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ON THE CRITICAL PROBLEM FOR A TWO-REGION SPHERE*

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Abstract—The one-speed critical problem is solved for a two-region sphere, where the total cross section is common, but different multiplication parameters in the two zones are permitted. A transform procedure is used to reduce the integral transport equation to a pseudo problem in slab geometry. This, in turn, is solved by Case's method of normal modes. The analysis leads to a Fredholm integral equation for the only unknown expansion coefficient. In addition, a critical condition, from which the allowable parameters of the problem are to be determined, is obtained.

1. INTRODUCTION

THE normal-mode expansion technique introduced by CASE (1960) has frequently been used to develop exact solutions to problems in neutron transport theory and also to radiative transfer problems in astrophysical applications; see, for example, CASE and ZWEIFEL (1967), SIEWERT and ZWEIFEL (1966), SIEWERT and FRALEY (1967), and MCCORMICK and KUŠČER (1965). Since the class of problems that have been solved explicitly has basically been restricted to those involving plane symmetry, there is a need for more comprehensive solutions to problems with spherical or cylindrical symmetry.

DAVISON (1945), in one of his classic papers, found the eigenfunctions of the homogeneous one-speed transport equation in spherical geometry. Hence he was able to construct an exact solution to the infinite-medium point-source problem. More recently MITSIS (1963) developed an ingenious transform technique which he used to generate solutions to single-region critical problems for spheres and cylinders. LEONARD and MULLIKIN (1964) as well as ERDMANN and SIEWERT (1968) have also solved several single-region problems for spheres.

Inherent in the above, however, is the need to note a similarity between the spherical problem of interest and a judiciously chosen slab problem. Once a correspondence between spherical and slab problems has been established, the solution to the latter can usually be constructed. Though similarities exist between transform procedures already successfully employed, the technique for solving exactly an arbitrary problem in spherical geometry is yet desired.

The problem considered here is a special case of a general two-region sphere; viz. we assume that the total cross sections in the two zones are identical, though we do allow different multiplication properties in the two regions. It is thought, however, that the problem considered here is the most meaningful extension to two-region spheres for which semi-analytical solutions can currently be obtained.

In Section 2 we indicate a method by which the spherical problem of interest may be reduced to a corresponding pseudo-slab problem. In Section 3 the singular eigenfunction expansion technique (CASE, 1960) is used to obtain the necessary critical

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condition as well as a Fredholm integral equation for the remaining unknown expansion coefficient. Finally, the last section is devoted to an investigation of special cases and approximations that lead to considerably more tractable solutions.

2. THE TRANSFORM TECHNIQUE

We seek the solution to the integral transport equation for the neutron density in a spherical medium consisting of an inner core and a surrounding outer blanket; the two regions are assumed to have the same total cross section, but different multiplication properties are allowed. Within the assumptions of spherical symmetry, the one-speed transport equation for an isotropically scattering medium takes the form

$$r\rho(r) = \frac{c_1}{2}\int_0^{R_1} t\rho(t)\{E_1(|r-t|) - E_1(r+t)\}\,\mathrm{d}t + \frac{c_2}{2}\int_{R_1}^{R_2} t\rho(t)\{E_1(|r-t|) - E_1(r+t)\}\,\mathrm{d}t, \quad 0 \le r \le R_2 \tag{1}$$

where c_i denotes the mean number of secondary neutrons per collision in the i-th region; R_1 and R_2 are the radii of the inner and outer regions, respectively. In addition, we note that

$$E_1(x) = \int_0^1 \mathrm{e}^{-x/\mu}\,\frac{\mathrm{d}\mu}{\mu}. \tag{2}$$

As MITSIS (1963) did for single-region spheres, we extend equation (1) to include negative r, $-R_2 \le r \le R_2$, by defining $\rho(-r) = \rho(r)$; thus

$$r\rho(r) = \frac{c_2}{2}\int_{-R_2}^{-R_1} t\rho(t)E_1(|r-t|)\,\mathrm{d}t + \frac{c_1}{2}\int_{-R_1}^{R_1} t\rho(t)E_1(|r-t|)\,\mathrm{d}t + \frac{c_2}{2}\int_{R_1}^{R_2} t\rho(t)E_1(|r-t|)\,\mathrm{d}t, \quad -R_2 \le r \le R_2. \tag{3}$$

