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A GENERALIZED SPHERICAL HARMONICS SOLUTION FOR RADIATIVE TRANSFER MODELS THAT INCLUDE POLARIZATION EFFECTS

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Abstract—A generalized spherical harmonics solution for all components ($m \geqslant 0$) in a Fourier representation of the Stokes vector basic to the scattering of polarized light is developed, and computational aspects of the solution are discussed in detail. The established solution is used in regard to two test problems to obtain numerical results, accurate, in general, to five significant figures, for the four Stokes parameters.

INTRODUCTION

In a recent work[1] we reported a generalized spherical harmonics solution for the azimuthally symmetric component ($m = 0$) in a Fourier representation of the vector of Stokes parameters basic to the scattering of polarized light. Here we develop a solution for all ($m \geqslant 0$) components.

We let $\mathbf{I}(\tau, \mu, \varphi)$ denote the density vector with the four Stokes parameters $I(\tau, \mu, \varphi)$, $Q(\tau, \mu, \varphi)$, $U(\tau, \mu, \varphi)$ and $V(\tau, \mu, \varphi)$ as components and consider the equation of transfer[2–4]

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu, \varphi) + \mathbf{I}(\tau, \mu, \varphi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \mathbf{P}(\mu, \mu', \varphi - \varphi') \mathbf{I}(\tau, \mu', \varphi') \, \mathrm{d}\mu' \, \mathrm{d}\varphi' \tag{1}$$

where we use the analytical representation[5]

$$\mathbf{P}(\mu, \mu', \varphi - \varphi') = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) [\mathbf{C}^m(\mu, \mu') \cos m(\varphi - \varphi') + \mathbf{S}^m(\mu, \mu') \sin m(\varphi - \varphi')] \tag{2}$$

for the phase matrix. Here[5]

$$\mathbf{C}^m(\mu, \mu') = \mathbf{A}^m(\mu, \mu') + \mathbf{D}\mathbf{A}^m(\mu, \mu')\mathbf{D} \tag{3a}$$

and

$$\mathbf{S}^m(\mu, \mu') = \mathbf{A}^m(\mu, \mu')\mathbf{D} - \mathbf{D}\mathbf{A}^m(\mu, \mu') \tag{3b}$$

where

$$\mathbf{A}^m(\mu, \mu') = \sum_{l=m}^{L} \mathbf{\Pi}_l^m(\mu) \mathbf{B}_l \mathbf{\Pi}_l^m(\mu') \tag{4a}$$

and

$$\mathbf{D} = \mathrm{diag}\{1, 1, -1, -1\}. \tag{4b}$$

In addition

$$\mathbf{\Pi}_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} \begin{vmatrix} P_l^m(\mu) & 0 & 0 & 0 \\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0 \\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0 \\ 0 & 0 & 0 & P_l^m(\mu) \end{vmatrix} \tag{5}$$

where

$$P_l^m(\mu) = (1 - \mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_l(\mu) \tag{6}$$

is used to denote the associated Legendre functions, and the functions $R_l^m(\mu)$ and $T_l^m(\mu)$ are as defined and used in Refs [3, 5]. Note that, in contrast to Ref. [5], we have, as a matter of computational convenience, included the factor $[(l-m)!/(l+m)!]^{1/2}$ in our current definition of the matrices $\mathbf{\Pi}_l^m(\mu)$. Continuing to follow Ref. [1], we note that $\tau \in [0, \tau_0]$ is the optical variable, μ is the direction cosine of the propagating radiation, ω is the single-scattering albedo and the scattering law is defined by the "Greek constants" $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \epsilon_l, \zeta_l\}$ so that

$$\mathbf{B}_l = \begin{vmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\epsilon_l \\ 0 & 0 & \epsilon_l & \delta_l \end{vmatrix} . \tag{7}$$

As we seek a solution of equation (1) subject to the boundary conditions

$$\mathbf{I}(0, \mu, \varphi) = \pi\delta(\mu - \mu_0)\delta(\varphi - \varphi_0)\mathbf{F} \tag{8a}$$

and

$$\mathbf{I}(\tau_0, -\mu, \varphi) = \frac{\lambda_0}{\pi}\mathbf{L}\int_0^{2\pi}\int_0^1 \mathbf{I}(\tau_0, \mu', \varphi')\mu'\,d\mu'\,d\varphi' \tag{8b}$$

for $\mu \in [0, 1]$ and $\varphi \in [0, 2\pi]$, we write

$$\mathbf{I}(\tau, \mu, \varphi) = \pi\delta(\mu - \mu_0)\delta(\varphi - \varphi_0)\mathbf{F}\exp(-\tau/\mu) + \mathbf{I}*(\tau, \mu, \varphi) \tag{9}$$

and find that the diffuse field $\mathbf{I}*(\tau, \mu, \varphi)$ is defined by

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}*(\tau, \mu, \varphi) + \mathbf{I}*(\tau, \mu, \varphi) = \frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^1 \mathbf{P}(\mu, \mu', \varphi - \varphi')\mathbf{I}*(\tau, \mu', \varphi')\,d\mu'\,d\varphi' + \mathbf{F}(\tau, \mu, \varphi), \tag{10}$$

where

$$\mathbf{F}(\tau, \mu, \varphi) = \frac{\omega}{4}\mathbf{P}(\mu, \mu_0, \varphi - \varphi_0)\mathbf{F}\exp(-\tau/\mu_0) \tag{11}$$

and the boundary conditions

$$\mathbf{I}*(0, \mu, \varphi) = \mathbf{0} \tag{12a}$$

and

$$\mathbf{I}*(\tau_0, -\mu, \varphi) = \lambda_0\mu_0\mathbf{L}\mathbf{F}\exp(-\tau_0/\mu_0) + \frac{\lambda_0}{\pi}\mathbf{L}\int_0^{2\pi}\int_0^1 \mathbf{I}*(\tau_0, \mu', \varphi')\mu'\,d\mu'\,d\varphi' \tag{12b}$$

for $\mu \in [0, 1]$ and $\varphi \in [0, 2\pi]$. We note that λ_0 is the coefficient for Lambert reflection, $\mathbf{L} = \mathrm{diag}\{1, 0, 0, 0\}$ and that the flux vector $\mathbf{F}$ has entries F_I, F_Q, F_U and F_V that are presumed given.

FOURIER DECOMPOSITION

As equations (10), (11) and (12) define the diffuse radiation field we seek, we now wish to utilize the Fourier decomposition of the Stokes vector that was developed in Ref. [3] and subsequently used by McCormick and Sanchez in a work[6] concerning the inverse problem in radiative transfer with polarization. First of all we define

$$\boldsymbol{\phi}_1^m(\chi) = (2 - \delta_{0,m})\mathrm{diag}\{\cos m\chi, \cos m\chi, \sin m\chi, \sin m\chi\} \tag{13a}$$

and

$$\boldsymbol{\phi}_2^m(\chi) = (2 - \delta_{0,m})\mathrm{diag}\{-\sin m\chi, -\sin m\chi, \cos m\chi, \cos m\chi\} \tag{13b}$$

and note that we can write the phase matrix as

$$\mathbf{P}(\mu, \mu', \varphi - \varphi') = \sum_{m=0}^{L} [\boldsymbol{\phi}_1^m(\varphi - \varphi')\mathbf{A}^m(\mu, \mu')\mathbf{D}_1 + \boldsymbol{\phi}_2^m(\varphi - \varphi')\mathbf{A}^m(\mu, \mu')\mathbf{D}_2] \tag{14}$$

where $\mathbf{A}^m(\mu, \mu')$ is given by equation (4a) and where

$$\mathbf{D}_1 = \text{diag}\{1, 1, 0, 0\} \quad \text{and} \quad \mathbf{D}_2 = \text{diag}\{0, 0, 1, 1\}. \tag{15a, b}$$

Now we can substitute

$$\mathbf{I}*(\tau, \mu, \varphi) = \sum_{m=0}^{L} [\boldsymbol{\phi}_1^m(\varphi - \varphi_0)\mathbf{I}_1^m(\tau, \mu) + \boldsymbol{\phi}_2^m(\varphi - \varphi_0)\mathbf{I}_2^m(\tau, \mu)] \tag{16}$$

into equation (10) and conclude that $\mathbf{I}_\beta^m(\tau, \mu)$, for $\beta = 1$ and 2, must satisfy

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}_\beta^m(\tau, \mu) + \mathbf{I}_\beta^m(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} \mathbf{K}^m(\mu' \to \mu)\mathbf{I}_\beta^m(\tau, \mu')\, d\mu' + \mathbf{S}_\beta^m(\tau, \mu) \tag{17}$$

where

$$\mathbf{K}^m(\mu' \to \mu) = \sum_{l=m}^{L} \boldsymbol{\Pi}_l^m(\mu)\mathbf{B}_l\boldsymbol{\Pi}_l^m(\mu') \tag{18}$$

and

$$\mathbf{S}_\beta^m(\tau, \mu) = \frac{\omega}{4} \sum_{l=m}^{L} \boldsymbol{\Pi}_l^m(\mu)\mathbf{B}_l\boldsymbol{\Pi}_l^m(\mu_0)\mathbf{D}_\beta\mathbf{F}\exp(-\tau/\mu_0). \tag{19}$$

Substituting equation (16) into equations (12), we deduce the following boundary conditions:

$$\mathbf{I}_\beta^m(0, \mu) = \mathbf{0}, \quad \beta = 1 \text{ and } 2,\ m \geqslant 1, \tag{20a}$$

$$\mathbf{I}_\beta^m(\tau_0, -\mu) = \mathbf{0}, \quad \beta = 1 \text{ and } 2, m \geqslant 1, \tag{20b}$$

$$\mathbf{I}_\beta^0(0, \mu) = \mathbf{0}, \quad \beta = 1 \text{ and } 2, \tag{21a}$$

$$\mathbf{I}_1^0(\tau_0, -\mu) = \lambda_0\mu_0\mathbf{L}\mathbf{F}\exp(-\tau_0/\mu_0) + 2\lambda_0\mathbf{L}\int_0^1 \mathbf{I}_1^0(\tau_0, \mu')\mu' d\mu' \tag{21b}$$

and

$$\mathbf{I}_2^0(\tau_0, -\mu) = \mathbf{0} \tag{21c}$$

for $\mu \in [0, 1]$. We note that $m = 0$ is a special case in that a given four-vector problem can be decomposed into two two-vector problems, as was done, say, in Ref. [1]; however, in order to keep our notation compact and to provide some computational checks on our general development, we choose not to use this decomposition here.

BASIC SOLUTION

Following and generalizing previously reported works by Chandrasekhar,[2] Deuze *et al.*[7] and Benassi *et al.*,[1,8] we consider the homogeneous version of equation (17) and seek a generalized spherical harmonics (approximate) solution of the form

$$\mathbf{H}^m(\tau, \mu) = \sum_{l=m}^{M} \left(\frac{2l+1}{2}\right)\boldsymbol{\Pi}_l^m(\mu)\mathbf{G}_l^m(\xi)\mathbf{M}(\xi)\exp(-\tau/\xi) \tag{22}$$

where $M = m + N$ and the order of the approximation N is taken to be odd. If we now substitute equation (22) into the homogeneous version of equation (17), multiply the resulting equation by $\boldsymbol{\Pi}_{l'}^m(\mu)$, *for* $l' = m, m+1, \ldots, M$, and integrate over μ from -1 to 1, we find that equation (22) will be a solution of the resulting set of moment equations if the 4×4 polynomial matrices $\mathbf{G}_l^m(\xi)$ satisfy, for $l = m, m+1, \ldots,$ the three-term recursion relation

$$\mathbf{A}_l^m\mathbf{G}_{l-1}^m(\xi) + \mathbf{B}_l^m\mathbf{G}_l^m(\xi) + \mathbf{C}_l^m\mathbf{G}_{l+1}^m(\xi) = \xi\mathbf{G}_l^m(\xi), \tag{23}$$

where

$$\mathbf{A}_l^m = \mathbf{h}_l^{-1}\mathbf{U}_l^m, \tag{24a}$$

$$\mathbf{B}_l^m = -\mathbf{h}_l^{-1}\mathbf{V}_l^m \tag{24b}$$

and

$$\mathbf{C}_l^m = \mathbf{h}_l^{-1}\mathbf{U}_{l+1}^m, \tag{24c}$$

and ξ and the four-vector $\mathbf{M}(\xi)$ are such that

$$\mathbf{G}_{M+1}^m(\xi)\mathbf{M}(\xi) = \mathbf{0}. \tag{25}$$

Here

$$\mathbf{h}_l = (2l+1)\mathbf{I} - \omega\mathbf{B}_l, \tag{26}$$

$$\mathbf{U}_l^m = \operatorname{diag}\{(l^2-m^2)^{1/2}, [(l^2-m^2)(l^2-4)]^{1/2}/l, [(l^2-m^2)(l^2-4)]^{1/2}/l, (l^2-m^2)^{1/2}\} \tag{27}$$

and

$$\mathbf{V}_l^m = \frac{2m(2l+1)}{l(l+1)}\begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \tag{28}$$

In verifying that equation (22) satisfies the mentioned moment equations we have used the fact[5] that the matrices $\mathbf{\Pi}_l^m(\mu)$ satisfy the three-term recursion relation

$$\mathbf{U}_l^m\mathbf{\Pi}_{l-1}^m(\mu) - \mathbf{V}_l^m\mathbf{\Pi}_l^m(\mu) + \mathbf{U}_{l+1}^m\mathbf{\Pi}_{l+1}^m(\mu) = (2l+1)\mu\mathbf{\Pi}_l^m(\mu). \tag{29}$$

In regard to the $\mathbf{G}$ matrices, we note that we require three starting values for the case $m = 0$, two starting values for $m = 1$ and only one starting value for $m \geqslant 2$, and so to normalize the $\mathbf{G}$ matrices, we use

$$\mathbf{G}_0^0(\xi) = \operatorname{diag}\{1, 0, 0, 1\}, \tag{30a}$$

$$\mathbf{G}_1^0(\xi) = \operatorname{diag}\{(1-\omega)\xi, 0, 0, (1-\omega\delta_0)\xi\}, \tag{30b}$$

$$\mathbf{G}_2^0(\xi) = \operatorname{diag}\{\tfrac{1}{2}[(1-\omega)(3-\omega\beta_1)\xi^2-1], 1, 1, \tfrac{1}{2}[(1-\omega\delta_0)(3-\omega\delta_1)\xi^2-1)]\}, \tag{30c}$$

$$\mathbf{G}_1^1(\xi) = 2^{-1/2}\operatorname{diag}\{1, 0, 0, 1\}, \tag{30d}$$

$$\mathbf{G}_2^1(\xi) = \operatorname{diag}\{6^{-1/2}(3-\omega\beta_1)\xi, \xi, \xi, 6^{-1/2}(3-\omega\delta_1)\xi\} \tag{30e}$$

and, for $m \geqslant 2$,

$$\mathbf{G}_m^m(\xi) = S_m \operatorname{diag}\{1, R_m, R_m, 1\}, \tag{30f}$$

where

$$S_m = (2m-1)!![(2m)!]^{-1/2} \tag{31}$$

and

$$R_m = \left[\frac{m(m-1)}{(m+2)(m+1)}\right]^{1/2}. \tag{32}$$

We can deduce from equations (23) and (30) that

$$\mathbf{D}\mathbf{G}_l^m(-\xi)\mathbf{D} = (-1)^{l-m}\mathbf{G}_l^m(\xi), \tag{33}$$

where $\mathbf{D}$ is given by equation (4b), and since the zeros of det $\mathbf{G}_{M+1}^m(\xi)$ must therefore occur in $\pm$ pairs, we let $\{\xi_j\}$ for $j = 1, 2, \ldots, J$ denote the J zeros of det $\mathbf{G}_{M+1}^m(\xi)$ in the right-half plane. We now write our generalized spherical harmonics solution as

$$\mathbf{I}_h^m(\tau,\mu)=\sum_{l=m}^{M}\left(\frac{2l+1}{2}\right)\mathbf{\Pi}_l^m(\mu)\sum_{j=1}^{J}\{A_j\exp(-\tau/\xi_j)+(-1)^{l-m}\mathbf{D}B_j\exp[-(\tau_0-\tau)/\xi_j]\}\mathbf{T}_l^m(\xi_j) \tag{34}$$

where

$$\mathbf{T}_l^m(\xi_j)=\mathbf{G}_l^m(\xi_j)\mathbf{M}(\xi_j) \tag{35}$$

and $\mathbf{M}(\xi_j)$ is a null vector of $\mathbf{G}_{M+1}^m(\xi_j)$. The arbitrary constants $\{A_j\}$ and $\{B_j\}$ are to be determined from appropriate boundary conditions.

