)	2.40718(-2)	5.82338(-2)	8.01154(-2)	9.43192(-2)
0.6		5.11030(-3)	1.04251(-2)	2.12819(-2)	5.23831(-2)	7.37544(-2)	8.93298(-2)
0.7		4.79703(-3)	9.73440(-3)	1.97631(-2)	4.87196(-2)	6.93677(-2)	8.54626(-2)
0.8		4.71115(-3)	9.50506(-3)	1.91501(-2)	4.68396(-2)	6.68825(-2)	8.31200(-2)
0.9		4.80640(-3)	9.64243(-3)	1.92635(-2)	4.64998(-2)	6.62130(-2)	8.24990(-2)
1.0		5.07113(-3)	1.01191(-2)	2.00435(-2)	4.76083(-2)	6.73492(-2)	8.37579(-2)

Table 16. The Intensity $I_*(\tau,\mu,\phi)$ for the Haze L Phase Function with $\tau_0 = 1$, $\omega = 0.9$, $\mu_0 = 0.5$ and $\phi - \phi_0 = \pi$.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	2.28190(-2)	2.14170(-2)	1.99920(-2)	1.71574(-2)	9.34719(-3)	4.02513(-3)	
-0.9	2.61852(-2)	2.46217(-2)	2.30415(-2)	1.99042(-2)	1.11589(-2)	4.97196(-3)	
-0.8	3.20368(-2)	3.01730(-2)	2.82831(-2)	2.45151(-2)	1.38928(-2)	6.24448(-3)	
-0.7	3.74817(-2)	3.54358(-2)	3.33331(-2)	2.90811(-2)	1.67591(-2)	7.61010(-3)	
-0.6	4.12139(-2)	3.92700(-2)	3.72011(-2)	3.28685(-2)	1.95280(-2)	9.02300(-3)	
-0.5	4.99464(-2)	4.77799(-2)	4.54343(-2)	4.04517(-2)	2.46595(-2)	1.16636(-2)	
-0.4	5.50134(-2)	5.32394(-2)	5.11567(-2)	4.64342(-2)	2.98815(-2)	1.47743(-2)	
-0.3	6.44934(-2)	6.30132(-2)	6.10801(-2)	5.64129(-2)	3.85985(-2)	2.04385(-2)	
-0.2	7.08868(-2)	7.03673(-2)	6.91450(-2)	6.55321(-2)	4.91145(-2)	2.92042(-2)	
-0.1	6.95600(-2)	7.10895(-2)	7.13778(-2)	7.01215(-2)	5.90302(-2)	4.25269(-2)	
-0.0	5.47761(-2)	6.23324(-2)	6.55280(-2)	6.81201(-2)	6.39396(-2)	5.39412(-2)	
0.0		6.23324(-2)	6.55280(-2)	6.81201(-2)	6.39396(-2)	5.39412(-2)	3.42189(-2)
0.1		1.99925(-2)	3.42555(-2)	5.10065(-2)	6.25009(-2)	5.80166(-2)	4.66801(-2)
0.2		9.41145(-3)	1.79861(-2)	3.18930(-2)	5.27234(-2)	5.56537(-2)	5.06958(-2)
0.3		5.52988(-3)	1.10120(-2)	2.10538(-2)	4.15040(-2)	4.87980(-2)	4.91257(-2)
0.4		3.67310(-3)	7.45174(-3)	1.48051(-2)	3.24697(-2)	4.12526(-2)	4.48014(-2)
0.5		2.67314(-3)	5.46407(-3)	1.10690(-2)	2.59261(-2)	3.47946(-2)	3.99910(-2)
0.6		2.11016(-3)	4.31933(-3)	8.82241(-3)	2.14934(-2)	2.99722(-2)	3.59328(-2)
0.7		1.81107(-3)	3.69988(-3)	7.56877(-3)	1.88391(-2)	2.69599(-2)	3.33330(-2)
0.8		1.72244(-3)	3.50577(-3)	7.15364(-3)	1.79613(-2)	2.61129(-2)	3.29719(-2)
0.9		1.93240(-3)	3.91318(-3)	7.93848(-3)	1.98733(-2)	2.90534(-2)	3.70748(-2)
1.0		5.07113(-3)	1.01191(-2)	2.00435(-2)	4.76083(-2)	6.73492(-2)	8.37579(-2)