Substituting the definition of $E_1(x)$ into equation (3) and interchanging the order of integration, we obtain

$$r\rho_i(r) = \frac{1}{2}\int_{-1}^{1} \Psi_i(r,\mu)\,\mathrm{d}\mu, \quad i = 1 \text{ or } 2, \tag{4}$$

where $\rho_i(r)$ is the density in the i-th region; in addition, we have defined

$$\Psi_1(r,\mu) \triangleq \begin{Bmatrix} \dfrac{c_2}{\mu}\displaystyle\int_{-R_2}^{-R_1} \mathrm{d}t\, t\rho_2(t)\mathrm{e}^{-(r-t)/\mu} + \dfrac{c_1}{\mu}\int_{-R_1}^{r} \mathrm{d}t\, t\rho_1(t)\mathrm{e}^{-(r-t)/\mu}, & r\epsilon(0,R_1), \\ & \mu\epsilon(0,1) \\ -\dfrac{c_1}{\mu}\displaystyle\int_{r}^{R_1} \mathrm{d}t\, t\rho_1(t)\mathrm{e}^{(t-r)/\mu} - \dfrac{c_2}{\mu}\int_{R_1}^{R_2} \mathrm{d}t\, t\rho_2(t)\mathrm{e}^{(t-r)/\mu} & r\epsilon(0,R_1), \\ & \mu\epsilon(-1,0) \end{Bmatrix} \tag{5a}$$

and

$$\Psi_2(r,\mu) \triangleq \begin{Bmatrix} \dfrac{c_2}{\mu}\displaystyle\int_{-R_2}^{-R_1} \mathrm{d}t\, t\rho_2(t)\mathrm{e}^{-(r-t)/\mu} + \dfrac{c_1}{\mu}\int_{-R_1}^{R_1} \mathrm{d}t\, t\rho_1(t)\mathrm{e}^{-(r-t)/\mu} \\ + \dfrac{c_2}{\mu}\displaystyle\int_{R_1}^{r} \mathrm{d}t\, t\rho_2(t)\mathrm{e}^{-(r-t)/\mu}, \quad r\epsilon(R_1,R_2), \quad \mu\epsilon(0,1) \\ -\dfrac{c_2}{\mu}\displaystyle\int_{r}^{R_2} \mathrm{d}t\, t\rho_2(t)\mathrm{e}^{(t-r)/\mu}, \quad r\epsilon(R_1,R_2), \quad \mu\epsilon(-1,0) \end{Bmatrix}. \tag{5b}$$

Selecting similar definitions for negative r, we find that $\Psi_i(r, \mu)$ satisfies the transport equation for plane geometry, i.e.

$$\mu \frac{\partial \Psi_i}{\partial r}(r, \mu) + \Psi_i(r, \mu) = \frac{c_i}{2}\int_{-1}^{1} \Psi_i(r, \mu)\, d\mu, \quad r\epsilon(-R_2, R_2) \text{ and } \mu\epsilon(-1, 1). \tag{6}$$

The boundary conditions which uniquely determine the solution of equation (6) may be found from equations (5) and their analogues for negative r:

$$\Psi_i(r, \mu) = -\Psi_i(-r, -\mu), \qquad r\epsilon(-R_2, R_2), \qquad \mu\epsilon(-1, 1) \tag{7a}$$

$$\Psi_1(R_1, \mu) = \Psi_2(R_1, \mu), \qquad \mu\epsilon(-1, 1) \tag{7b}$$

and

$$\Psi_2(R_2, -\mu) = 0, \qquad \mu\epsilon(0, 1). \tag{7c}$$

We turn now to the solution of equation (6), from which the density for the considered problem may readily be obtained, as indicated by equation (4).