Following Ref. [1], we now write our solution to equation (17) as

$$\mathbf{I}_\beta^m(\tau,\mu)=\mathbf{I}_h^m(\tau,\mu)+\mathbf{I}_p^m(\tau,\mu) \tag{36}$$

where

$$\mathbf{I}_p^m(\tau,\mu)=\frac{1}{2}\sum_{l=m}^{M}\left(\frac{2l+1}{2}\right)\mathbf{\Pi}_l^m(\mu)\mathbf{H}_l^m(\mu_0)\mathbf{D}_\beta\mathbf{F}\exp(-\tau/\mu_0), \tag{37}$$

with

$$\mathbf{H}_l^m(\mu_0)=\mathbf{G}_l^m(\mu_0)\gamma-\mathbf{\Pi}_l^m(\mu_0) \tag{38}$$

and

$$\gamma=[\mathbf{G}_{M+1}^m(\mu_0)]^{-1}\mathbf{\Pi}_{M+1}^m(\mu_0), \tag{39}$$

is our particular solution.

We note from equations (20) and (21) that only the $m=0$ case with $\beta=1$ requires us to consider inhomogeneous boundary conditions. For this case, then, we write

$$\mathbf{I}_1^0(\tau,\mu)=\boldsymbol{\Phi}(\tau,\mu)+\alpha\boldsymbol{\Psi}(\tau,\mu) \tag{40}$$

where $\boldsymbol{\Phi}(\tau,\mu)$ and $\boldsymbol{\Psi}(\tau,\mu)$ are defined by

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{\Phi}(\tau,\mu)+\boldsymbol{\Phi}(\tau,\mu)=\frac{\omega}{2}\int_{-1}^{1}\mathbf{K}^0(\mu'\to\mu)\boldsymbol{\Phi}(\tau,\mu')\,\mathrm{d}\mu'+\mathbf{S}_1^0(\tau,\mu) \tag{41a}$$

with

$$\boldsymbol{\Phi}(0,\mu)=\mathbf{0} \tag{41b}$$

and

$$\boldsymbol{\Phi}(\tau_0,-\mu)=\mathbf{0}, \tag{41c}$$

for $\mu\in[0,1]$, and

$$\mu\frac{\partial}{\partial\tau}\boldsymbol{\Psi}(\tau,\mu)+\boldsymbol{\Psi}(\tau,\mu)=\frac{\omega}{2}\int_{-1}^{1}\mathbf{K}^0(\mu'\to\mu)\boldsymbol{\Psi}(\tau,\mu')\,\mathrm{d}\mu' \tag{42a}$$

with

$$\boldsymbol{\Psi}(0,\mu)=\mathbf{0} \tag{42b}$$

and

$$\boldsymbol{\Psi}(\tau_0,-\mu)=\mathbf{E}=\begin{vmatrix}1\\0\\0\\0\end{vmatrix} \tag{42c}$$

for $\mu\in[0,1]$. Considering equations (17), (20) and (21), we find that equation (40) yields the desired result if

$$\alpha=\left[1-2\lambda_0\int_0^1\mu\psi_1(\tau_0,\mu)\,\mathrm{d}\mu\right]^{-1}\left[\lambda_0\mu_0F_I\exp(-\tau_0/\mu_0)+2\lambda_0\int_0^1\mu\phi_1(\tau_0,\mu)\,\mathrm{d}\mu\right] \tag{43}$$

where $\phi_1(\tau,\mu)$ and $\psi_1(\tau,\mu)$ denote, respectively, the first components in $\boldsymbol{\Phi}(\tau,\mu)$ and $\boldsymbol{\Psi}(\tau,\mu)$.

Now following the discussion of our generalized Mark boundary conditions given in the Appendix, we let $\{\mu_\alpha\}$ denote the J positive zeros of det $\mathbf{\Pi}^m_{M+1}(\mu)$ and $\mathbf{N}(\mu_\alpha)$ denote an appropriate null vector of $\mathbf{\Pi}^m_{M+1}(\mu_\alpha)$, i.e.

$$\mathbf{\Pi}^m_{M+1}(\mu_\alpha)\mathbf{N}(\mu_\alpha) = \mathbf{0}, \tag{44}$$

and constrain our approximate solution to satisfy, for $\beta = 2$ with $m = 0$ and for $\beta = 1$ and 2 with $m \geqslant 1$,

$$\mathbf{N}^T(\mu_\alpha)\mathbf{I}^m_\beta(0, \mu_\alpha) = 0 \tag{45a}$$

and

$$\mathbf{N}^T(\mu_\alpha)\mathbf{D}\mathbf{I}^m_\beta(\tau_0, -\mu_\alpha) = 0 \tag{45b}$$

for $\alpha = 1, 2, \ldots, J$. Using equations (34) and (37) and substituting equation (36) into equations (45), we find the system of linear algebraic equations we must solve, in general, to find the constants $\{A_j\}$ and $\{B_j\}$ required to complete our solution, viz.

$$\mathbf{N}^T(\mu_\alpha)\sum_{l=m}^{M}(2l+1)\mathbf{\Pi}^m_l(\mu_\alpha)\sum_{j=1}^{J}[A_j + (-1)^{l-m}\mathbf{D}B_j\exp(-\tau_0/\xi_j)]\mathbf{T}^m_l(\xi_j) = R_{1,\alpha} \tag{46a}$$

and

$$\mathbf{N}^T(\mu_\alpha)\sum_{l=m}^{M}(2l+1)\mathbf{\Pi}^m_l(\mu_\alpha)\sum_{j=1}^{J}[B_j + (-1)^{l-m}\mathbf{D}A_j\exp(-\tau_0/\xi_j)]\mathbf{T}^m_l(\xi_j) = R_{2,\alpha} \tag{46b}$$

where

$$R_{1,\alpha} = -\tfrac{1}{2}\mathbf{N}^T(\mu_\alpha)\sum_{l=m}^{M}(2l+1)\mathbf{\Pi}^m_l(\mu_\alpha)\mathbf{H}^m_l(\mu_0)\mathbf{D}_\beta\mathbf{F} \tag{47a}$$

and

$$R_{2,\alpha} = -\tfrac{1}{2}\mathbf{N}^T(\mu_\alpha)\sum_{l=m}^{M}(2l+1)(-1)^{l-m}\mathbf{\Pi}^m_l(\mu_\alpha)\mathbf{D}\mathbf{H}^m_l(\mu_0)\mathbf{D}_\beta\mathbf{F}\exp(-\tau_0/\mu_0). \tag{47b}$$

Equations (46) and (47) are valid for $m = 0$ with $\beta = 2$ and all $m \geqslant 1$ with $\beta = 1$ and 2. The $\boldsymbol{\Phi}$ problem defined by equations (41) is also included in equations (46) and (47), if we put $m = 0$ and $\beta = 1$; however to solve the $\boldsymbol{\Psi}$ problem defined by equations (42) we must use equations (46) with

$$R_{1,\alpha} = 0 \tag{48a}$$

and

$$R_{2,\alpha} = 2\mathbf{N}^T(\mu_\alpha)\mathbf{E}. \tag{48b}$$

As in Ref. [1] we consider that we can solve equations (46) to find the constants $\{A_j\}$ and $\{B_j\}$. Finally we make an iteration by substituting equations (36), with equations (34) and (37), into the right-hand side of equation (17), and we then solve the resulting equation to find our final result for $\mathbf{I}^m_\beta(\tau, \mu)$, viz. for $m = 0$ with $\beta = 2$ and all $m \geqslant 1$ with $\beta = 1$ and 2,

$$\mathbf{I}^m_\beta(\tau, -\mu) = \frac{\omega}{2}\mathbf{D}[\mu_0 S(\tau_0 - \tau : \mu, \mu_0)\boldsymbol{\Xi}^m_\beta(-\mu)\exp(-\tau/\mu_0) + \boldsymbol{\Upsilon}^m_\beta(\tau, -\mu)] \tag{49a}$$

and

$$\mathbf{I}^m_\beta(\tau, \mu) = \frac{\omega}{2}[\mu_0 C(\tau : \mu, \mu_0)\boldsymbol{\Xi}^m_\beta(\mu) + \boldsymbol{\Upsilon}^m_\beta(\tau, \mu)], \tag{49b}$$

for $\mu \in [0, 1]$, where

$$S(a : x, y) = \frac{1 - \exp(-a/x)\exp(-a/y)}{x + y} \tag{50a}$$

and

$$C(a : x, y) = \frac{\exp(-a/x) - \exp(-a/y)}{x - y}. \tag{50b}$$

In addition,

$$\boldsymbol{\Xi}_{\beta}^{m}(-\mu)=\frac{1}{2}\left[\sum_{l=m}^{M}(-1)^{l-m}\boldsymbol{\Pi}_{l}^{m}(\mu)\mathbf{D}\mathbf{B}_{l}\mathbf{G}_{l}^{m}(\mu_{0})\gamma+\sum_{l=M+1}^{L}(-1)^{l-m}\boldsymbol{\Pi}_{l}^{m}(\mu)\mathbf{D}\mathbf{B}_{l}\boldsymbol{\Pi}_{l}^{m}(\mu_{0})\right]\mathbf{D}_{\beta}\mathbf{F}, \tag{51a}$$

$$\boldsymbol{\Xi}_{\beta}^{m}(\mu)=\frac{1}{2}\left[\sum_{l=m}^{M}\boldsymbol{\Pi}_{l}^{m}(\mu)\mathbf{B}_{l}\mathbf{G}_{l}^{m}(\mu_{0})\gamma+\sum_{l=M+1}^{L}\boldsymbol{\Pi}_{l}^{m}(\mu)\mathbf{B}_{l}\boldsymbol{\Pi}_{l}^{m}(\mu_{0})\right]\mathbf{D}_{\beta}\mathbf{F}, \tag{51b}$$

$$\boldsymbol{\Upsilon}_{\beta}^{m}(\tau,-\mu)=\sum_{l=m}^{M}\boldsymbol{\Pi}_{l}^{m}(\mu)\mathbf{B}_{l}\sum_{j=1}^{J}\xi_{j}[(-1)^{l-m}\mathbf{D}A_{j}\exp(-\tau/\xi_{j})S(\tau_{0}-\tau:\mu,\xi_{j})+B_{j}C(\tau_{0}-\tau:\mu,\xi_{j})]\mathbf{T}_{l}^{m}(\xi_{j}) \tag{52a}$$

and

$$\boldsymbol{\Upsilon}_{\beta}^{m}(\tau,\mu)=\sum_{l=m}^{M}\boldsymbol{\Pi}_{l}^{m}(\mu)\mathbf{B}_{l}\sum_{j=1}^{J}\xi_{j}\{A_{j}C(\tau:\mu,\xi_{j})+(-1)^{l-m}\mathbf{D}B_{j}\exp[-(\tau_{0}-\tau)/\xi_{j}]S(\tau:\mu,\xi_{j})\}\mathbf{T}_{l}^{m}(\xi_{j}). \tag{52b}$$

Again the $\boldsymbol{\Phi}$ problem defined by equations (41) is also included in equations (49), if we put $m = 0$ and $\beta = 1$; for the $\boldsymbol{\Psi}$ problem defined by equations (42) we find

$$\boldsymbol{\Psi}(\tau,-\mu)=\mathbf{E}\exp[-(\tau_{0}-\tau)/\mu]+\frac{\omega}{2}\mathbf{D}\boldsymbol{\Upsilon}_{1}^{0}(\tau,-\mu) \tag{53a}$$

and

$$\boldsymbol{\Psi}(\tau,\mu)=\frac{\omega}{2}\boldsymbol{\Upsilon}_{1}^{0}(\tau,\mu) \tag{53b}$$

for $\mu \in [0, 1]$.

To conclude this section we note that equations (40), (49) and (53) can now be used in equation (16) to yield the desired final result.

COMPUTATIONAL METHODS

To begin our calculation we must find the eigenvalues $\{\xi_j\}$ defined as the J zeros in the right-half plane of det $\mathbf{G}_{M+1}^{m}(\xi)$. Multiplying equation (23), for $l = m, m+1, \ldots, M$, by $\mathbf{M}(\xi)$ and using the truncation condition given by equation (25), we obtain the system of equations

$$\mathbf{B}_{m}^{m}\mathbf{T}_{m}^{m}(\xi)+\mathbf{C}_{m}^{m}\mathbf{T}_{m+1}^{m}(\xi)=\xi\mathbf{T}_{m}^{m}(\xi) \tag{54a}$$

$$\mathbf{A}_{l}^{m}\mathbf{T}_{l-1}^{m}(\xi)+\mathbf{B}_{l}^{m}\mathbf{T}_{l}^{m}(\xi)+\mathbf{C}_{l}^{m}\mathbf{T}_{l+1}^{m}(\xi)=\xi\mathbf{T}_{l}^{m}(\xi), \tag{54b}$$

for $l = m+1, m+2, \ldots, M-1$, and

$$\mathbf{A}_{M}^{m}\mathbf{T}_{M-1}^{m}(\xi)+\mathbf{B}_{M}^{m}\mathbf{T}_{M}^{m}(\xi)=\xi\mathbf{T}_{M}^{m}(\xi) \tag{54c}$$

where, in general,

$$\mathbf{T}_{l}^{m}(\xi)=\mathbf{G}_{l}^{m}(\xi)\mathbf{M}(\xi). \tag{55}$$

We clearly can write equations (54) as the eigenvalue problem

$$\mathbf{W}\mathbf{T}(\xi)=\xi\mathbf{T}(\xi) \tag{56}$$

where $\mathbf{T}(\xi)$ is a vector with elements $\mathbf{T}_{l}^{m}(\xi)$, $l = m, m+1, \ldots, M$, and

$$\mathbf{W}=\begin{vmatrix} \mathbf{B}_{m}^{m} & \mathbf{C}_{m}^{m} & & \\ \mathbf{A}_{m+1}^{m} & \mathbf{B}_{m+1}^{m} & \ddots & \\ & \ddots & \mathbf{B}_{M-1}^{m} & \mathbf{C}_{M-1}^{m} \\ & & \mathbf{A}_{M}^{m} & \mathbf{B}_{M}^{m} \end{vmatrix}. \tag{57}$$