Table 17. The Legendre Coefficients for the Cloud C_1 Phase Function.

l	β_l	β_{l+35}	β_{l+70}	β_{l+105}	β_{l+140}	β_{l+175}	β_{l+210}	β_{l+245}	β_{l+280}
0	1	19.884	16.144	6.990	2.025	0.440	0.079	0.012	0.002
1	2.544	20.024	15.883	6.785	1.940	0.422	0.074	0.011	0.002
2	3.883	20.145	15.606	6.573	1.869	0.401	0.071	0.011	0.001
3	4.568	20.251	15.338	6.377	1.790	0.384	0.067	0.010	0.001
4	5.235	20.330	15.058	6.173	1.723	0.364	0.064	0.009	0.001
5	5.887	20.401	14.784	5.986	1.649	0.349	0.060	0.009	0.001
6	6.457	20.444	14.501	5.790	1.588	0.331	0.057	0.008	0.001
7	7.177	20.477	14.225	5.612	1.518	0.317	0.054	0.008	0.001
8	7.859	20.489	13.941	5.424	1.461	0.301	0.052	0.008	0.001
9	8.494	20.483	13.662	5.255	1.397	0.288	0.049	0.007	0.001
10	9.286	20.467	13.378	5.075	1.344	0.273	0.047	0.007	0.001
11	9.856	20.427	13.098	4.915	1.284	0.262	0.044	0.006	0.001
12	10.615	20.382	12.816	4.744	1.235	0.248	0.042	0.006	0.001
13	11.229	20.310	12.536	4.592	1.179	0.238	0.039	0.006	0.001
14	11.851	20.236	12.257	4.429	1.134	0.225	0.038	0.005	0.001
15	12.503	20.136	11.978	4.285	1.082	0.215	0.035	0.005	0.001
16	13.058	20.036	11.703	4.130	1.040	0.204	0.034	0.005	0.001
17	13.626	19.909	11.427	3.994	0.992	0.195	0.032	0.005	0.001
18	14.209	19.785	11.156	3.847	0.954	0.185	0.030	0.004	0.001
19	14.660	19.632	10.884	3.719	0.909	0.177	0.029	0.004	0.001
20	15.231	19.486	10.618	3.580	0.873	0.167	0.027	0.004	
21	15.641	19.311	10.350	3.459	0.832	0.160	0.026	0.004	
22	16.126	19.145	10.090	3.327	0.799	0.151	0.024	0.003	
23	16.539	18.949	9.827	3.214	0.762	0.145	0.023	0.003	
24	16.934	18.764	9.574	3.090	0.731	0.137	0.022	0.003	
25	17.325	18.551	9.318	2.983	0.696	0.131	0.021	0.003	
26	17.673	18.348	9.072	2.866	0.668	0.124	0.020	0.003	
27	17.999	18.119	8.822	2.766	0.636	0.118	0.018	0.003	
28	18.329	17.901	8.584	2.656	0.610	0.112	0.018	0.002	
29	18.588	17.659	8.340	2.562	0.581	0.107	0.017	0.002	
30	18.885	17.428	8.110	2.459	0.557	0.101	0.016	0.002	
31	19.103	17.174	7.874	2.372	0.530	0.097	0.015	0.002	
32	19.345	16.931	7.652	2.274	0.508	0.091	0.014	0.002	
33	19.537	16.668	7.424	2.193	0.483	0.087	0.013	0.002	
34	19.721	16.415	7.211	2.102	0.463	0.082	0.013	0.002	

Table 18. The Discrete Eigenvalues for the Cloud C_1 Phase Function.