3. SOLUTION TO THE PSEUDO-SLAB PROBLEM

As was previously shown, we must solve equation (6) subject to boundary conditions given by equation (7). It is sufficient, however, to consider only positive r and the conditions,

$$\Psi_1(0, \mu) = -\Psi_1(0, -\mu), \quad \mu\epsilon(-1, 1) \tag{8a}$$

$$\Psi_1(R_1, \mu) = \Psi_2(R_1, \mu), \quad \mu\epsilon(-1, 1) \tag{8b}$$

and

$$\Psi_2(R_2, -\mu) = 0, \quad \mu\epsilon(0, 1) \tag{8c}$$

with the proviso that the solution for negative r (although it is not needed explicitly) is to be obtained in the manner indicated by equation (7a).

Were it not for the anti-symmetry condition, equation (8a), the solutions for $\Psi_i(r, \mu)$ would be those found by KUSZELL (1961) for a finite two-region slab. This condition clearly indicates that a negative flux is allowed; hence, the terminology 'pseudo-slab problem' is used. The solution here is constructed in a manner similar to that used by KUSZELL (1961); we thus write the solutions in terms of the normal modes introduced by CASE (1960), i.e.

$$\Psi_1(r, \mu) = A_+[\phi_+^{(1)}(\mu)e^{-r/\nu_{01}} - \phi_-^{(1)}(\mu)e^{r/\nu_{01}}] + \int_0^1 A(\nu)[\phi_\nu^{(1)}(\mu)e^{-r/\nu} - \phi_{-\nu}^{(1)}(\mu)e^{r/\nu}]\, d\nu, \; 0 \leq r \leq R_1 \tag{9a}$$

and

$$\Psi_2(r, \mu) = B_+\phi_{+(\mu)}^{(2)}e^{-r/\nu_{02}} + B_-\phi_-^{(2)}(\mu)e^{r/\nu_{02}} + \int_{-1}^1 B(\nu)\phi_\nu^{(2)}(\mu)e^{-r/\nu}\, d\nu, \quad R_1 \leq r \leq R_2. \tag{9b}$$

Here

$$\phi_\pm^{(i)}(\mu) = \frac{c_i\nu_{0i}}{2}\frac{1}{\nu_{0i} \pm \mu} \tag{10a}$$

$$\phi_\nu^{(i)}(\mu) = \frac{c_i\nu}{2}\frac{P}{\nu - \mu} + \lambda_i(\nu)\delta(\nu - \mu) \tag{10b}$$

$$\lambda_i(\nu) = 1 - c_i\nu \tanh^{-1}(\nu) \tag{10c}$$

and ν_{0i} is the positive zero of

$$\Lambda_i(z) = 1 - c_i z \tanh^{-1}\left(\frac{1}{z}\right). \tag{10d}$$

In addition, the symbol P indicates that integrals involving these functions are to be evaluated in the Cauchy principal-value sense, and $\delta(x)$ denotes the Dirac delta function.

The first boundary condition is clearly satisfied by equation (9a). The remaining expansion coefficients, A_+, $A(\nu)$, B_-, and $B(\nu)$ are to be determined from the second two conditions. (One parameter is arbitrary; we take $B_+ = -1$.) Application of the free-surface boundary condition yields

$$\phi_-^{(2)}(\mu)e^{-R_2/\nu_{02}} - \int_0^1 B(\nu)e^{-R_2/\nu}\,\phi_{-\nu}^{(2)}(\mu)\,d\nu = B_-\,e^{R_2/\nu_{02}}\,\phi_+^{(2)}(\mu) + \int_0^1 B(-\nu)e^{R_2/\nu}\,\phi_\nu^{(2)}(\mu)\,d\nu, \quad \mu\epsilon(0, 1). \tag{11}$$