Thus it is clear that the $4(N+1)$ eigenvalues of $\mathbf{W}$ include the $2J$ eigenvalues we seek $\{\pm\xi_j\}$. Rather than compute the eigenvalues of $\mathbf{W}$, we prefer to develop an eigenvalue problem of smaller size for the set $\{\xi_j^2\}$. To this end we note that we can write

$$\mathbf{A}_l^m = \begin{vmatrix} {}^1\mathbf{A}_l^m & \mathbf{0} \\ \mathbf{0} & {}^2\mathbf{A}_l^m \end{vmatrix}, \tag{58a}$$

$$\mathbf{B}_l^m = \begin{vmatrix} \mathbf{0} & {}^1\mathbf{B}_l^m \\ {}^2\mathbf{B}_l^m & \mathbf{0} \end{vmatrix} \tag{58b}$$

and

$$\mathbf{C}_l^m = \begin{vmatrix} {}^1\mathbf{C}_l^m & \mathbf{0} \\ \mathbf{0} & {}^2\mathbf{C}_l^m \end{vmatrix} \tag{58c}$$

where the partition elements are 2×2 matrices. If we now carry out a 2×2 row and column shuffle (by writing down the 2×2 rows and columns in the order $1, 4, 5, 8, 9, 12 \ldots$ and then $2, 3, 6, 7, 10, 11 \ldots$) we can write equation (56) in the equivalent form

$$\begin{vmatrix} \mathbf{0} & \mathbf{U} \\ \mathbf{L} & \mathbf{0} \end{vmatrix} \mathbf{T}_*(\xi) = \xi \mathbf{T}_*(\xi) \tag{59}$$

where $\mathbf{T}_*(\xi)$ is a rearrangement of $\mathbf{T}(\xi)$,

$$\mathbf{U} = \begin{vmatrix} {}^1\mathbf{B}_m^m & {}^1\mathbf{C}_m^m & & & & & \\ {}^2\mathbf{A}_{m+1}^m & {}^2\mathbf{B}_{m+1}^m & {}^2\mathbf{C}_{m+1}^m & \cdots & & & \\ & {}^1\mathbf{A}_{m+2}^m & {}^1\mathbf{B}_{m+2}^m & \cdots & {}^1\mathbf{C}_{M-3}^m & & \\ & & {}^2\mathbf{A}_{m+3}^m & \cdots & {}^2\mathbf{B}_{M-2}^m & {}^2\mathbf{C}_{M-2}^m & \\ & & & \cdots & {}^1\mathbf{A}_{M-1}^m & {}^1\mathbf{B}_{M-1}^m & {}^1\mathbf{C}_{M-1}^m \\ & & & & & {}^2\mathbf{A}_M^m & {}^2\mathbf{B}_M^m \end{vmatrix} \tag{60}$$

and

$$\mathbf{L} = \begin{vmatrix} {}^2\mathbf{B}_m^m & {}^2\mathbf{C}_m^m & & & & & \\ {}^1\mathbf{A}_{m+1}^m & {}^1\mathbf{B}_{m+1}^m & {}^1\mathbf{C}_{m+1}^m & \cdots & & & \\ & {}^2\mathbf{A}_{m+2}^m & {}^2\mathbf{B}_{m+2}^m & \cdots & {}^2\mathbf{C}_{M-3}^m & & \\ & & {}^1\mathbf{A}_{m+3}^m & \cdots & {}^1\mathbf{B}_{M-2}^m & {}^1\mathbf{C}_{M-2}^m & \\ & & & \cdots & {}^2\mathbf{A}_{M-1}^m & {}^2\mathbf{B}_{M-1}^m & {}^2\mathbf{C}_{M-1}^m \\ & & & & & {}^1\mathbf{A}_M^m & {}^1\mathbf{B}_M^m \end{vmatrix}. \tag{61}$$

It is clear from equation (59) that the squares ξ_j^2, $j = 1, 2, \ldots, J$, of the eigenvalues we seek will be eigenvalues of

$$\mathbf{M}_1 = \mathbf{UL} \tag{62a}$$

and

$$\mathbf{M}_2 = \mathbf{LU}. \tag{62b}$$

It follows that for $m \geqslant 2$ the $2(N+1)$ eigenvalues of $\mathbf{M}_1$ or $\mathbf{M}_2$ provide the squares of the $J = 2(N+1)$ eigenvalues we seek. For $m = 1$ we find that $\lambda = 1$ is an eigenvalue of $\mathbf{M}_1$ and $\mathbf{M}_2$, but we also find, with the normalization we used, that $\det \mathbf{G}_{M+1}^1(\pm 1) \neq 0$; it follows that the remaining $2N+1$ eigenvalues of $\mathbf{M}_1$ or $\mathbf{M}_2$ provide the squares of the $J = 2N+1$ eigenvalues we seek. The case $m = 0$ is also special in that $\lambda = 0$ is an eigenvalue (with multiplicity two) of $\mathbf{M}_1$ and $\mathbf{M}_2$, but $\det \mathbf{G}_{M+1}^0(0) \neq 0$. Thus the non zero eigenvalues of $\mathbf{M}_1$ or $\mathbf{M}_2$ provide the squares of the $J = 2N$ required eigenvalues for the case $m = 0$. Of course, since $\mathbf{M}_1$ and $\mathbf{M}_2$ are

$2(N+1) \times 2(N+1)$ matrices, in contrast to $\mathbf{W}$ which is $4(N+1) \times 4(N+1)$, the computational work required to find the desired eigenvalues has been reduced dramatically. Although $\mathbf{M}_1$ and $\mathbf{M}_2$ are banded matrices (and sparse for large N) we have not made use of this structure here and have simply used the driver program **RG** in the EISPACK collection[9,10] to compute the eigenvalues $\{\xi_j^2\}$. We note, however, that Cullum[11] has used a Lanczos procedure (works within the bands[12]) to obtain good results for several of our eigenvalue problems.

Considering that the eigenvalues $\{\xi_j\}$ are now available, we proceed to compute the vectors $\mathbf{T}_l^m(\xi_j)$ required in equations (46) and (52). For $|\xi_j| \leqslant 1$ we find that the matrices $\mathbf{G}_l^m(\xi_j)$ can be computed by forward recursion without significant loss of accuracy. For $|\xi_j| \leqslant 1$ we therefore use the starting values given by equations (30) and the recursion formula given by equation (23) to compute $\mathbf{G}_l^m(\xi_j)$ for $l = m, m+1, \ldots, M+1$. Considering first the case of real $\xi_j \leqslant 1$, we turn to the LINPACK collection[13] and use the subroutine **DSVDC** to compute the singular value decomposition (SVD) of $\mathbf{G}_{M+1}^m(\xi_j)$. When the rank of $\mathbf{G}_{M+1}^m(\xi_j)$ is (numerically) clearly three, we use the appropriate singular vector from the SVD as $\mathbf{M}(\xi_j)$ and subsequently find the required $\mathbf{T}_l^m(\xi_j)$ from equation (35). For ξ_j complex and $|\xi_j| \leqslant 1$, we compute $\mathbf{G}_{M+1}^m(\xi)$ by forward recursion, use the SVD subroutine to solve

$$\begin{bmatrix} \mathrm{Re}\{\mathbf{G}_{M+1}^m(\xi_j)\} & -\mathrm{Im}\{\mathbf{G}_{M+1}^m(\xi_j)\} \\ \mathrm{Im}\{\mathbf{G}_{M+1}^m(\xi_j)\} & \mathrm{Re}\{\mathbf{G}_{M+1}^m(\xi_j)\} \end{bmatrix} \begin{bmatrix} \mathrm{Re}\{\mathbf{M}(\xi_j)\} \\ \mathrm{Im}\{\mathbf{M}(\xi_j)\} \end{bmatrix} = \mathbf{0} \tag{63}$$

and find the $\mathbf{T}_l^m(\xi_j)$ from equation (35). As an alternative to using equation (63), we have also used the SVD subroutine **ZSVDC** from the LINPACK collection[13] to find the required null vector of $\mathbf{G}_{M+1}^m(\xi_j)$ for complex ξ_j.

For $|\xi_j| > 1$ we let

$$\mathbf{T}_{l+1}^m(\xi_j) = \mathbf{R}_l^m(\xi_j)\mathbf{T}_l^m(\xi_j) \tag{64}$$

for $l = m, m+1, \ldots, M$, and define

$$\mathbf{R}_M^m(\xi_j) \equiv \mathbf{0}. \tag{65}$$

We can now multiply equation (23) by $\mathbf{M}(\xi_j)$ and use equation (64) to obtain

$$\mathbf{R}_{l-1}^m(\xi_j) = [\xi_j \mathbf{h}_l + \mathbf{V}_l^m - \mathbf{U}_{l+1}^m \mathbf{R}_l^m(\xi_j)]^{-1}\mathbf{U}_l^m \tag{66}$$

for $l = M, M-1, \ldots, m+1$. It is apparent that equations (65) and (66) define the matrices $\mathbf{R}_l^m(\xi_j)$ for $l = m, m+1, \ldots, M$, and so we require only the starting value $\mathbf{T}_m^m(\xi_j)$ to use equation (64). Since $\mathbf{U}_m^m \equiv \mathbf{0}$, we see from equations (23), (35) and (64) that

$$\mathbf{U}_{m+1}^m \mathbf{T}_{m+1}^m(\xi_j) = (\xi_j \mathbf{h}_m + \mathbf{V}_m^m)\mathbf{T}_m^m(\xi_j) \tag{67}$$

and

$$\mathbf{T}_{m+1}^m(\xi_j) = \mathbf{R}_m^m(\xi_j)\mathbf{T}_m^m(\xi_j). \tag{68}$$

Thus we let

$$\mathbf{W}_m^m(\xi_j) = \mathbf{U}_{m+1}^m \mathbf{R}_m^m(\xi_j) - \xi_j \mathbf{h}_m - \mathbf{V}_m^m \tag{69}$$

and eliminate between equations (67) and (68) to find

$$\mathbf{W}_m^m(\xi_j)\mathbf{T}_m^m(\xi_j) = \mathbf{0}. \tag{70}$$

For real values of ξ_j we now use **DSVDC** to compute an SVD of $\mathbf{W}_m^m(\xi_j)$, and when the rank is (numerically) clearly three we use the appropriate singular vector as $\mathbf{T}_m^m(\xi_j)$. For complex values of ξ_j, we use **DSVDC** to solve

$$\begin{bmatrix} \mathrm{Re}\{\mathbf{W}_m^m(\xi_j)\} & -\mathrm{Im}\{\mathbf{W}_m^m(\xi_j)\} \\ \mathrm{Im}\{\mathbf{W}_m^m(\xi_j)\} & \mathrm{Re}\{\mathbf{W}_m^m(\xi_j)\} \end{bmatrix} \begin{bmatrix} \mathrm{Re}\{\mathbf{T}_m^m(\xi_j)\} \\ \mathrm{Im}\{\mathbf{T}_m^m(\xi_j)\} \end{bmatrix} = \mathbf{0}, \tag{71}$$

or as an alternative procedure we can use the subroutine **ZSVDC** to find a null vector of $\mathbf{W}_m^m(\xi_j)$. Finally the remaining $\mathbf{T}_l^m(\xi_j)$ can be found from equation (64).

In the foregoing discussion we have considered all of those cases for which the rank of $\mathbf{G}_{M+1}^m(\xi_j)$, $j = 1, 2, \ldots, J$, is three; however, there are cases when $\mathbf{G}_{M+1}^m(\xi_j)$ can, from a numerical point of view, have only rank two and then something special must be done. For any m if ω is sufficiently small, or for any ω if m is sufficiently close to L, some of the eigenvalues can nearly coalesce in pairs; it follows that the rank of $\mathbf{G}^m_{M+1}(\xi_j)$ for these eigenvalues is, in regard to a machine computation, only two. In our work we have considered two eigenvalues, say ξ_j and $\xi_{j'}$, to be identical when they are sufficiently close to one another, and then we have used two linearly independent vectors from a singular value decomposition of $\mathbf{G}^m_{M+1}(\xi_j)$ as appropriate null vectors $\mathbf{M}^{(1)}(\xi_j)$ and $\mathbf{M}^{(2)}(\xi_j)$.

Having found that we can have complex eigenvalues ξ_j, we now go back and rewrite equation (34) in terms of real quantities. We let J_r = the number of real eigenvalues and J_c = the number of complex pairs of eigenvalues so that $J = J_r + 2J_c$. We next define

$$\mathbf{X}^m_l(\tau, \xi) = \mathrm{Re}[\exp(-\tau/\xi)]\mathrm{Re}[\mathbf{T}^m_l(\xi)] - \mathrm{Im}[\exp(-\tau/\xi)]\mathrm{Im}[\mathbf{T}^m_l(\xi)] \tag{72a}$$

and

$$\mathbf{Y}^m_l(\tau, \xi) = \mathrm{Im}[\exp(-\tau/\xi)]\mathrm{Re}[\mathbf{T}^m_l(\xi)] + \mathrm{Re}[\exp(-\tau/\xi)]\mathrm{Im}[\mathbf{T}^m_l(\xi)] \tag{72b}$$

and write equation (34) as

$$\mathbf{I}^m_h(\tau, \mu) = \sum_{l=m}^{M} \left(\frac{2l+1}{2}\right)\mathbf{\Pi}^m_l(\mu)[\boldsymbol{\theta}^m_{r,l}(\tau) + \boldsymbol{\theta}^m_{c,l}(\tau)] \tag{73}$$

where

$$\boldsymbol{\theta}^m_{r,l}(\tau) = \sum_{j=1}^{J_r} \{A_j \exp(-\tau/\xi_j) + (-1)^{l-m}\mathbf{D}B_j \exp[-(\tau_0-\tau)/\xi_j]\}\mathbf{T}^m_l(\xi_j) \tag{74a}$$

and

$$\boldsymbol{\theta}^m_{c,l}(\tau) = \sum_{j=1}^{J_c} \{A^{(1)}_j\mathbf{X}^m_l(\tau, \xi_j) + A^{(2)}_j\mathbf{Y}^m_l(\tau, \xi_j) + (-1)^{l-m}\mathbf{D} \times [B^{(1)}_j\mathbf{X}^m_l(\tau_0-\tau, \xi_j) + B^{(2)}_j\mathbf{Y}^m_l(\tau_0-\tau, \xi_j)]\}. \tag{74b}$$

In writing equation (73) we have regrouped some constants so that now our unknowns are the real constants A_j and B_j, for $j = 1, 2, \ldots, J_r$, and $A^{(1)}_j$, $A^{(2)}_j$, $B^{(1)}_j$ and $B^{(2)}_j$, for $j = 1, 2, \ldots, J_c$. To find these constants we must solve equations (46) rewritten as

$$\mathbf{N}^T(\mu_\alpha) \sum_{l=m}^{M} (2l+1)\mathbf{\Pi}^m_l(\mu_\alpha)[\boldsymbol{\theta}^m_{r,l}(0) + \boldsymbol{\theta}^m_{c,l}(0)] = R_{1,\alpha} \tag{75a}$$

and

$$\mathbf{N}^T(\mu_\alpha) \sum_{l=m}^{M} (2l+1)\mathbf{\Pi}^m_l(\mu_\alpha)(-1)^{l-m}\mathbf{D}[\boldsymbol{\theta}^m_{r,l}(\tau_0) + \boldsymbol{\theta}^m_{c,l}(\tau_0)] = R_{2,\alpha}. \tag{75b}$$