$\omega = 0.9$	$\omega = 1$
1.000389979099	1.001089977491
1.002932994824	1.004455958544
1.007484963089	1.009794688092
1.013879322804	1.016992993207
1.022052177312	1.026013974309
1.031988750346	1.036860767247
1.043709272680	1.049569734514
1.057261138191	1.064203095402
1.072717289782	1.080848711296
1.090178937016	1.099621107967
1.109775208699	1.120666102959
1.131670715243	1.144165473952
1.156074203314	1.170346326639
1.183252816545	1.199490155026
1.213551335403	1.231958509514
1.247430064830	1.268217657885
1.285516474695	1.308897146570
1.328721070560	1.354857325[illegible]98
1.378439970833	1.407354457250
1.437038878133	1.468360492288
1.509302317539	1.541322572005
1.610657591351	1.633766541474
1.928954632197	1.776999320660
4.282156025437	2.405084010173
	∞

$$q(\tau) = \int_{-1}^{1} \int_{0}^{2\pi} \mu I(\tau,\mu,\phi) d\phi d\mu \tag{95}$$

for the five test problems.

At this point we record several observations concerning the numerical implementation of our F_N solution of the problems investigated here. First of all, the methods outlined in Section IV for evaluating all of the basic aspects of the solution were found stable and reliable for the three classes of problems considered here: the L=8 model, the haze model and the cloud model. Secondly, we found that $N \approx 20-50$ was adequate to solve

Table 19. The Intensity $I_*(\tau,\mu,\phi)$ for the Cloud C_1 Phase Function with $\tau_0 = 64$, $\omega = 1$ and $\mu_0 = 1$.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	1.06370	1.00624	9.63206(-1)	8.58242(-1)	5.24533(-1)	2.46002(-1)	
-0.9	9.53090(-1)	9.95662(-1)	9.69724(-1)	8.69390(-1)	5.35989(-1)	2.57405(-1)	
-0.8	9.54076(-1)	9.98283(-1)	9.77766(-1)	8.80528(-1)	5.47444(-1)	2.68836(-1)	
-0.7	8.82542(-1)	9.88506(-1)	9.83519(-1)	8.91568(-1)	5.58900(-1)	2.80281(-1)	
-0.6	8.24712(-1)	9.79099(-1)	9.88904(-1)	9.02556(-1)	5.70355(-1)	2.91734(-1)	
-0.5	7.72606(-1)	9.69772(-1)	9.93991(-1)	9.13497(-1)	5.81811(-1)	3.03190(-1)	
-0.4	7.11439(-1)	9.58004(-1)	9.98330(-1)	9.24376(-1)	5.93266(-1)	3.14647(-1)	
-0.3	6.40311(-1)	9.43425(-1)	1.00178	9.35179(-1)	6.04721(-1)	3.26105(-1)	
-0.2	5.58482(-1)	9.25834(-1)	1.00421	9.45894(-1)	6.16177(-1)	3.37563(-1)	
-0.1	4.58734(-1)	9.04593(-1)	1.00548	9.56509(-1)	6.27632(-1)	3.49021(-1)	
-0.0	2.51582(-1)	8.79520(-1)	1.00548	9.67014(-1)	6.39087(-1)	3.60478(-1)	
0.0		8.79520(-1)	1.00548	9.67014(-1)	6.39087(-1)	3.60478(-1)	3.92639(-2)
0.1		8.50696(-1)	1.00421	9.77407(-1)	6.50543(-1)	3.71935(-1)	7.20391(-2)
0.2		8.18718(-1)	1.00188	9.87701(-1)	6.61998(-1)	3.83392(-1)	8.89490(-2)
0.3		7.85215(-1)	9.99016(-1)	9.97941(-1)	6.73453(-1)	3.94849(-1)	1.04140(-1)
0.4		7.54599(-1)	9.96788(-1)	1.00823	6.84908(-1)	4.06306(-1)	1.18384(-1)
0.5		7.35162(-1)	9.97603(-1)	1.01877	6.96363(-1)	4.17763(-1)	1.31990(-1)
0.6		7.37655(-1)	1.00596	1.03000	7.07819(-1)	4.29219(-1)	1.45139(-1)
0.7		7.77924(-1)	1.03006	1.04278	7.19275(-1)	4.40676(-1)	1.57958(-1)
0.8		8.87046(-1)	1.08595	1.05891	7.30733(-1)	4.52132(-1)	1.70543(-1)
0.9		1.15394	1.21413	1.08265	7.42194(-1)	4.63588(-1)	1.82954(-1)
1.0		8.07460(1)	1.17861(1)	1.26060	7.53663(-1)	4.75044(-1)	1.95231(-1)