The half-range completeness theorem (CASE and ZWEIFEL, 1967) states that equation (11) is a valid expansion; B_- and $B(-\nu)$ can thus be found in terms of $B(\nu)$, $\nu > 0$, by taking half-range scalar products (KUŠČER *et al.*, 1964). We find

$$B_- = \frac{-e^{-R_2/\nu_{02}}}{\left(\frac{c_2\nu_{02}}{2}\right)^2 X_2(\nu_{02})}\int_0^1 F(\mu)W_2(\mu)\phi_+^{(2)}(\mu)\,d\mu \tag{12a}$$

and

$$B(-\nu) = \frac{e^{-R_2/\nu}}{W_2(\nu)}g(c_2, \nu)\int_0^1 F(\mu)W_2(\mu)\phi_\nu^{(2)}(\mu)\,d\mu \tag{12b}$$

where

$$F(\mu) \triangleq \phi_-^{(2)}(\mu)e^{-R_2/\nu_{02}} - \int_0^1 B(\nu')e^{-R_2/\nu'}\,\phi_{-\nu'}^{(2)}(\mu)\,d\nu'. \tag{13}$$

Also

$$X_2(z) = \frac{1}{1-z}\exp\left\{\frac{1}{\pi}\int_0^1 \tan^{-1}\left(\frac{\pi c_2\nu}{2\lambda_2(\nu)}\right)\frac{d\nu}{\nu - z}\right\}, \tag{14a}$$

$$W_2(\nu) = \frac{c_2\nu}{2(1-c_2)(\nu_0+\nu)\,X_2(-\nu)}, \tag{14b}$$

and

$$g(c_i, \nu) = \left\{\lambda_i^2(\nu) + \left(\frac{c_i\nu\pi}{2}\right)^2\right\}^{-1}, \quad i = 1 \text{ or } 2. \tag{14c}$$

If $B(\nu)$ were known, B_- and $B(-\nu)$ would also be determined, as is indicated by equations (12).

Considering the final boundary condition, equation (8b), we are led to the following full-range expansion:

$$G(\mu) = A_+[\phi_+^{(1)}(\mu)e^{-R_1/\nu_{01}} - \phi_-^{(1)}(\mu)e^{R_1/\nu_{01}}] + \int_0^1 A(\nu)[\phi_\nu^{(1)}(\mu)e^{-R_1/\nu} - \phi_{-\nu}^{(1)}(\mu)\,e^{R_1/\nu}]\,d\nu, \quad \mu\epsilon(-1, 1), \tag{15}$$

where

$$G(\mu) \triangleq -\phi_+^{(2)}(\mu)e^{-R_1/\nu_{02}} + B_-\phi_-^{(2)}(\mu)e^{R_1/\nu_{02}} + \int_{-1}^{1} B(\nu)\phi_\nu^{(2)}(\mu)e^{-R_1/\nu}\,d\nu. \quad (16)$$

In order for equation (15) to be a valid full-range expansion, two restrictions must be imposed on the expansion function, $G(\mu)$, viz.

$$\int_{-1}^{1} \mu G(\mu)\phi_+^{(1)}(\mu)\,d\mu = e^{-2R_1/\nu_{01}}\int_{-1}^{1} \mu G(\mu)\phi_-^{(1)}(\mu)\,d\mu \quad (17a)$$

and

$$\int_{-1}^{1} \mu G(\mu)\phi_\nu^{(1)}(\mu)\,d\mu = e^{-2R_1/\nu}\int_{-1}^{1} \mu G(\mu)\phi_{-\nu}^{(1)}(\mu)\,d\nu, \quad \nu\epsilon(0, 1). \quad (17b)$$