Considering that we have found the desired constants from equations (75), we must rewrite equations (52) so that $\mathbf{I}^m_\beta(\tau, \mu)$ can be evaluated from equations (49). We find

$$\boldsymbol{\Upsilon}^m_\beta(\tau, -\mu) = \sum_{l=m}^{M} \mathbf{\Pi}^m_l(\mu)\mathbf{B}_l[\boldsymbol{\Gamma}^m_{r,l}(\tau, -\mu) + \boldsymbol{\Gamma}^m_{c,l}(\tau, -\mu)] \tag{76a}$$

and

$$\boldsymbol{\Upsilon}^m_\beta(\tau, \mu) = \sum_{l=m}^{M} \mathbf{\Pi}^m_l(\mu)\mathbf{B}_l[\boldsymbol{\Gamma}^m_{r,l}(\tau, \mu) + \boldsymbol{\Gamma}^m_{c,l}(\tau, \mu)] \tag{76b}$$

where

$$\boldsymbol{\Gamma}^m_{r,l}(\tau, -\mu) = \sum_{j=1}^{J_r} \xi_j[(-1)^{l-m}\mathbf{D}A_j S(\tau_0-\tau:\mu, \xi_j)\exp(-\tau/\xi_j) + B_j C(\tau_0-\tau:\mu, \xi_j)]\mathbf{T}^m_l(\xi_j), \tag{77a}$$

$$\boldsymbol{\Gamma}^m_{r,l}(\tau, \mu) = \sum_{j=1}^{J_r} \xi_j\{A_j C(\tau:\mu, \xi_j) + (-1)^{l-m}\mathbf{D}B_j S(\tau:\mu, \xi_j)\exp[-(\tau_0-\tau)/\xi_j]\}\mathbf{T}^m_l(\xi_j), \tag{77b}$$

$$\boldsymbol{\Gamma}^m_{c,l}(\tau, -\mu) = \sum_{j=1}^{J_c} \{(-1)^{l-m}\mathbf{D}[A_j^{(1)}\mathbf{Z}_l^m(\tau_0-\tau:\mu,\xi_j) + A_j^{(2)}\mathbf{W}_l^m(\tau_0-\tau:\mu,\xi_j)] + [B_j^{(1)}\mathbf{U}_l^m(\tau_0-\tau:\mu,\xi_j) + B_j^{(2)}\mathbf{V}_l^m(\tau_0-\tau:\mu,\xi_j)]\} \quad (78a)$$

and

$$\boldsymbol{\Gamma}^m_{c,l}(\tau, \mu) = \sum_{j=1}^{J_c} \{A_j^{(1)}\mathbf{U}_l^m(\tau:\mu,\xi_j) + A_j^{(2)}\mathbf{V}_l^m(\tau:\mu,\xi_j) + (-1)^{l-m}\mathbf{D}[B_j^{(1)}\mathbf{Z}_l^m(\tau:\mu,\xi_j) + B_j^{(2)}\mathbf{W}_l^m(\tau:\mu,\xi_j)]\} \quad (78b)$$

where

$$\mathbf{Z}_l^m(\tau:\mu,\xi_j) = \mathrm{Re}[\xi_j \exp[-(\tau_0-\tau)/\xi_j]S(\tau:\mu,\xi_j)]\mathrm{Re}[\mathbf{T}_l^m(\xi_j)] - \mathrm{Im}[\xi_j \exp[-(\tau_0-\tau)/\xi_j]S(\tau:\mu,\xi_j)]\mathrm{Im}[\mathbf{T}_l^m(\xi_j)] \quad (79a)$$

$$\mathbf{W}_l^m(\tau:\mu,\xi_j) = \mathrm{Im}[\xi_j \exp[-(\tau_0-\tau)/\xi_j]S(\tau:\mu,\xi_j)]\mathrm{Re}[\mathbf{T}_l^m(\xi_j)] + \mathrm{Re}[\xi_j \exp[-(\tau_0-\tau)/\xi_j]S(\tau:\mu,\xi_j)]\mathrm{Im}[\mathbf{T}_l^m(\xi_j)] \quad (79b)$$

$$\mathbf{U}_l^m(\tau:\mu,\xi_j) = \mathrm{Re}[\xi_j C(\tau:\mu,\xi_j)]\mathrm{Re}[\mathbf{T}_l^m(\xi_j)] - \mathrm{Im}[\xi_j C(\tau:\mu,\xi_j)]\mathrm{Im}[\mathbf{T}_l^m(\xi_j)] \quad (79c)$$

$$\mathbf{V}_l^m(\tau:\mu,\xi_j) = \mathrm{Im}[\xi_j C(\tau:\mu,\xi_j)]\mathrm{Re}[\mathbf{T}_l^m(\xi_j)] + \mathrm{Re}[\xi_j C(\tau:\mu,\xi_j)]\mathrm{Im}[\mathbf{T}_l^m(\xi_j)]. \quad (79d)$$

NUMERICAL RESULTS

In order to demonstrate that our generalized spherical harmonics solution and the numerical techniques developed do, in fact, define a computationally viable method, we now wish to report some numerical results for two test cases. The "Greek constants" for these two model problems were computed by de Rooij and van der Stap,[14] were used in Refs [1, 15, 16] and are tabulated in Ref. [15]; we refer to the two test problems as the $L = 13$ model and the $L = 60$ model. We note that these "Greek constants" were calculated for the case $\omega = 1$; however, to allow some absorption and to avoid recomputing these constants from the basic Mie theory, we use the constants from Ref. [15] with $\omega = 0.99$.

As in Ref. [1], we again have followed a suggestion of de Haan and Hovenier[17] and have considered the following four cases that clearly will allow us to construct by superposition the solution for any flux vector **F**:

$$\mathbf{F} = \begin{vmatrix}1\\0\\0\\0\end{vmatrix}, \begin{vmatrix}1\\1\\0\\0\end{vmatrix}, \begin{vmatrix}1\\0\\1\\0\end{vmatrix} \text{ and } \begin{vmatrix}1\\0\\0\\1\end{vmatrix}. \quad (80)$$

To report some specific results we take $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\mathbf{F} = |1\,0\,0\,0|^T$ and report in Tables 1–8, for the $L = 13$ model, and Tables 9–16, for the $L = 60$ model, our results for the diffuse field

$$\mathbf{I}*(\tau,\mu,\varphi) = \mathbf{I}(\tau,\mu,\varphi) - \pi\delta(\mu-\mu_0)\delta(\varphi-\varphi_0)\mathbf{F}\exp(-\tau/\mu). \quad (81)$$

We note that the results listed in Tables 1–16 were obtained with a maximum value of $N = 449$.

Though we have no definitive evidence about the accuracy of our reported results, we do believe that the entries in Tables 1–16 are correct, typically, but not always, to within ± 1 in the last digits given. To establish this confidence in our results we have observed what appears to be convergence as N increases and we have found excellent agreement with a solution communicated by de Haan.[18] The reference results made available by de Haan were obtained from his own variants of the doubling and adding methods, and so the results clearly provide a basis for comparison that shares no common numerical technique with our own solution.

Table 1. The diffuse intensity $I_*(\tau, \mu, \varphi)$ for the $L = 13$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = 0$

μ	τ = 0	τ = τ₀/20	τ = τ₀/10	τ = τ₀/5	τ = τ₀/2	τ = 3τ₀/4	τ = τ₀
-1.0	5.4956(-2)	4.8734(-2)	4.3346(-2)	3.4749(-2)	1.9591(-2)	1.2820(-2)	8.7469(-3)
-0.9	9.0491(-2)	7.8183(-2)	6.7724(-2)	5.1529(-2)	2.5242(-2)	1.4781(-2)	8.7469(-3)
-0.8	1.2560(-1)	1.0751(-1)	9.2214(-2)	6.8726(-2)	3.1506(-2)	1.7132(-2)	8.7469(-3)
-0.7	1.6781(-1)	1.4281(-1)	1.2173(-1)	8.9532(-2)	3.9282(-2)	2.0170(-2)	8.7469(-3)
-0.6	2.1934(-1)	1.8592(-1)	1.5781(-1)	1.1504(-1)	4.9047(-2)	2.4148(-2)	8.7469(-3)
-0.5	2.8294(-1)	2.3915(-1)	2.0236(-1)	1.4661(-1)	6.1458(-2)	2.9467(-2)	8.7469(-3)
-0.4	3.6268(-1)	3.0588(-1)	2.5822(-1)	1.8623(-1)	7.7489(-2)	3.6802(-2)	8.7469(-3)
-0.3	4.6523(-1)	3.9162(-1)	3.2989(-1)	2.3697(-1)	9.8619(-2)	4.7393(-2)	8.7469(-3)
-0.2	6.0287(-1)	5.0647(-1)	4.2552(-1)	3.0408(-1)	1.2693(-1)	6.3636(-2)	8.7469(-3)
-0.1	8.0223(-1)	6.7267(-1)	5.6301(-1)	3.9860(-1)	1.6445(-1)	8.9460(-2)	8.7469(-3)
-0.0	1.1163	9.4905(-1)	7.9285(-1)	5.5431(-1)	2.1668(-1)	1.1988(-1)	8.7469(-3)
0.0		9.4905(-1)	7.9285(-1)	5.5431(-1)	2.1668(-1)	1.1988(-1)	6.9468(-2)
0.1		4.1261(-1)	5.9668(-1)	6.3733(-1)	3.0992(-1)	1.6337(-1)	9.5789(-2)
0.2		2.3293(-1)	3.7720(-1)	4.9365(-1)	3.6418(-1)	2.2056(-1)	1.3351(-1)
0.3		1.5865(-1)	2.6784(-1)	3.8067(-1)	3.5342(-1)	2.4880(-1)	1.6680(-1)
0.4		1.1644(-1)	2.0096(-1)	2.9879(-1)	3.1832(-1)	2.4937(-1)	1.8322(-1)
0.5		8.8096(-2)	1.5420(-1)	2.3610(-1)	2.7609(-1)	2.3363(-1)	1.8446(-1)
0.6		6.7032(-2)	1.1855(-1)	1.8550(-1)	2.3266(-1)	2.0898(-1)	1.7476(-1)
0.7		5.0297(-2)	8.9720(-2)	1.4293(-1)	1.9003(-1)	1.7944(-1)	1.5756(-1)
0.8		3.6351(-2)	6.5371(-2)	1.0592(-1)	1.4871(-1)	1.4707(-1)	1.3506(-1)
0.9		2.4158(-2)	4.3852(-2)	7.2433(-2)	1.0801(-1)	1.1226(-1)	1.0806(-1)
1.0		9.8920(-3)	1.8391(-2)	3.1837(-2)	5.4252(-2)	6.2253(-2)	6.5318(-2)

Table 2. The Stokes parameter $Q(\tau, \mu, \varphi)$ for the $L = 13$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = 0$

μ	τ = 0	τ = τ₀/20	τ = τ₀/10	τ = τ₀/5	τ = τ₀/2	τ = 3τ₀/4	τ = τ₀
-1.0	-2.1609(-2)	-1.7634(-2)	-1.4400(-2)	-9.6652(-3)	-3.0410(-3)	-1.0147(-3)	
-0.9	-3.2581(-2)	-2.6486(-2)	-2.1568(-2)	-1.4427(-2)	-4.5929(-3)	-1.6006(-3)	
-0.8	-3.5048(-2)	-2.8256(-2)	-2.2837(-2)	-1.5073(-2)	-4.7119(-3)	-1.6889(-3)	
-0.7	-3.4950(-2)	-2.7860(-2)	-2.2281(-2)	-1.4420(-2)	-4.3578(-3)	-1.6107(-3)	
-0.6	-3.2768(-2)	-2.5680(-2)	-2.0204(-2)	-1.2662(-2)	-3.5748(-3)	-1.3741(-3)	
-0.5	-2.8664(-2)	-2.1825(-2)	-1.6677(-2)	-9.8179(-3)	-2.3333(-3)	-9.5506(-4)	
-0.4	-2.2754(-2)	-1.6349(-2)	-1.1714(-2)	-5.8540(-3)	-5.5838(-4)	-3.0068(-4)	
-0.3	-1.5241(-2)	-9.3458(-3)	-5.3416(-3)	-7.1879(-4)	1.8756(-3)	6.9319(-4)	
-0.2	-6.6429(-3)	-1.1063(-3)	2.2863(-3)	5.5873(-3)	5.1500(-3)	2.2430(-3)	
-0.1	1.4355(-3)	7.4216(-3)	1.0570(-2)	1.2818(-2)	9.3853(-3)	4.7931(-3)	
-0.0	-8.0429(-4)	1.1852(-2)	1.6909(-2)	1.9913(-2)	1.4287(-2)	8.3382(-3)	
0.0		1.1852(-2)	1.6909(-2)	1.9913(-2)	1.4287(-2)	8.3382(-3)	2.5038(-3)
0.1		6.1279(-3)	1.2308(-2)	1.9981(-2)	1.8922(-2)	1.2106(-2)	6.0100(-3)
0.2		3.7767(-3)	8.0641(-3)	1.4947(-2)	1.9304(-2)	1.4587(-2)	8.8222(-3)
0.3		1.7927(-3)	4.3695(-3)	9.4072(-3)	1.5788(-2)	1.3951(-2)	9.8508(-3)
0.4		-8.7378(-5)	8.6389(-4)	3.7871(-3)	1.0138(-2)	1.0607(-2)	8.5415(-3)
0.5		-1.8736(-3)	-2.4720(-3)	-1.7340(-3)	3.3790(-3)	5.3899(-3)	5.1805(-3)
0.6		-3.5200(-3)	-5.5610(-3)	-6.9655(-3)	-3.8308(-3)	-9.7975(-4)	2.7808(-4)
0.7		-4.9546(-3)	-8.2779(-3)	-1.1681(-2)	-1.0969(-2)	-7.8932(-3)	-5.6324(-3)
0.8		-6.0785(-3)	-1.0450(-2)	-1.5595(-2)	-1.7533(-2)	-1.4787(-2)	-1.2024(-2)
0.9		-6.7285(-3)	-1.1792(-2)	-1.8259(-2)	-2.2898(-2)	-2.1036(-2)	-1.8336(-2)
1.0		-5.9155(-3)	-1.0610(-2)	-1.7182(-2)	-2.4508(-2)	-2.4801(-2)	-2.3440(-2)

Table 3. The intensity $I(\tau, \mu, \varphi)$ for the $L = 13$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi/2$