Table 20. The Intensity $I_*(\tau,\mu,\phi)$ for the Cloud C_1 Phase Function with $\tau_0 = 64$, $\omega = 0.9$ and $\mu_0 = 1$.

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	2.09773(-1)	8.66126(-2)	4.13435(-2)	9.51105(-3)	1.08267(-4)	2.57858(-6)	
-0.9	1.33057(-1)	7.99160(-2)	4.16423(-2)	9.88858(-3)	1.13004(-4)	2.69195(-6)	
-0.8	1.55857(-1)	8.49795(-2)	4.37516(-2)	1.04152(-2)	1.19270(-4)	2.84172(-6)	
-0.7	1.22477(-1)	8.30448(-2)	4.53146(-2)	1.10714(-2)	1.27346(-4)	3.03460(-6)	
-0.6	1.06131(-1)	8.38166(-2)	4.77055(-2)	1.19091(-2)	1.37561(-4)	3.27843(-6)	
-0.5	1.00372(-1)	8.72627(-2)	5.11025(-2)	1.29658(-2)	1.50302(-4)	3.58246(-6)	
-0.4	9.36361(-2)	9.18819(-2)	5.54028(-2)	1.42755(-2)	1.66031(-4)	3.95768(-6)	
-0.3	8.61098(-2)	9.77875(-2)	6.07302(-2)	1.58826(-2)	1.85298(-4)	4.41725(-6)	
-0.2	7.84675(-2)	1.05214(-1)	6.72524(-2)	1.78411(-2)	2.08767(-4)	4.97698(-6)	
-0.1	6.76112(-2)	1.14041(-1)	7.51246(-2)	2.02153(-2)	2.37239(-4)	5.65602(-6)	
-0.0	4.06393(-2)	1.24429(-1)	8.45706(-2)	2.30841(-2)	2.71692(-4)	6.47767(-6)	
0.0		1.24429(-1)	8.45706(-2)	2.30841(-2)	2.71692(-4)	6.47767(-6)	7.64446(-8)
0.1		1.36598(-1)	9.58751(-2)	2.65448(-2)	3.13320(-4)	7.47040(-6)	1.36418(-7)
0.2		1.50947(-1)	1.09418(-1)	3.07180(-2)	3.63592(-4)	8.66925(-6)	1.74370(-7)
0.3		1.68265(-1)	1.25733(-1)	3.57557(-2)	4.24324(-4)	1.01175(-5)	2.15257(-7)
0.4		1.90217(-1)	1.45622(-1)	4.18534(-2)	4.97777(-4)	1.18692(-5)	2.61674(-7)
0.5		2.20190(-1)	1.70416(-1)	4.92710(-2)	5.86781(-4)	1.39916(-5)	3.15782(-7)
0.6		2.63712(-1)	2.02431(-1)	5.83725(-2)	6.94910(-4)	1.65699(-5)	3.79933(-7)
0.7		3.31269(-1)	2.46092(-1)	6.97143(-2)	8.26710(-4)	1.97124(-5)	4.56930(-7)
0.8		4.46037(-1)	3.10866(-1)	8.42687(-2)	9.88030(-4)	2.35575(-5)	5.50243(-7)
0.9		6.76716(-1)	4.22683(-1)	1.04174(-1)	1.18650(-3)	2.82847(-5)	6.64252(-7)
1.0		6.94794(1)	8.97566	2.23085(-1)	1.43269(-3)	3.41286(-5)	8.04618(-7)

Table 21. Basic Data.