Once the above two conditions have been satisfied, the expansion coefficients A_+ and $A(\nu)$ are found trivially by taking, this time, full-range scalar products (CASE and ZWEIFEL, 1967). Thus,

$$A_+ = \frac{e^{R_1/\nu_{01}}}{N_+}\int_{-1}^{1} \mu G(\mu)\phi_+^{(1)}(\mu)\,d\mu \quad (18a)$$

and

$$A(\nu) = \frac{e^{R_1/\nu}}{\nu} g(c_1, \nu)\int_{-1}^{1} \mu G(\mu)\phi_\nu^{(1)}(\mu)\,d\mu \quad (18b)$$

where

$$N_+ = \frac{c_1}{2}\nu_{01}^3\left[\frac{c_1}{\nu_{01}^2 - 1} - \frac{1}{\nu_{01}^2}\right]. \quad (18c)$$

The restrictions on $G(\mu)$ given by equations (17) form the basis for the remaining analysis needed to complete the solution we seek. The first, equation (17a), is the critical condition from which the allowable parameters of the problem (c_1, c_2, R_1 and R_2) are to be determined. Equation (17b), as shall be shown, can be reduced to a Fredholm integral equation for the remaining unkown expansion coefficient, $B(\nu)$, $\nu > 0$. The function $G(\mu)$, we note, may be written explicitly in terms of only $B(\nu)$, $\nu > 0$, by utilizing equations (12) and (13). However, since at this point to do so only complicates the equations involved, we prefer to keep B_- and $B(-\nu)$, $\nu > 0$, in the equations until the majority of the simplifying manipulations have been performed.

To reduce equations (17) to an explicit form, we make use of the cross-product integrals,

$$\int_{-1}^{1} \mu\phi_\xi^{(1)}(\mu)\phi_{\xi'}^{(2)}(\mu)\,d\mu = \xi\xi'\frac{P}{\xi' - \xi}\frac{c_2 - c_1}{2} + M(\xi)\delta(\xi - \xi')\xi, \quad (19)$$

where

$$M(\xi) = \lambda_1(\xi)\lambda_2(\xi) + \frac{\pi^2}{4}c_1c_2\xi^2, \quad \xi = \pm\nu_{01} \text{ or } \epsilon(-1, 1),$$
$$\xi' = \pm\nu_{02} \text{ or } \epsilon(-1, 1),$$

to write equation (17a) as

$$-H(R_1, \nu_{01}, \nu_{02}) + B_-H(-R_1, \nu_{01}, \nu_{02}) = -\int_0^1 d\eta\,\{B(\eta)H(R_1, \nu_{01}, \eta) + B(-\eta)H(-R_1, \nu_{01}, \eta)\}, \quad (20)$$

where we have defined

$$H(R_1, \xi, \xi') = (c_2 - c_1)\, e^{-R_1/\xi'}\, \xi' \frac{P}{\xi'^2 - \xi^2}\, [\xi' \cosh R_1/\xi + \xi \sinh R_1/\xi] + M(\xi)\delta(\xi - \xi').$$

In a similar manner, we write equation (17b) in the form

$$-H(R_1, \nu', \nu_{02}) + B_{\_}H(-R_1, \nu', \nu_{02}) = -\int_0^1 d\eta [B(\eta)H(R_1, \nu', \eta) + B(-\eta)H(-R_1, \nu'_i, \eta)],\ \nu', \eta > 0. \quad (21)$$

The final form of the critical condition may be determined by substituting the expressions for $B_{\_}$ and $B(-\eta)$ given by equations (12) into equation (20). In expanded form, equations (12) become

$$B_{\_} = e^{-2R_2'/\nu_{02}} + \frac{e^{-R_2/\nu_{02}}}{\nu_{02}X(\nu_{02})} \int_0^1 d\eta B(\eta)\, e^{-R_2/\eta}\, \eta X_2(-\eta) \quad (22a)$$

and

$$B(-\eta) = F(\eta, \nu_{02}) - \int_0^1 d\eta' B(\eta') F(\eta, \eta'), \quad (22b)$$

where

$$F(\xi, \xi') = \tfrac{1}{2}\, e^{-R_2/\xi}\, c_2(1 - c_2)X_2(-\xi)g(c_2, \xi)(\nu_{02} + \xi)e^{-R_2/\xi'}\, X_2(-\xi') \left(\frac{\nu_{02} + \xi'}{\xi' + \xi}\right) \xi'$$

and

$$R_2' = R_2 + \delta,$$

δ being the Milne problem extrapolated endpoint (CASE and ZWEIFEL, 1967).