μ	τ = 0	τ = τ₀/20	τ = τ₀/10	τ = τ₀/5	τ = τ₀/2	τ = 3τ₀/4	τ = τ₀
-1.0	5.4956(-2)	4.8734(-2)	4.3346(-2)	3.4749(-2)	1.9591(-2)	1.2820(-2)	8.7469(-3)
-0.9	6.2210(-2)	5.5302(-2)	4.9251(-2)	3.9489(-2)	2.1907(-2)	1.3797(-2)	8.7469(-3)
-0.8	7.0553(-2)	6.2904(-2)	5.6123(-2)	4.5066(-2)	2.4736(-2)	1.5041(-2)	8.7469(-3)
-0.7	8.0201(-2)	7.1747(-2)	6.4162(-2)	5.1662(-2)	2.8219(-2)	1.6641(-2)	8.7469(-3)
-0.6	9.1434(-2)	8.2105(-2)	7.3630(-2)	5.9518(-2)	3.2552(-2)	1.8734(-2)	8.7469(-3)
-0.5	1.0461(-1)	9.4331(-2)	8.4865(-2)	6.8949(-2)	3.8016(-2)	2.1538(-2)	8.7469(-3)
-0.4	1.2018(-1)	1.0887(-1)	9.8298(-2)	8.0355(-2)	4.5020(-2)	2.5423(-2)	8.7469(-3)
-0.3	1.3868(-1)	1.2626(-1)	1.1444(-1)	9.4196(-2)	5.4123(-2)	3.1065(-2)	8.7469(-3)
-0.2	1.6070(-1)	1.4714(-1)	1.3385(-1)	1.1088(-1)	6.5901(-2)	3.9743(-2)	8.7469(-3)
-0.1	1.8701(-1)	1.7261(-1)	1.5752(-1)	1.3086(-1)	7.9960(-2)	5.3361(-2)	8.7469(-3)
-0.0	2.0939(-1)	2.0376(-1)	1.8814(-1)	1.5687(-1)	9.5106(-2)	6.7331(-2)	8.7469(-3)
0.0		2.0376(-1)	1.8814(-1)	1.5687(-1)	9.5106(-2)	6.7331(-2)	4.4302(-2)
0.1		8.1918(-2)	1.2672(-1)	1.5457(-1)	1.1394(-1)	8.0964(-2)	5.7382(-2)
0.2		4.5911(-2)	7.9064(-2)	1.1581(-1)	1.1852(-1)	9.3157(-2)	6.9646(-2)
0.3		3.1711(-2)	5.6791(-2)	8.9643(-2)	1.1057(-1)	9.6737(-2)	7.8074(-2)
0.4		2.4136(-2)	4.4086(-2)	7.2404(-2)	9.9903(-2)	9.4557(-2)	8.1600(-2)
0.5		1.9436(-2)	3.5897(-2)	6.0374(-2)	8.9636(-2)	8.9813(-2)	8.1711(-2)
0.6		1.6247(-2)	3.0196(-2)	5.1543(-2)	8.0491(-2)	8.4147(-2)	7.9801(-2)
0.7		1.3954(-2)	2.6014(-2)	4.4807(-2)	7.2513(-2)	7.8311(-2)	7.6772(-2)
0.8		1.2239(-2)	2.2835(-2)	3.9520(-2)	6.5576(-2)	7.2638(-2)	7.3160(-2)
0.9		1.0922(-2)	2.0356(-2)	3.5283(-2)	5.9533(-2)	6.7268(-2)	6.9280(-2)
1.0		9.8920(-3)	1.8391(-2)	3.1837(-2)	5.4252(-2)	6.2253(-2)	6.5318(-2)

Table 4. The Stokes parameter $Q(\tau, \mu, \varphi)$ for the $L = 13$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi/2$

μ	τ = 0	τ = τ₀/20	τ = τ₀/10	τ = τ₀/5	τ = τ₀/2	τ = 3τ₀/4	τ = τ₀
-1.0	2.1609(-2)	1.7634(-2)	1.4400(-2)	9.6652(-3)	3.0410(-3)	1.0147(-3)	
-0.9	2.5704(-2)	2.1194(-2)	1.7472(-2)	1.1931(-2)	3.8794(-3)	1.2806(-3)	
-0.8	3.0469(-2)	2.5368(-2)	2.1099(-2)	1.4645(-2)	4.9434(-3)	1.6464(-3)	
-0.7	3.6046(-2)	3.0285(-2)	2.5399(-2)	1.7906(-2)	6.2914(-3)	2.1440(-3)	
-0.6	4.2632(-2)	3.6126(-2)	3.0536(-2)	2.1850(-2)	8.0063(-3)	2.8212(-3)	
-0.5	5.0505(-2)	4.3146(-2)	3.6740(-2)	2.6665(-2)	1.0209(-2)	3.7544(-3)	
-0.4	6.0066(-2)	5.1713(-2)	4.4342(-2)	3.2621(-2)	1.3082(-2)	5.0737(-3)	
-0.3	7.1913(-2)	6.2373(-2)	5.3829(-2)	4.0105(-2)	1.6899(-2)	7.0184(-3)	
-0.2	8.6986(-2)	7.5994(-2)	6.5957(-2)	4.9675(-2)	2.2032(-2)	1.0059(-2)	
-0.1	1.0690(-1)	9.4203(-2)	8.2148(-2)	6.2284(-2)	2.8743(-2)	1.5008(-2)	
-0.0	1.2996(-1)	1.1987(-1)	1.0571(-1)	8.0636(-2)	3.7303(-2)	2.1056(-2)	
0.0		1.1987(-1)	1.0571(-1)	8.0636(-2)	3.7303(-2)	2.1056(-2)	1.0282(-2)
0.1		4.9307(-2)	7.4084(-2)	8.4891(-2)	4.9745(-2)	2.8504(-2)	1.6031(-2)
0.2		2.7634(-2)	4.6336(-2)	6.4373(-2)	5.5147(-2)	3.6466(-2)	2.2525(-2)
0.3		1.9064(-2)	3.3253(-2)	4.9908(-2)	5.2670(-2)	3.9963(-2)	2.7672(-2)
0.4		1.4486(-2)	2.5759(-2)	4.0237(-2)	4.7939(-2)	4.0001(-2)	3.0323(-2)
0.5		1.1644(-2)	2.0919(-2)	3.3431(-2)	4.2981(-2)	3.8295(-2)	3.1037(-2)
0.6		9.7174(-3)	1.7546(-2)	2.8407(-2)	3.8373(-2)	3.5814(-2)	3.0507(-2)
0.7		8.3355(-3)	1.5074(-2)	2.4564(-2)	3.4242(-2)	3.3040(-2)	2.9224(-2)
0.8		7.3068(-3)	1.3202(-2)	2.1546(-2)	3.0584(-2)	3.0210(-2)	2.7501(-2)
0.9		6.5225(-3)	1.1750(-2)	1.9134(-2)	2.7354(-2)	2.7444(-2)	2.5531(-2)
1.0		5.9155(-3)	1.0610(-2)	1.7182(-2)	2.4508(-2)	2.4801(-2)	2.3440(-2)

Table 5. The Stokes parameter $U(\tau, \mu, \varphi)$ for the $L = 13$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi/2$

μ	τ = 0	τ = τ₀/20	τ = τ₀/10	τ = τ₀/5	τ = τ₀/2	τ = 3τ₀/4	τ = τ₀
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	-5.9894(-3)	-5.2850(-3)	-4.6505(-3)	-3.5962(-3)	-1.6040(-3)	-6.4291(-4)	
-0.8	-9.1368(-3)	-8.0894(-3)	-7.1421(-3)	-5.5617(-3)	-2.5418(-3)	-1.0449(-3)	
-0.7	-1.2109(-2)	-1.0756(-2)	-9.5290(-3)	-7.4735(-3)	-3.5047(-3)	-1.4811(-3)	
-0.6	-1.5187(-2)	-1.3536(-2)	-1.2032(-2)	-9.5062(-3)	-4.5840(-3)	-1.9991(-3)	
-0.5	-1.8526(-2)	-1.6568(-2)	-1.4778(-2)	-1.1764(-2)	-5.8510(-3)	-2.6484(-3)	
-0.4	-2.2261(-2)	-1.9973(-2)	-1.7874(-2)	-1.4339(-2)	-7.3879(-3)	-3.5043(-3)	
-0.3	-2.6534(-2)	-2.3880(-2)	-2.1435(-2)	-1.7323(-2)	-9.2969(-3)	-4.6975(-3)	
-0.2	-3.1534(-2)	-2.8447(-2)	-2.5587(-2)	-2.0793(-2)	-1.1668(-2)	-6.4698(-3)	
-0.1	-3.7631(-2)	-3.4014(-2)	-3.0593(-2)	-2.4854(-2)	-1.4382(-2)	-9.1487(-3)	
-0.0	-4.4797(-2)	-4.1432(-2)	-3.7344(-2)	-3.0184(-2)	-1.7218(-2)	-1.1692(-2)	
0.0		-4.1432(-2)	-3.7344(-2)	-3.0184(-2)	-1.7218(-2)	-1.1692(-2)	-7.5588(-3)
0.1		-1.6932(-2)	-2.5633(-2)	-3.0265(-2)	-2.0804(-2)	-1.4106(-2)	-9.6641(-3)
0.2		-9.4017(-3)	-1.5848(-2)	-2.2513(-2)	-2.1619(-2)	-1.6246(-2)	-1.1687(-2)
0.3		-6.3655(-3)	-1.1153(-2)	-1.7070(-2)	-1.9815(-2)	-1.6633(-2)	-1.2933(-2)
0.4		-4.6901(-3)	-8.3776(-3)	-1.3338(-2)	-1.7340(-2)	-1.5780(-2)	-1.3140(-2)
0.5		-3.5969(-3)	-6.4967(-3)	-1.0593(-2)	-1.4832(-2)	-1.4305(-2)	-1.2569(-2)
0.6		-2.7990(-3)	-5.0898(-3)	-8.4285(-3)	-1.2429(-2)	-1.2518(-2)	-1.1469(-2)
0.7		-2.1612(-3)	-3.9462(-3)	-6.6031(-3)	-1.0114(-2)	-1.0531(-2)	-9.9745(-3)
0.8		-1.6021(-3)	-2.9322(-3)	-4.9409(-3)	-7.7863(-3)	-8.3248(-3)	-8.1005(-3)
0.9		-1.0431(-3)	-1.9112(-3)	-3.2352(-3)	-5.2103(-3)	-5.6911(-3)	-5.6631(-3)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 6. The Stokes parameter $V(\tau, \mu, \varphi)$ for the $L = 13$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi/2$

μ	τ = 0	τ = τ₀/20	τ = τ₀/10	τ = τ₀/5	τ = τ₀/2	τ = 3τ₀/4	τ = τ₀
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	-5.6876(-5)	-5.5464(-5)	-5.2440(-5)	-4.5015(-5)	-2.3997(-5)	-1.0204(-5)	
-0.8	-6.8062(-5)	-6.8194(-5)	-6.5829(-5)	-5.8326(-5)	-3.3205(-5)	-1.4958(-5)	
-0.7	-6.7491(-5)	-7.0240(-5)	-6.9703(-5)	-6.4263(-5)	-3.9418(-5)	-1.8829(-5)	
-0.6	-5.8655(-5)	-6.4803(-5)	-6.6941(-5)	-6.5100(-5)	-4.3626(-5)	-2.2178(-5)	
-0.5	-4.2700(-5)	-5.2905(-5)	-5.8443(-5)	-6.1518(-5)	-4.6087(-5)	-2.5118(-5)	
-0.4	-1.9781(-5)	-3.4652(-5)	-4.4285(-5)	-5.3539(-5)	-4.6751(-5)	-2.7692(-5)	
-0.3	1.0762(-5)	-9.3949(-6)	-2.3866(-5)	-4.0648(-5)	-4.5257(-5)	-2.9880(-5)	
-0.2	5.0591(-5)	2.4497(-5)	4.2796(-6)	-2.1653(-5)	-4.0670(-5)	-3.1460(-5)	
-0.1	1.0277(-4)	7.0344(-5)	4.3061(-5)	5.5194(-6)	-3.1043(-5)	-3.0935(-5)	
-0.0	1.5768(-4)	1.3440(-4)	9.9770(-5)	4.5965(-5)	-1.5508(-5)	-2.2047(-5)	
0.0		1.3440(-4)	9.9770(-5)	4.5965(-5)	-1.5508(-5)	-2.2047(-5)	-2.8510(-5)
0.1		5.3973(-5)	7.6646(-5)	7.2327(-5)	8.0388(-6)	-8.5767(-6)	-1.2612(-5)
0.2		2.5592(-5)	4.1709(-5)	5.1509(-5)	2.3313(-5)	6.6404(-6)	1.0869(-6)
0.3		1.3086(-5)	2.2728(-5)	3.1653(-5)	2.3106(-5)	1.4392(-5)	1.1751(-5)
0.4		5.6464(-6)	1.0363(-5)	1.5722(-5)	1.5642(-5)	1.4440(-5)	1.6710(-5)
0.5		6.4985(-7)	1.5776(-6)	3.0183(-6)	5.5206(-6)	9.6015(-6)	1.6515(-5)
0.6		-2.7951(-6)	-4.7427(-6)	-6.8685(-6)	-4.8601(-6)	2.3096(-6)	1.2670(-5)
0.7		-5.0073(-6)	-8.9705(-6)	-1.3936(-5)	-1.3883(-5)	-5.4865(-6)	6.7672(-6)
0.8		-6.0167(-6)	-1.1042(-5)	-1.7744(-5)	-1.9969(-5)	-1.1929(-5)	4.3110(-7)
0.9		-5.5224(-6)	-1.0307(-5)	-1.6970(-5)	-2.0637(-5)	-1.4480(-5)	-4.3098(-6)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 7. The intensity $I(\tau, \mu, \varphi)$ for the $L = 13$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi$

μ	τ = 0	τ = τ₀/20	τ = τ₀/10	τ = τ₀/5	τ = τ₀/2	τ = 3τ₀/4	τ = τ₀
-1.0	5.4956(-2)	4.8734(-2)	4.3346(-2)	3.4749(-2)	1.9591(-2)	1.2820(-2)	8.7469(-3)
-0.9	5.3085(-2)	4.7982(-2)	4.3428(-2)	3.5868(-2)	2.1199(-2)	1.3716(-2)	8.7469(-3)
-0.8	5.8688(-2)	5.3327(-2)	4.8470(-2)	4.0279(-2)	2.3821(-2)	1.4966(-2)	8.7469(-3)
-0.7	6.5653(-2)	5.9887(-2)	5.4598(-2)	4.5573(-2)	2.6974(-2)	1.6513(-2)	8.7469(-3)
-0.6	7.3678(-2)	6.7452(-2)	6.1674(-2)	5.1718(-2)	3.0765(-2)	1.8464(-2)	8.7469(-3)
-0.5	8.2754(-2)	7.6055(-2)	6.9762(-2)	5.8822(-2)	3.5387(-2)	2.0996(-2)	8.7469(-3)
-0.4	9.2933(-2)	8.5783(-2)	7.8970(-2)	6.7036(-2)	4.1119(-2)	2.4413(-2)	8.7469(-3)
-0.3	1.0421(-1)	9.6675(-2)	8.9359(-2)	7.6459(-2)	4.8319(-2)	2.9264(-2)	8.7469(-3)
-0.2	1.1641(-1)	1.0861(-1)	1.0080(-1)	8.6949(-2)	5.7242(-2)	3.6576(-2)	8.7469(-3)
-0.1	1.2913(-1)	1.2144(-1)	1.1312(-1)	9.8051(-2)	6.7054(-2)	4.7736(-2)	8.7469(-3)
-0.0	1.3629(-1)	1.3476(-1)	1.2697(-1)	1.1059(-1)	7.5935(-2)	5.7979(-2)	8.7469(-3)
0.0		1.3476(-1)	1.2697(-1)	1.1059(-1)	7.5935(-2)	5.7979(-2)	4.0250(-2)
0.1		5.2018(-2)	8.1507(-2)	1.0260(-1)	8.5069(-2)	6.6024(-2)	4.9823(-2)
0.2		2.8077(-2)	4.9006(-2)	7.3926(-2)	8.3659(-2)	7.1451(-2)	5.7172(-2)
0.3		1.8603(-2)	3.3832(-2)	5.5090(-2)	7.4837(-2)	7.0701(-2)	6.0906(-2)
0.4		1.3541(-2)	2.5181(-2)	4.2805(-2)	6.5198(-2)	6.6488(-2)	6.1102(-2)
0.5		1.0421(-2)	1.9653(-2)	3.4356(-2)	5.6628(-2)	6.1161(-2)	5.9213(-2)
0.6		8.3578(-3)	1.5909(-2)	2.8356(-2)	4.9511(-2)	5.5893(-2)	5.6424(-2)
0.7		6.9821(-3)	1.3364(-2)	2.4126(-2)	4.3927(-2)	5.1312(-2)	5.3557(-2)
0.8		6.1648(-3)	1.1815(-2)	2.1454(-2)	4.0087(-2)	4.7969(-2)	5.1346(-2)
0.9		6.0329(-3)	1.1507(-2)	2.0794(-2)	3.8922(-2)	4.7023(-2)	5.1033(-2)
1.0		9.8920(-3)	1.8391(-2)	3.1837(-2)	5.4252(-2)	6.2253(-2)	6.5318(-2)