Case	Model	τ_0	ω	μ_0
1	Haze L	1	1	1
2	Haze L	1	0.9	1
3	Haze L	1	0.9	0.5
4	Cloud C_1	64	1	1
5	Cloud C_1	64	0.9	1

(with six figure accuracy) the L=8 problem, that N≈100-150 was adequate for the haze problems and that N≈300-500 was required for the cloud problem. In solving the haze and cloud problems, where N had to be rather large (N>100, at least) to obtain really good results, we found that the choice of the collocation scheme based on the zeros of the Chebyshev polynomials was an especially good one. In fact all other choices that we tried (with the exception of the one[9] based on the zeros of the Chebyshev polynomials of the first kind) failed completely. We even found that a slight perturbation (say by 1%) of the Chebyshev points was enough to make the method fail for those problems based on many-term phase functions (haze and cloud). We have found that our basic collocation scheme based on the discrete eigenvalues and the Chebyshev points provides excellent results for $I(\tau,\pm\mu,\phi)$ for general values of $\mu\epsilon[0,1]$; however, improved results, for some given value of N, can be obtained for specific values of μ, say the eleven values used in our accompanying tables, by including those values in the collocation scheme.

We note that Gaussian elimination proved adequate for the L=8 problem and the haze problems (even for N=500); however, we could solve the cloud problems only after following a suggestion of Karp[24] and using the singular-value decomposition method[25] to deal

Table 22. The Fluxes.

Case	Quantity	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
	$q_+(\tau)$	3.14159	3.13794	3.13349	3.12286	3.07948	3.03087	2.96837
1	$q_-(\tau)$	1.73223(-1)	1.69570(-1)	1.65124(-1)	1.54494(-1)	1.11113(-1)	6.24993(-2)	0.0
	$q(\tau)$	2.96837	2.96837	2.96837	2.96837	2.96837	2.96837	2.96837
	$q_+(\tau)$	3.14159	3.12151	3.10074	3.05777	2.92065	2.79923	2.67127
2	$q_-(\tau)$	1.23665(-1)	1.20901(-1)	1.17603(-1)	1.09841(-1)	7.88688(-2)	4.45453(-2)	0.0
	$q(\tau)$	3.01793	3.00061	2.98314	2.94793	2.84178	2.75469	2.67127
	$q_+(\tau)$	1.57080	1.54485	1.51680	1.45804	1.28063	1.14320	1.01588
3	$q_-(\tau)$	2.25487(-1)	2.19149(-1)	2.10953(-1)	1.91599(-1)	1.23848(-1)	6.40822(-2)	0.0
	$q(\tau)$	1.34531	1.32570	1.30585	1.26644	1.15678	1.07912	1.01588
	$q_+(\tau)$	3.14159	3.55742	3.56929	3.29179	2.24768	1.37243	4.79852(-1)
4	$q_-(\tau)$	2.66174	3.07757	3.08944	2.81194	1.76783	8.92573(-1)	0.0
	$q(\tau)$	4.79852(-1)	4.79852(-1)	4.79852(-1)	4.79852(-1)	4.79852(-1)	4.79852(-1)	4.79852(-1)
	$q_+(\tau)$	3.14159	1.87572	9.80914(-1)	2.33984(-1)	2.67761(-3)	6.38345(-5)	1.47334(-6)
5	$q_-(\tau)$	3.75305(-1)	2.70485(-1)	1.51045(-1)	3.74253(-2)	4.31818(-4)	1.02904(-5)	0.0
	$q(\tau)$	2.76629	1.60524	8.29869(-1)	1.96558(-1)	2.24579(-3)	5.35441(-5)	1.47334(-6)

with the ill-conditioned system of linear algebraic equations we encountered (note that L=299 and that there are 24 or 25 pairs of discrete eigenvalues). More specifically, we used the singular-value decomposition of the matrix of coefficients in our linear system to construct a pseudo inverse (as discussed in Ref. 20). We elected to consider a singular value to be zero when the ratio of it to the largest singular value was less than $\varepsilon=10^{-7}$. For the two cloud problems solved we found (for $\varepsilon=10^{-7}$) that our linear system was rank deficient by five (independent of N). This clearly suggests that some of the equations derived from the 24 or 25 pairs of eigenvalues do not provide sufficiently different information.