Effecting the above substitutions, we write the critical condition as

$$-H(R_1, \nu_{01}, \nu_{02}) + e^{-2R_2'/\nu_{02}}\, H(-R_1, \nu_{01}, \nu_{02}) + \int_0^1 d\eta H(-R_1, \nu_{01}, \eta)F(\eta, \nu_{02})$$
$$= -\int_0^1 d\eta B(\eta) \left\{ \frac{e^{-R_2/\eta}\, \eta X_2(-\eta)}{e^{R_2/\nu_{02}}\, \nu_{02}X_2(\nu_{02})} H(-R_1, \nu_{01}, \nu_{02}) + H(R_1, \nu_{01}, \eta) - \int_0^1 d\eta' H(-R_1, \nu_{01}, \eta')F(\eta', \eta) \right\}. \quad (23)$$

The equation from which $B(\eta)$ is determined follows similarly

$$\int_0^1 d\eta B(\eta) e^{-R_1/\eta}\, T(\eta, \nu') = e^{-R_1/\nu'} \{H(R_1, \nu', \nu_{02}) + J(\nu_{02}, \nu') - \int_0^1 d\eta' B(\eta') [e^{-R_1/\nu'} \frac{c_2 - c_1}{2} \frac{\eta'}{\eta' + \nu'}\, e^{-R_1/\eta'} + J(\eta', \nu')]\} \quad (24)$$

where

$$J(\xi, \xi') = \frac{e^{-R_2/\xi} X_2(-\xi)\xi}{e^{R_2/\nu_{02}}\, X_2(\nu_{02})\nu_{02}} H(-R_1, \xi', \nu_{02}) - \int_0^1 d\eta e^{R_1/\eta} \left[\frac{c_2 - c_1}{2}\, e^{R_1/\xi'} \frac{\eta}{\eta + \xi'} + e^{-R_1/\xi'}\, T(\eta, \xi')\right] F(\eta, \xi) \quad (25a)$$

and

$$T(\eta, \nu') = \frac{c_2 - c_1}{2} \eta \frac{P}{\eta - \nu'} + M(\eta)\delta(\eta - \nu'). \tag{25b}$$

We note that equation (24) is a singular integral equation for $B(\eta)$. We follow a procedure similar to that used by KUSZELL (1961) for a related slab problem to reduce equation (24) to the form,

$$B(\eta) = -\int_0^1 \mathrm{d}\eta' B(\eta') \hat{L}(\eta, \nu') \left\{ \mathrm{e}^{-2R_1/\nu'} \frac{c_2 - c_1}{2} \frac{\eta'}{\eta' + \nu'} \mathrm{e}^{-R_1/\eta'} + J(\eta', \nu') \mathrm{e}^{-R_1/\nu'} \right\}$$
$$+ L(\eta, \nu') \left\{ \mathrm{e}^{-R_1/\nu'} H(R_1, \nu', \nu_{02}) + \mathrm{e}^{-R_1/\nu'} J(\nu_{02}, \nu') \right\}. \tag{26}$$

Here $\hat{L}(\eta, \nu')$ operates on a function $f(\nu')$ as follows:

$$\hat{L}(\eta, \nu') f(\nu') \triangleq \frac{\mathrm{e}^{R_1/\eta}}{\left[M^2(\eta) + \left(\frac{c_2 - c_1}{2} \eta \pi \right)^2 \right] \gamma(\eta)} \int_0^1 \mathrm{d}\nu' \gamma(\nu') T(\eta, \nu') f(\nu') \, \mathrm{d}\nu', \tag{27}$$

where $\gamma(\nu)$ is the appropriate half-range weight function used by KUSZELL (1961).