Table 8. The Stokes parameter $Q(\tau, \mu, \varphi)$ for the $L = 13$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi$

μ	τ = 0	τ = τ₀/20	τ = τ₀/10	τ = τ₀/5	τ = τ₀/2	τ = 3τ₀/4	τ = τ₀
-1.0	-2.1609(-2)	-1.7634(-2)	-1.4400(-2)	-9.6652(-3)	-3.0410(-3)	-1.0147(-3)	
-0.9	-8.9018(-3)	-6.8803(-3)	-5.3060(-3)	-3.1415(-3)	-6.6006(-4)	-2.0316(-4)	
-0.8	-3.3079(-3)	-2.0038(-3)	-1.0740(-3)	2.9874(-5)	5.8785(-4)	2.1848(-4)	
-0.7	1.2155(-3)	2.0097(-3)	2.4609(-3)	2.7446(-3)	1.7087(-3)	6.0300(-4)	
-0.6	5.2168(-3)	5.6216(-3)	5.6863(-3)	5.2778(-3)	2.8046(-3)	9.8767(-4)	
-0.5	8.8753(-3)	8.9897(-3)	8.7397(-3)	7.7334(-3)	3.9226(-3)	1.3921(-3)	
-0.4	1.2230(-2)	1.2162(-2)	1.1670(-2)	1.0159(-2)	5.0972(-3)	1.8349(-3)	
-0.3	1.5202(-2)	1.5099(-2)	1.4462(-2)	1.2562(-2)	6.3636(-3)	2.3440(-3)	
-0.2	1.7500(-2)	1.7615(-2)	1.6992(-2)	1.4890(-2)	7.7583(-3)	2.9750(-3)	
-0.1	1.8225(-2)	1.9142(-2)	1.8862(-2)	1.6923(-2)	9.2679(-3)	3.8572(-3)	
-0.0	1.2126(-2)	1.7278(-2)	1.8577(-2)	1.7954(-2)	1.0629(-2)	5.0147(-3)	
0.0		1.7278(-2)	1.8577(-2)	1.7954(-2)	1.0629(-2)	5.0147(-3)	-3.3509(-4)
0.1		4.9735(-3)	9.0430(-3)	1.3375(-2)	1.1092(-2)	5.8342(-3)	1.0120(-3)
0.2		1.8316(-3)	3.9103(-3)	7.2009(-3)	8.6356(-3)	5.3337(-3)	1.3502(-3)
0.3		4.1023(-4)	1.2558(-3)	3.0354(-3)	4.8313(-3)	3.1495(-3)	4.1279(-4)
0.4		-5.0411(-4)	-5.2359(-4)	-3.8704(-5)	9.2004(-4)	8.5468(-5)	-1.6013(-3)
0.5		-1.2140(-3)	-1.9206(-3)	-2.5485(-3)	-2.8342(-3)	-3.3473(-3)	-4.2977(-3)
0.6		-1.8394(-3)	-3.1473(-3)	-4.7761(-3)	-6.4423(-3)	-6.9420(-3)	-7.4125(-3)
0.7		-2.4513(-3)	-4.3323(-3)	-6.9110(-3)	-9.9978(-3)	-1.0649(-2)	-1.0812(-2)
0.8		-3.1177(-3)	-5.5978(-3)	-9.1401(-3)	-1.3657(-2)	-1.4520(-2)	-1.4462(-2)
0.9		-3.9586(-3)	-7.1538(-3)	-1.1780(-2)	-1.7737(-2)	-1.8750(-2)	-1.8438(-2)
1.0		-5.9155(-3)	-1.0610(-2)	-1.7182(-2)	-2.4508(-2)	-2.4801(-2)	-2.3440(-2)

Table 9. The diffuse intensity $I_*(\tau, \mu, \varphi)$ for the $L = 60$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = 0$

μ	τ = 0	τ = τ₀/20	τ = τ₀/10	τ = τ₀/5	τ = τ₀/2	τ = 3τ₀/4	τ = τ₀
-1.0	3.8783(-2)	3.5700(-2)	3.2831(-2)	2.7869(-2)	1.7890(-2)	1.2986(-2)	9.8757(-3)
-0.9	6.3881(-2)	5.7475(-2)	5.1589(-2)	4.1648(-2)	2.2973(-2)	1.4696(-2)	9.8757(-3)
-0.8	9.3567(-2)	8.3382(-2)	7.4038(-2)	5.8359(-2)	2.9490(-2)	1.7042(-2)	9.8757(-3)
-0.7	1.3570(-1)	1.2016(-1)	1.0591(-1)	8.2090(-2)	3.8835(-2)	2.0484(-2)	9.8757(-3)
-0.6	1.9652(-1)	1.7324(-1)	1.5189(-1)	1.1632(-1)	5.2432(-2)	2.5612(-2)	9.8757(-3)
-0.5	2.8490(-1)	2.5035(-1)	2.1867(-1)	1.6603(-1)	7.2433(-2)	3.3382(-2)	9.8757(-3)
-0.4	4.1401(-1)	3.6300(-1)	3.1622(-1)	2.3875(-1)	1.0233(-1)	4.5509(-2)	9.8757(-3)
-0.3	6.0620(-1)	5.3148(-1)	4.6242(-1)	3.4804(-1)	1.4867(-1)	6.5700(-2)	9.8757(-3)
-0.2	9.3026(-1)	8.2127(-1)	7.1530(-1)	5.3672(-1)	2.2873(-1)	1.0420(-1)	9.8757(-3)
-0.1	1.7498	1.5632	1.3600	1.0048	4.0483(-1)	1.9398(-1)	9.8757(-3)
-0.0	4.4038	4.0655	3.5237	2.5354	9.0099(-1)	4.0840(-1)	9.8757(-3)
0.0		4.0655	3.5237	2.5354	9.0099(-1)	4.0840(-1)	1.8125(-1)
0.1		3.3143	4.8211	5.1552	2.2686	9.6412(-1)	4.3026(-1)
0.2		2.4110	3.8836	5.0198	3.4113	1.7930	8.8786(-1)
0.3		1.2325	2.0713	2.9236	2.6101	1.7142	1.0399
0.4		4.0496(-1)	7.0650(-1)	1.0752	1.2118	9.6972(-1)	7.1287(-1)
0.5		1.2146(-1)	2.2138(-1)	3.6524(-1)	5.1312(-1)	4.8624(-1)	4.1812(-1)
0.6		6.2809(-2)	1.1625(-1)	1.9734(-1)	3.0197(-1)	3.0835(-1)	2.8612(-1)
0.7		4.0662(-2)	7.5525(-2)	1.2938(-1)	2.0549(-1)	2.1809(-1)	2.1120(-1)
0.8		2.4381(-2)	4.5679(-2)	7.9634(-2)	1.3347(-1)	1.4815(-1)	1.4989(-1)
0.9		1.3164(-2)	2.4943(-2)	4.4441(-2)	7.9330(-2)	9.2561(-2)	9.8153(-2)
1.0		4.4260(-3)	8.5732(-3)	1.5903(-2)	3.1619(-2)	3.9954(-2)	4.5507(-2)

Table 10. The Stokes parameter $Q(\tau, \mu, \varphi)$ for the $L = 60$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = 0$

μ	τ = 0	τ = τ₀/20	τ = τ₀/10	τ = τ₀/5	τ = τ₀/2	τ = 3τ₀/4	τ = τ₀
-1.0	3.2087(-3)	2.7780(-3)	2.3949(-3)	1.7705(-3)	6.8164(-4)	2.4523(-4)	
-0.9	5.6437(-3)	4.8476(-3)	4.1411(-3)	3.0025(-3)	1.0965(-3)	3.8288(-4)	
-0.8	7.8901(-3)	6.7396(-3)	5.7261(-3)	4.1090(-3)	1.4690(-3)	5.1454(-4)	
-0.7	9.9943(-3)	8.4924(-3)	7.1814(-3)	5.1110(-3)	1.8044(-3)	6.4030(-4)	
-0.6	1.1613(-2)	9.8047(-3)	8.2469(-3)	5.8181(-3)	2.0333(-3)	7.3574(-4)	
-0.5	1.2199(-2)	1.0205(-2)	8.5198(-3)	5.9395(-3)	2.0489(-3)	7.6195(-4)	
-0.4	1.1091(-2)	9.1359(-3)	7.5255(-3)	5.1267(-3)	1.7092(-3)	6.6290(-4)	
-0.3	8.3270(-3)	6.6457(-3)	5.2926(-3)	3.3582(-3)	9.3111(-4)	3.8472(-4)	
-0.2	7.3504(-3)	5.5801(-3)	4.1297(-3)	2.1066(-3)	-5.6998(-6)	-6.7468(-5)	
-0.1	1.9182(-2)	1.4601(-2)	1.0880(-2)	5.6637(-3)	-1.1044(-4)	-5.8119(-4)	
-0.0	5.1144(-2)	3.8733(-2)	2.8994(-2)	1.5703(-2)	9.2600(-4)	-1.1452(-3)	
0.0		3.8733(-2)	2.8994(-2)	1.5703(-2)	9.2600(-4)	-1.1452(-3)	-5.5048(-4)
0.1		1.0270(-2)	1.3306(-2)	1.0637(-2)	-8.5349(-4)	-2.7655(-3)	-2.1606(-3)
0.2		-1.0881(-4)	-6.3630(-4)	-2.2565(-3)	-5.3840(-3)	-5.1023(-3)	-3.7978(-3)
0.3		4.2377(-3)	6.7709(-3)	8.4488(-3)	3.9659(-3)	2.6108(-4)	-1.3577(-3)
0.4		5.2372(-3)	8.7403(-3)	1.2141(-2)	9.9143(-3)	5.5286(-3)	2.4465(-3)
0.5		2.1125(-3)	3.6348(-3)	5.3620(-3)	5.3343(-3)	3.6559(-3)	2.1691(-3)
0.6		1.1218(-3)	2.0131(-3)	3.1885(-3)	3.8688(-3)	3.2173(-3)	2.4361(-3)
0.7		1.5602(-3)	2.8208(-3)	4.5563(-3)	6.0030(-3)	5.4675(-3)	4.5991(-3)
0.8		1.7477(-3)	3.1832(-3)	5.2401(-3)	7.4047(-3)	7.1942(-3)	6.4575(-3)
0.9		1.4458(-3)	2.6610(-3)	4.4822(-3)	6.8310(-3)	7.0809(-3)	6.7679(-3)
1.0		6.1502(-4)	1.1621(-3)	2.0623(-3)	3.6652(-3)	4.2854(-3)	4.5723(-3)

Table 11. The intensity $I(\tau, \mu, \varphi)$ for the $L = 60$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi/2$

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	3.8783(-2)	3.5700(-2)	3.2831(-2)	2.7869(-2)	1.7890(-2)	1.2986(-2)	9.8757(-3)
-0.9	4.3702(-2)	4.0236(-2)	3.6982(-2)	3.1298(-2)	1.9614(-2)	1.3700(-2)	9.8757(-3)
-0.8	4.9701(-2)	4.5806(-2)	4.2108(-2)	3.5576(-2)	2.1829(-2)	1.4640(-2)	9.8757(-3)
-0.7	5.7037(-2)	5.2668(-2)	4.8465(-2)	4.0943(-2)	2.4702(-2)	1.5897(-2)	9.8757(-3)
-0.6	6.6034(-2)	6.1156(-2)	5.6385(-2)	4.7721(-2)	2.8477(-2)	1.7611(-2)	9.8757(-3)
-0.5	7.7098(-2)	7.1701(-2)	6.6311(-2)	5.6349(-2)	3.3522(-2)	2.0011(-2)	9.8757(-3)
-0.4	9.0697(-2)	8.4827(-2)	7.8797(-2)	6.7421(-2)	4.0438(-2)	2.3523(-2)	9.8757(-3)
-0.3	1.0709(-1)	1.0095(-1)	9.4369(-2)	8.1613(-2)	5.0216(-2)	2.9052(-2)	9.8757(-3)
-0.2	1.2517(-1)	1.1954(-1)	1.1280(-1)	9.9108(-2)	6.4101(-2)	3.8566(-2)	9.8757(-3)
-0.1	1.3935(-1)	1.3687(-1)	1.3134(-1)	1.1806(-1)	8.1727(-2)	5.5333(-2)	9.8757(-3)
-0.0	1.2246(-1)	1.3936(-1)	1.4087(-1)	1.3337(-1)	9.8722(-2)	7.4373(-2)	9.8757(-3)
0.0		1.3936(-1)	1.4087(-1)	1.3337(-1)	9.8722(-2)	7.4373(-2)	4.6512(-2)
0.1		4.7507(-2)	8.0031(-2)	1.1240(-1)	1.1052(-1)	8.8951(-2)	6.6935(-2)
0.2		2.4118(-2)	4.5002(-2)	7.5278(-2)	1.0276(-1)	9.4546(-2)	7.8705(-2)
0.3		1.5713(-2)	3.0170(-2)	5.3656(-2)	8.6812(-2)	8.9272(-2)	8.1514(-2)
0.4		1.1571(-2)	2.2446(-2)	4.0954(-2)	7.2378(-2)	8.0000(-2)	7.8226(-2)
0.5		9.0991(-3)	1.7738(-2)	3.2794(-2)	6.0981(-2)	7.0624(-2)	7.2472(-2)
0.6		7.4653(-3)	1.4580(-2)	2.7146(-2)	5.2112(-2)	6.2303(-2)	6.6190(-2)
0.7		6.3251(-3)	1.2349(-2)	2.3054(-2)	4.5140(-2)	5.5192(-2)	6.0160(-2)
0.8		5.5007(-3)	1.0719(-2)	2.0006(-2)	3.9605(-2)	4.9192(-2)	5.4663(-2)
0.9		4.8890(-3)	9.5009(-3)	1.7689(-2)	3.5178(-2)	4.4160(-2)	4.9779(-2)
1.0		4.4260(-3)	8.5732(-3)	1.5903(-2)	3.1619(-2)	3.9954(-2)	4.5507(-2)

Table 12. The Stokes parameter $Q(\tau, \mu, \varphi)$ for the $L = 60$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi/2$