We have solved to a high degree of accuracy, we believe, the problems we considered; however we would like to point out two interesting points (in regard to numerical linear algebra) that have evolved from this work and that deserve additional study. Firstly, if the numerical linear dependence encountered in the cloud problem could be eliminated analytically we could use Gaussian elimination rather than SVD to solve our linear system (note that SVD is considerably more time-consuming than Gaussian elimination). Finally, in monitoring the condition number estimates returned by the Gaussian elimination routines we used, we found for the m>0 components of the L=8 problem and the Haze L problem with $\mu_0=0.5$ (Case 3) that our results were, we believe, better than the estimates suggested.

ACKNOWLEDGEMENT

The authors would like to express their gratitude to A. H. Karp and the IBM Scientific Center (Palo Alto) for the kind hospitality extended to them during the course of this investigation. They would also like to thank A. A. Dubrulle and A. H. Karp for the interest shown in this work and for two especially helpful suggestions relevant to numerical linear algebra.

This work was also supported in part by the National Science Foundation.

REFERENCES

1. J. Lenoble, Ed., *Standard Procedures to Compute Atmospheric Radiative Transfer in a Scattering Atmosphere*, National Center for Atmospheric Research, Boulder, Colorado (1977).

2. C. E. Siewert and P. Benoist, *Nucl. Sci. Eng.*, **69**, 156 (1979).

3. C. Devaux and C. E. Siewert, *Z. angew. Math. Phys.*, **31**, 592 (1980).

4. C. Devaux, C. E. Siewert and Y. L. Yuan, *Astrophys. J.*, **253**, 773 (1982).

5. S. Chandrasekhar, *Radiative Transfer*, Oxford University Press, London (1950).

6. R. D. M. Garcia and C. E. Siewert, *Nucl. Sci. Eng.*, **78**, 315 (1981).

7. N. J. McCormick and I. Kuščer, *J. Math. Phys.*, **7**, 2036 (1966).

8. R. D. M. Garcia and C. E. Siewert, *Nucl. Sci. Eng.*, **81**, 474 (1982).

9. R. D. M. Garcia and C. E. Siewert, *J. Comp. Phys.*, **46**, 237 (1982).

10. J. K. Shultis and T. R. Hill, *Nucl. Sci. Eng.*, **59**, 53 (1976).

11. C. E. Siewert, *J. Math. Phys.*, **21**, 2468 (1980).

12. B. Davison, *Neutron Transport Theory*, Oxford University Press, London (1957).

13. A. H. Karp, J. Greenstadt and J. A. Filmore, *J. Quant. Spectrosc. Rad. Transf.*, **24**, 391 (1980).

14. M. Benassi, R. D. M. Garcia, A. H. Karp and C. E. Siewert, *Astrophys. J.*, **280**, 853 (1984).

15. R. D. M. Garcia and C. E. Siewert, *Z. angew. Math. Phys.*, **33**, 801 (1982).

16. E. İnönü, *J. Math. Phys.*, **11**, 568 (1970).

17. B. T. Smith, J. M. Boyle, J. J. Dongarra, B. S. Garbow, Y. Ikebe, V. C. Klema, and C. B. Moler, *Matrix Eigensystem Routines - EISPACK Guide*, Springer-Verlag, Berlin (1976).

18. W. Gautschi, *SIAM Review*, **9**, 24 (1967).

19. A. A. Dubrulle, personal communication (1984).

20. J. J. Dongarra, J. R. Bunch, C. B. Moler, and G. W. Stewart, *LINPACK User's Guide*, SIAM, Philadelphia (1979).

21. G. H. Golub, and C. Reinsch, *Numer. Math.*, **14**, 403 (1970).

22. J. F. de Haan, personal communication (1982).

23. A. H. Karp, personal communication (1982).

24. A. H. Karp, personal communication (1984).

25. G. H. Golub and C. F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore (1983).

Received: November 20, 1984
Revised: March 22, 1985