Equation (26) is a Fredholm integral equation for $B(\eta)$ for which closed-form solutions are not available. However, a numerical procedure should be applicable to effect a solution to any desired degree of accuracy. Once $B(\eta)$ and the critical parameters determined by equation (23) are known, the other coefficients follow readily from previously derived results. In the subsequent section we analyse these results by invoking several approximations which provide a check on the validity of the results obtained in this section and render an insight into the physical aspects of the problem.

4. ANALYSIS OF RESULTS

From the results of the previous section it is clear that the expansion coefficients cannot be written in explicit form. However, the formulation of the Fredholm integral equation for $B(\eta)$ permits a systematic approximation to the solution, whereby any desired degree of accuracy may be obtained. Analogous to the well-known Neumann series method, a technique may be employed which uses the free term in equation (26) as an initial estimate of $B(\eta)$; an iteration scheme may then be used to improve the accuracy. Because of the complexity of the integrals involved, analytical results become unwieldy for the first-order correction; thus a numerical procedure is preferable.

A zeroth-order solution may be obtained, however, permitting analytical formulations for the coefficients and providing an insight into the basic form of the solution. In this approximation we take $B(\eta) = B(-\eta) = 0$. From equations (18a) and (22a) the discrete coefficients take the forms ($\Delta c = c_2 - c_1$)

$$B_- = \mathrm{e}^{-2R_2'/\nu_{02}} \tag{28a}$$

and

$$A_+ = -\frac{\Delta c}{N_+} \mathrm{e}^{-R_2'/\nu_{02}} \mathrm{e}^{R_1/\nu_{01}} \frac{\nu_{01}\nu_{02}}{\nu_{02}^2 - \nu_{01}^2} [\nu_{02} \sinh \Delta R/\nu_{02} + \nu_{01} \cosh \Delta R/\nu_{02}], \tag{28b}$$

where $\Delta R = R_2 + \delta - R_1$.

The critical condition reduces to the diffusion theory results, i.e.

$$\frac{1}{\nu_{01}} \coth R_1/\nu_{01} = -\frac{1}{\nu_{02}} \coth \Delta R/\nu_{02} \tag{29}$$

where ν_{0i}^2 is analogous to the buckling in the i-th region. Also, the diffusion theory density is obtained from equations (9):

$$\rho_1(r) = -A_+ \frac{\sinh r/\nu_{01}}{r} \quad (30a)$$

and

$$\rho_2(r) = -e^{-R_2'/\nu_{02}} \sinh (R_2' - r)/\nu_{02}. \quad (30b)$$

We note from equation (30b) that the density goes to zero at the extrapolated boundary. In addition, using equations (30), we find

$$\rho_1(R_1)/\rho_2(R_1) = \frac{\nu_{01}\nu_{02}^2 \Delta c}{(\nu_{02}^2 - \nu_{01}^2)N_+}. \quad (31)$$

For weakly absorbing media, the right-hand side of equation (31) approaches one; hence, the density is continuous at the inner boundary, and diffusion theory results thus provide a good approximation to the true transport solution for this limiting case.

As previously mentioned, the next higher order approximation, i.e. letting $B(\eta)$ equal the free term in equation (26), leads to results too cumbersome to permit a simple statement of the physical implications. However, as a check on the validity of the analytical results, we let c_2 approach c_1 which reduces equations (23) and (26) to those obtained by MITSIS (1963) for homogeneous spheres.

In conclusion, even though the expansion coefficients cannot be determined explicitly, the reduction of the initial integral equation for the density to a Fredholm equation for the expansion coefficients has greatly enhanced the possibility of obtaining a rapidly converging solution. The integrals involved in the final formulation, although formidable analytically, present no special problems for modern day computer techniques. Numerical calculations for this problem have been initiated and will be reported at a later date.

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