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	-3.2087(-3)	-2.7780(-3)	-2.3949(-3)	-1.7705(-3)	-6.8164(-4)	-2.4523(-4)	
-0.9	-3.7776(-3)	-3.2885(-3)	-2.8492(-3)	-2.1247(-3)	-8.3077(-4)	-2.9821(-4)	
-0.8	-4.5173(-3)	-3.9591(-3)	-3.4512(-3)	-2.6019(-3)	-1.0408(-3)	-3.7526(-4)	
-0.7	-5.4695(-3)	-4.8316(-3)	-4.2420(-3)	-3.2395(-3)	-1.3350(-3)	-4.8763(-4)	
-0.6	-6.6841(-3)	-5.9574(-3)	-5.2721(-3)	-4.0842(-3)	-1.7443(-3)	-6.5160(-4)	
-0.5	-8.2210(-3)	-7.3990(-3)	-6.6042(-3)	-5.1960(-3)	-2.3134(-3)	-8.9292(-4)	
-0.4	-1.0153(-2)	-9.2351(-3)	-8.3201(-3)	-6.6590(-3)	-3.1160(-3)	-1.2580(-3)	
-0.3	-1.2551(-2)	-1.1559(-2)	-1.0526(-2)	-8.5922(-3)	-4.2782(-3)	-1.8433(-3)	
-0.2	-1.5362(-2)	-1.4398(-2)	-1.3289(-2)	-1.1104(-2)	-5.9798(-3)	-2.8651(-3)	
-0.1	-1.7919(-2)	-1.7374(-2)	-1.6373(-2)	-1.4086(-2)	-8.2838(-3)	-4.7240(-3)	
-0.0	-1.6086(-2)	-1.8445(-2)	-1.8447(-2)	-1.6868(-2)	-1.0826(-2)	-7.0972(-3)	
0.0		-1.8445(-2)	-1.8447(-2)	-1.6868(-2)	-1.0826(-2)	-7.0972(-3)	-3.5485(-3)
0.1		-6.2936(-3)	-1.0564(-2)	-1.4572(-2)	-1.2930(-2)	-9.2802(-3)	-6.0875(-3)
0.2		-3.1733(-3)	-5.8971(-3)	-9.7206(-3)	-1.2316(-2)	-1.0408(-2)	-7.8327(-3)
0.3		-2.0530(-3)	-3.9226(-3)	-6.8742(-3)	-1.0426(-2)	-1.0016(-2)	-8.4672(-3)
0.4		-1.5043(-3)	-2.8997(-3)	-5.2050(-3)	-8.6420(-3)	-8.9955(-3)	-8.2527(-3)
0.5		-1.1791(-3)	-2.2785(-3)	-4.1328(-3)	-7.2098(-3)	-7.8920(-3)	-7.6532(-3)
0.6		-9.6792(-4)	-1.8678(-3)	-3.3975(-3)	-6.0905(-3)	-6.8870(-3)	-6.9396(-3)
0.7		-8.2586(-4)	-1.5871(-3)	-2.8794(-3)	-5.2218(-3)	-6.0263(-3)	-6.2351(-3)
0.8		-7.2870(-4)	-1.3926(-3)	-2.5111(-3)	-4.5538(-3)	-5.3136(-3)	-5.5935(-3)
0.9		-6.6159(-4)	-1.2569(-3)	-2.2487(-3)	-4.0460(-3)	-4.7385(-3)	-5.0370(-3)
1.0		-6.1502(-4)	-1.1621(-3)	-2.0623(-3)	-3.6652(-3)	-4.2854(-3)	-4.5723(-3)

Table 13. The Stokes parameter $U(\tau, \mu, \varphi)$ for the $L = 60$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi/2$

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	8.2963(-4)	7.4835(-4)	6.7250(-4)	5.3963(-4)	2.5758(-4)	1.0493(-4)	
-0.8	1.3068(-3)	1.1882(-3)	1.0757(-3)	8.7500(-4)	4.3242(-4)	1.8071(-4)	
-0.7	1.7878(-3)	1.6389(-3)	1.4951(-3)	1.2332(-3)	6.3169(-4)	2.7130(-4)	
-0.6	2.3127(-3)	2.1382(-3)	1.9659(-3)	1.6450(-3)	8.7512(-4)	3.8737(-4)	
-0.5	2.9041(-3)	2.7093(-3)	2.5116(-3)	2.1339(-3)	1.1834(-3)	5.4252(-4)	
-0.4	3.5776(-3)	3.3704(-3)	3.1526(-3)	2.7241(-3)	1.5870(-3)	7.6079(-4)	
-0.3	4.3260(-3)	4.1224(-3)	3.8965(-3)	3.4347(-3)	2.1332(-3)	1.0920(-3)	
-0.2	5.0624(-3)	4.9024(-3)	4.6963(-3)	4.2427(-3)	2.8696(-3)	1.6403(-3)	
-0.1	5.5033(-3)	5.4948(-3)	5.3721(-3)	5.0044(-3)	3.7273(-3)	2.5585(-3)	
-0.0	4.6677(-3)	5.3515(-3)	5.5206(-3)	5.4484(-3)	4.4180(-3)	3.4720(-3)	
0.0		5.3515(-3)	5.5206(-3)	5.4484(-3)	4.4180(-3)	3.4720(-3)	2.2125(-3)
0.1		1.8018(-3)	3.0643(-3)	4.4228(-3)	4.7419(-3)	4.0168(-3)	3.1368(-3)
0.2		9.1134(-4)	1.7085(-3)	2.9114(-3)	4.2458(-3)	4.0898(-3)	3.5339(-3)
0.3		5.8849(-4)	1.1304(-3)	2.0350(-3)	3.4687(-3)	3.7057(-3)	3.4936(-3)
0.4		4.2609(-4)	8.2332(-4)	1.5109(-3)	2.7801(-3)	3.1721(-3)	3.1865(-3)
0.5		3.2614(-4)	6.2991(-4)	1.1624(-3)	2.2227(-3)	2.6425(-3)	2.7741(-3)
0.6		2.5597(-4)	4.9238(-4)	9.0709(-4)	1.7660(-3)	2.1554(-3)	2.3340(-3)
0.7		2.0114(-4)	3.8455(-4)	7.0401(-4)	1.3770(-3)	1.7088(-3)	1.8917(-3)
0.8		1.5295(-4)	2.9041(-4)	5.2696(-4)	1.0257(-3)	1.2848(-3)	1.4446(-3)
0.9		1.0286(-4)	1.9396(-4)	3.4843(-4)	6.7049(-4)	8.4281(-4)	9.5746(-4)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 14. The Stokes parameter $V(\tau, \mu, \varphi)$ for the $L = 60$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi/2$

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	-2.3298(-5)	-2.4283(-5)	-2.4454(-5)	-2.3302(-5)	-1.5163(-5)	-7.3764(-6)	
-0.8	-3.3159(-5)	-3.4906(-5)	-3.5383(-5)	-3.4004(-5)	-2.2487(-5)	-1.1109(-5)	
-0.7	-4.0605(-5)	-4.3269(-5)	-4.4195(-5)	-4.2870(-5)	-2.8859(-5)	-1.4524(-5)	
-0.6	-4.6329(-5)	-5.0126(-5)	-5.1646(-5)	-5.0599(-5)	-3.4708(-5)	-1.7869(-5)	
-0.5	-5.0166(-5)	-5.5327(-5)	-5.7572(-5)	-5.6959(-5)	-3.9709(-5)	-2.0987(-5)	
-0.4	-5.2027(-5)	-5.8668(-5)	-6.1628(-5)	-6.1318(-5)	-4.2756(-5)	-2.3060(-5)	
-0.3	-5.3865(-5)	-6.1292(-5)	-6.4271(-5)	-6.3135(-5)	-4.2098(-5)	-2.2329(-5)	
-0.2	-6.1686(-5)	-6.7053(-5)	-6.7778(-5)	-6.2812(-5)	-3.6556(-5)	-1.6775(-5)	
-0.1	-8.0698(-5)	-8.1217(-5)	-7.6679(-5)	-6.3376(-5)	-2.6925(-5)	-5.5507(-6)	
-0.0	-8.0023(-5)	-9.7167(-5)	-9.1212(-5)	-6.9350(-5)	-1.6181(-5)	7.9814(-6)	
0.0		-9.7167(-5)	-9.1212(-5)	-6.9350(-5)	-1.6181(-5)	7.9814(-6)	3.6945(-5)
0.1		-3.1108(-5)	-4.8016(-5)	-5.2857(-5)	-7.4220(-6)	2.1486(-5)	4.2753(-5)
0.2		-1.3015(-5)	-1.9350(-5)	-2.0089(-5)	9.6742(-6)	3.4518(-5)	5.4504(-5)
0.3		-6.4642(-6)	-7.8580(-6)	-3.2752(-6)	2.5393(-5)	4.6566(-5)	6.3452(-5)
0.4		-3.2441(-6)	-2.8056(-6)	3.4556(-6)	3.1940(-5)	5.1776(-5)	6.6338(-5)
0.5		-1.5463(-6)	-4.5883(-7)	5.7275(-6)	3.1803(-5)	5.0127(-5)	6.3234(-5)
0.6		-7.1022(-7)	4.9984(-7)	5.9465(-6)	2.8328(-5)	4.4441(-5)	5.6195(-5)
0.7		-3.6508(-7)	6.7862(-7)	5.0790(-6)	2.3287(-5)	3.6782(-5)	4.6966(-5)
0.8		-2.6324(-7)	4.7982(-7)	3.6989(-6)	1.7488(-5)	2.8067(-5)	3.6366(-5)
0.9		-2.2813(-7)	1.8224(-7)	2.1575(-6)	1.1116(-5)	1.8270(-5)	2.4114(-5)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 15. The intensity $I(\tau, \mu, \varphi)$ for the $L = 60$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi$

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	3.8783(-2)	3.5700(-2)	3.2831(-2)	2.7869(-2)	1.7890(-2)	1.2986(-2)	9.8757(-3)
-0.9	4.1409(-2)	3.8470(-2)	3.5685(-2)	3.0729(-2)	1.9962(-2)	1.4033(-2)	9.8757(-3)
-0.8	5.1943(-2)	4.8047(-2)	4.4360(-2)	3.7802(-2)	2.3520(-2)	1.5565(-2)	9.8757(-3)
-0.7	6.8133(-2)	6.2593(-2)	5.7377(-2)	4.8172(-2)	2.8466(-2)	1.7645(-2)	9.8757(-3)
-0.6	9.5937(-2)	8.6855(-2)	7.8507(-2)	6.4200(-2)	3.5357(-2)	2.0429(-2)	9.8757(-3)
-0.5	1.3265(-1)	1.1850(-1)	1.0570(-1)	8.4289(-2)	4.3565(-2)	2.3747(-2)	9.8757(-3)
-0.4	1.3780(-1)	1.2565(-1)	1.1386(-1)	9.2985(-2)	4.9920(-2)	2.7073(-2)	9.8757(-3)
-0.3	1.4338(-1)	1.3349(-1)	1.2297(-1)	1.0314(-1)	5.8557(-2)	3.2197(-2)	9.8757(-3)
-0.2	1.9176(-1)	1.7633(-1)	1.6073(-1)	1.3281(-1)	7.5158(-2)	4.2240(-2)	9.8757(-3)
-0.1	1.9322(-1)	1.8690(-1)	1.7596(-1)	1.5181(-1)	9.3579(-2)	5.8922(-2)	9.8757(-3)
-0.0	2.0949(-1)	2.2842(-1)	2.2211(-1)	1.9548(-1)	1.1994(-1)	8.0414(-2)	9.8757(-3)
0.0		2.2842(-1)	2.2211(-1)	1.9548(-1)	1.1994(-1)	8.0414(-2)	4.6169(-2)
0.1		1.0287(-1)	1.6334(-1)	2.0604(-1)	1.5248(-1)	1.0230(-1)	6.7736(-2)
0.2		4.6507(-2)	8.3007(-2)	1.2789(-1)	1.3968(-1)	1.0977(-1)	8.0614(-2)
0.3		2.3046(-2)	4.3243(-2)	7.3228(-2)	1.0261(-1)	9.5015(-2)	7.9438(-2)
0.4		1.3544(-2)	2.5891(-2)	4.5740(-2)	7.3588(-2)	7.6051(-2)	7.0228(-2)
0.5		8.7555(-3)	1.6934(-2)	3.0714(-2)	5.3939(-2)	6.0022(-2)	5.9540(-2)
0.6		5.9548(-3)	1.1640(-2)	2.1583(-2)	4.0568(-2)	4.7727(-2)	4.9999(-2)
0.7		4.2988(-3)	8.4626(-3)	1.5948(-2)	3.1594(-2)	3.8824(-2)	4.2454(-2)
0.8		3.3664(-3)	6.6436(-3)	1.2629(-2)	2.5920(-2)	3.2889(-2)	3.7159(-2)
0.9		2.9864(-3)	5.8780(-3)	1.1170(-2)	2.3247(-2)	3.0038(-2)	3.4658(-2)
1.0		4.4260(-3)	8.5732(-3)	1.5903(-2)	3.1619(-2)	3.9954(-2)	4.5507(-2)

Table 16. The Stokes parameter $Q(\tau, \mu, \varphi)$ for the $L = 60$ model with $\mathbf{F} = |1000|^T$, $\omega = 0.99$, $\lambda_0 = 0.1$, $\mu_0 = 0.2$, $\tau_0 = 1$ and $\varphi - \varphi_0 = \pi$

μ	$\tau = 0$	$\tau = \tau_0/20$	$\tau = \tau_0/10$	$\tau = \tau_0/5$	$\tau = \tau_0/2$	$\tau = 3\tau_0/4$	$\tau = \tau_0$
-1.0	3.2087(-3)	2.7780(-3)	2.3949(-3)	1.7705(-3)	6.8164(-4)	2.4523(-4)	
-0.9	3.4697(-3)	2.9568(-3)	2.5094(-3)	1.7979(-3)	6.3578(-4)	2.2023(-4)	
-0.8	3.9993(-3)	3.3370(-3)	2.7688(-3)	1.8892(-3)	5.6848(-4)	1.7603(-4)	
-0.7	3.2661(-3)	2.6628(-3)	2.1470(-3)	1.3644(-3)	2.9755(-4)	6.6522(-5)	
-0.6	1.0031(-3)	7.6856(-4)	5.5944(-4)	2.4535(-4)	-9.1278(-5)	-6.3163(-5)	
-0.5	3.4663(-3)	2.6052(-3)	1.9231(-3)	9.7408(-4)	-4.6090(-5)	-8.5086(-5)	
-0.4	1.1649(-2)	8.8727(-3)	6.6943(-3)	3.6710(-3)	2.8540(-4)	-8.2490(-5)	
-0.3	7.3601(-3)	5.2278(-3)	3.5605(-3)	1.3259(-3)	-6.6164(-4)	-4.2237(-4)	
-0.2	-3.5796(-3)	-3.7223(-3)	-3.7490(-3)	-3.5099(-3)	-1.9808(-3)	-7.8412(-4)	
-0.1	1.2076(-2)	8.3597(-3)	5.6621(-3)	2.2980(-3)	-5.4524(-4)	-3.2936(-4)	
-0.0	3.5724(-2)	2.7510(-2)	2.1107(-2)	1.2267(-2)	1.9637(-3)	3.5429(-4)	
0.0		2.7510(-2)	2.1107(-2)	1.2267(-2)	1.9637(-3)	3.5429(-4)	6.7533(-4)
0.1		5.4171(-3)	7.6962(-3)	7.6431(-3)	2.1094(-3)	4.6193(-4)	4.9547(-4)
0.2		3.3847(-4)	6.2807(-4)	8.8727(-4)	5.2878(-4)	3.5219(-4)	5.8320(-4)
0.3		7.4234(-4)	1.2708(-3)	1.8547(-3)	1.8977(-3)	1.5624(-3)	1.4320(-3)
0.4		1.1848(-3)	2.0652(-3)	3.1506(-3)	3.6688(-3)	3.1545(-3)	2.6755(-3)
0.5		1.0894(-3)	1.9538(-3)	3.1273(-3)	4.1115(-3)	3.8363(-3)	3.4138(-3)
0.6		8.5201(-4)	1.5629(-3)	2.6002(-3)	3.7883(-3)	3.8173(-3)	3.6106(-3)
0.7		6.4854(-4)	1.2103(-3)	2.0784(-3)	3.3090(-3)	3.5654(-3)	3.5644(-3)
0.8		5.0438(-4)	9.5489(-4)	1.6865(-3)	2.9076(-3)	3.3201(-3)	3.4809(-3)
0.9		4.2224(-4)	8.0808(-4)	1.4600(-3)	2.6872(-3)	3.2130(-3)	3.4955(-3)
1.0		6.1502(-4)	1.1621(-3)	2.0623(-3)	3.6652(-3)	4.2854(-3)	4.5723(-3)

COMMENTS

In this work the generalized spherical harmonics method has been used to solve the complete and general polarization problem for a plane parallel layer. The numerical methods developed have been implemented and shown to work—a task, we believe, any work on computational methods should complete.

It is worthwhile to note here that (for $m > 0$) we were unable to obtain correct results by using (as we did in Ref. [1], for $m = 0$) a generalization of the Marshak[19] boundary condition for this general problem; i.e. projecting the boundary conditions against the matrices $\mu \mathbf{\Pi}^m_{m+2\alpha}(\mu)$, $\alpha = 0, 1, 2, \ldots$, did not yield acceptable results (nor did several other variants of this idea that we tried). Our generalization of the Mark boundary condition,[19,20] on the other hand, did work well.

Our program runs in core on a large IBM mainframe computer (3081-KX) and in low order, say $N \leqslant 99$, and modest values of L, say $L = 13$, will produce results for all $m \leqslant L$ accurate to three or four figures, which are supposed adequate for most practical applications, in something like 15 min. We have observed that a large fraction of the CPU time is spent in the EISPACK routines used to compute the eigenvalues. For this reason we expect soon to investigate the merits of using some of Cullum's work with the Lanczos method for our problem.

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APPENDIX

A Generalization of the Mark Boundary Condition

We consider here problems (for $m = 0, 1, \ldots, L$) defined, in general, by the equation of transfer

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} \mathbf{K}^m(\mu' \to \mu)\, \mathbf{I}(\tau, \mu')\, d\mu' + \mathbf{S}^m(\tau, \mu), \tag{A1}$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$, and the boundary conditions

$$\mathbf{I}(0, \mu) = \mathbf{0} \tag{A2a}$$

and

$$\mathbf{I}(\tau_0, -\mu) = \mathbf{S} \tag{A2b}$$

for $\mu \in [0, 1]$. Here the four-vector $\mathbf{S}$ is a constant, considered given, and

$$\mathbf{K}^m(\mu' \to \mu) = \sum_{l=m}^{L} \mathbf{\Pi}_l^m(\mu)\mathbf{B}_l\mathbf{\Pi}_l^m(\mu'). \tag{A3}$$

Following an idea of Mark,[19,20] we now consider that there is source-free, purely absorbing material in the region $-\infty < \tau < 0$, and we write the generalized spherical harmonics approximation to $\mathbf{I}(\tau, \mu)$ in this region as

$$\mathbf{I}(\tau, \mu) = \sum_{l=m}^{M} \left(\frac{2l+1}{2}\right)(-1)^{l-m}\mathbf{\Pi}_l^m(\mu) \sum_{j=1}^{J} D_j \exp(\tau/\xi_j)\mathbf{D}\boldsymbol{\Gamma}_l^m(\xi_j)\mathbf{M}(\xi_j), \tag{A4}$$

where we have defined

$$\boldsymbol{\Gamma}_l^m(\xi) = \lim_{\omega \to 0} \mathbf{G}_l^m(\xi). \tag{A5}$$

We now write the recursion formulas for the $\mathbf{\Pi}$ and $\boldsymbol{\Gamma}$ matrices as

$$(2l+1)\mu\mathbf{\Pi}_l^m(\mu) = \mathbf{U}_{l+1}^m\mathbf{\Pi}_{l+1}^m(\mu) - \mathbf{V}_l^m\mathbf{\Pi}_l^m(\mu) + \mathbf{U}_l^m\mathbf{\Pi}_{l-1}^m(\mu) \tag{A6}$$

and

$$(2l+1)\xi\boldsymbol{\Gamma}_l^m(\xi) = \mathbf{U}_{l+1}^m\boldsymbol{\Gamma}_{l+1}^m(\xi) - \mathbf{V}_l^m\boldsymbol{\Gamma}_l^m(\xi) + \mathbf{U}_l^m\boldsymbol{\Gamma}_{l-1}^m(\xi) \tag{A7}$$

where $\mathbf{U}_l^m$ and $\mathbf{V}_l^m$ are given by equations (27) and (28). If we multiply equation (A6) by $\boldsymbol{\Gamma}_l^m(\xi)$, multiply equation (A7) by $\mathbf{\Pi}_l^m(\mu)$, subtract one of the two equations from the other and then sum the resulting equation from $l = m$ to $l = M$, we find

$$(\mu - \xi) \sum_{l=m}^{M} (2l+1)\mathbf{\Pi}_l^m(\mu)\boldsymbol{\Gamma}_l^m(\xi) = \mathbf{U}_{M+1}^m[\mathbf{\Pi}_{M+1}^m(\mu)\boldsymbol{\Gamma}_M^m(\xi) - \mathbf{\Pi}_M^m(\mu)\boldsymbol{\Gamma}_{M+1}^m(\xi)]. \tag{A8}$$

We therefore can rewrite equation (A4) as

$$\mathbf{I}(\tau, \mu) = -\tfrac{1}{2}\mathbf{U}_{M+1}^m\mathbf{\Pi}_{M+1}^m(\mu) \sum_{j=1}^{J} \left(\frac{1}{\mu + \xi_j}\right) D_j \exp(\tau/\xi_j)\mathbf{D}\boldsymbol{\Gamma}_M^m(\xi_j)\mathbf{M}(\xi_j), \tag{A9}$$

$\mu \in [-1, 1]$ and $\tau \in (-\infty, 0)$. We let $\{\mu_\alpha\}$ denote the J zeros of det $\mathbf{\Pi}_{M+1}^m(\mu)$ that are contained in the interval (0, 1) and $\mathbf{N}(\mu_\alpha)$ the corresponding null vector of $\mathbf{\Pi}_{M+1}^m(\mu_\alpha)$. It is thus apparent from equation (A9) that, for $\tau \in (-\infty, 0)$,

$$\mathbf{N}^T(\mu_\alpha)\mathbf{I}(\tau, \mu_\alpha) = 0, \quad \alpha = 1, 2, \ldots, J. \tag{A10}$$

We require that $\mathbf{I}(\tau, \mu)$ be continuous (except for $\mu = 0$) across the interface $\tau = 0$, and so in regard to our generalized spherical harmonics solution of equation (A1), we replace the boundary condition given by equation (A2a) with

$$\mathbf{N}^T(\mu_\alpha)\mathbf{I}(0, \mu_\alpha) = 0, \quad \alpha = 1, 2, \ldots, J. \tag{A11}$$

We also consider the material in the region $\tau_0 < \tau < \infty$ to be purely absorbing, but since the constant $\mathbf{S}$ in equation (A2b) is non zero, we include a source term $\mathbf{S}(\tau, \mu) = \mathbf{S}$ in the equation of transfer for $\tau > \tau_0$. We thus write our generalized spherical harmonics solution in this region as

$$\mathbf{I}(\tau, \mu) = \sum_{l=m}^{M} \left(\frac{2l+1}{2}\right)\mathbf{\Pi}_l^m(\mu) \sum_{j=1}^{J} C_j \exp(-\tau/\xi_j)\boldsymbol{\Gamma}_l^m(\xi_j)\mathbf{M}(\xi_j) + \mathbf{S}. \tag{A12}$$

Using equation (A8), we write equation (A12) as

$$\mathbf{I}(\tau, -\mu) = -\tfrac{1}{2}\mathbf{D}\mathbf{U}_{M+1}^m\mathbf{\Pi}_{M+1}^m(\mu) \sum_{j=1}^{J} \left(\frac{1}{\mu + \xi_j}\right) C_j \exp(-\tau/\xi_j)\mathbf{D}\boldsymbol{\Gamma}_M^m(\xi_j)\mathbf{M}(\xi_j) + \mathbf{S}. \tag{A13}$$

It thus is clear that, for $\tau \in (\tau_0, \infty)$,

$$\mathbf{N}^T(\mu_\alpha)\mathbf{D}\mathbf{I}(\tau, -\mu_\alpha) = \mathbf{N}^T(\mu_\alpha)\mathbf{D}\mathbf{S}. \tag{A14}$$

We require $\mathbf{I}(\tau, \mu)$ to be continuous (except for $\mu = 0$) across the interface $\tau = \tau_0$, and so we replace equation (A2b) with

$$\mathbf{N}^T(\mu_\alpha)\mathbf{D}\mathbf{I}(\tau_0, -\mu_\alpha) = \mathbf{N}^T(\mu_\alpha)\mathbf{D}\mathbf{S}, \quad \alpha = 1, 2, \ldots, J, \tag{A15}$$

for use with our generalized spherical harmonics solution of equation (A1).

To complete this appendix we discuss the way we compute the required zeros $\{\mu_\alpha\}$ of det $\mathbf{\Pi}_{M+1}^m(\mu)$ in the interval (0, 1). Considering first of all the case $m = 0$, we note[1] that

$$\mathbf{\Pi}_{N+1}^0(\mu) = \text{diag}\{P_{N+1}(\mu), R_{N+1}(\mu), R_{N+1}(\mu), P_{N+1}(\mu)\}, \tag{A16}$$

where $P_l(\mu)$ is used to denote the Legendre polynomials and

$$R_l(\mu) = \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} (1 - \mu^2) \frac{\mathrm{d}^2}{\mathrm{d}\mu^2} P_l(\mu). \tag{A17}$$

Note that $R_l(\mu)$ is a normalized version of the associated Legendre function $P_l^2(\mu)$. It follows that $N + 1$ of the required $2N$ zeros of det $\mathbf{\Pi}_{N+1}^0(\mu)$ are the $(N + 1)/2$ positive zeros, say $\{\nu_\alpha\}$, of $P_{N+1}(\mu)$ where each of these zeros has multiplicity two and each of these zeros has the associated null vectors

$$\mathbf{N}^{(1)}(\nu_\alpha) = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \end{vmatrix} \quad \text{and} \quad \mathbf{N}^{(2)}(\nu_\alpha) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix}. \tag{A18a, b}$$

Similarly, the $N-1$ remaining zeros we seek are the $(N-1)/2$ positive zeros, say $\{\eta_\alpha\}$, of $P^2_{N+1}(\mu)$ where again each zero has multiplicity two and each zero has associated null vectors

$$\mathbf{N}^{(1)}(\eta_\alpha) = \begin{vmatrix} 0 \\ 1 \\ 0 \\ 0 \end{vmatrix} \quad \text{and} \quad \mathbf{N}^{(2)}(\eta_\alpha) = \begin{vmatrix} 0 \\ 0 \\ 1 \\ 0 \end{vmatrix}. \tag{A19a, b}$$

From the three-term recursion formula for the associated Legendre functions and the truncation condition

$$P^m_{m+N'+1}(\mu) = 0, \tag{A20}$$

where N' is odd, we can deduce, in a manner identical to that used in Ref. [8] with regard to the eigenvalue calculation, that the $(N'+1)/2$ eigenvalues $\{\xi_j^2\}$ of the tridiagonal matrix $\mathbf{Q}$ of order $(N'+1)/2$ yield the $(N'+1)/2$ desired positive zeros $\{\xi_j\}$ of $P^m_{m+N'+1}(\mu)$. Here the elements of the $\mathbf{Q}$ matrix are given by

$$Q_{\alpha,\alpha+1} = \frac{2\alpha(2\alpha-1)}{(2m+4\alpha-3)(2m+4\alpha-1)}, \tag{A21a}$$

$$Q_{\alpha,\alpha} = \frac{1}{2m+4\alpha-3}\left[\frac{4(\alpha-1)(m+\alpha-1)}{2m+4\alpha-5} + \frac{(2\alpha-1)(2m+2\alpha-1)}{2m+4\alpha-1}\right] \tag{A21b}$$

and

$$Q_{\alpha+1,\alpha} = \frac{2(m+\alpha)(2m+2\alpha-1)}{(2m+4\alpha+1)(2m+4\alpha-1)} \tag{A21c}$$

where $\alpha \geqslant 1$. To find the $\{\nu_\alpha\}$ we take $m=0$, $N'=N$ and use the subroutines **FIGI** and **IMTQL1** from the EISPACK collection[9,10] to find the eigenvalues of $\mathbf{Q}$. A similar procedure with $m=2$ and $N'=N-2$ can be used to find the roots $\{\eta_\alpha\}$.

Turning now to the cases $m \geqslant 2$, we note from Ref. [3] that the matrices $\mathbf{\Pi}^m_l(\mu)$ can all be reduced to diagonal form with a similarity transformation based on constant matrices, i.e.

$$\mathbf{S}\mathbf{\Pi}^m_l(\mu)\mathbf{S}^{-1} = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} \operatorname{diag}\{P^m_l(\mu), R^m_l(\mu) - T^m_l(\mu), R^m_l(\mu) + T^m_l(\mu), P^m_l(\mu)\} \tag{A22}$$

where

$$\mathbf{S} = (2)^{-1/2}\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \tag{A23}$$

We note that

$$\left[\frac{(l-m)!}{(l+m)!}\right]^{1/2}[R^m_l(\mu) - T^m_l(\mu)] = -(i)^m P^l_{m,-2}(\mu) \tag{A24a}$$

and

$$\left[\frac{(l-m)!}{(l+m)!}\right]^{1/2}[R^m_l(\mu) + T^m_l(\mu)] = -(i)^m P^l_{m,2}(\mu) \tag{A24b}$$

where $P^l_{m,2}(\mu)$ and $P^l_{m,-2}(\mu)$ are the generalized spherical functions discussed by Gel'fand and Sapiro[21] and used, for example, in Ref. [5]. It follows that $N+1$ of the required $2(N+1)$ zeros of det $\mathbf{\Pi}^m_{M+1}(\mu)$ are the $(N+1)/2$ zeros, say $\{\nu_\alpha\}$, of $P^m_{M+1}(\mu)$, in the interval $(0, 1)$, where each zero has multiplicity two and each has the associated null vectors given by equations (A18). These zeros can be found from the eigenvalues of $\mathbf{Q}$ with $m=m$ and $N'=N$.

We note that

$$P^l_{m,2}(\mu) = (-1)^{l-m}P^l_{m,-2}(-\mu), \tag{A25}$$

and so we now compute the $N+1$ zeros of $P^{M+1}_{m,2}(\mu)$; with the positive zeros, say $\{n^{(1)}_\alpha\}$, we use the null vector

$$\mathbf{N}[\eta^{(1)}_\alpha] = \begin{vmatrix} 0 \\ 1 \\ -1 \\ 0 \end{vmatrix} \tag{A26a}$$

and with the negative zeros, say $\{-\eta^{(2)}_\alpha\}$, of $P^{M+1}_{m,2}(\mu)$, i.e. the positive zeros $\{\eta^{(2)}_\alpha\}$ of $P^{M+1}_{m,-2}(\mu)$, we use the null vector

$$\mathbf{N}[\eta^{(2)}_\alpha] = \begin{vmatrix} 0 \\ 1 \\ 1 \\ 0 \end{vmatrix}. \tag{A26b}$$

At this point the recursion formula[5]

$$A^l_m P^{l-1}_{m,2}(\xi) + B^l_m P^l_{m,2}(\xi) + C^l_m P^{l+1}_{m,2}(\xi) = \xi P^l_{m,2}(\xi) \tag{A27}$$

and the truncation condition

$$P^{M+1}_{m,2}(\xi) = 0 \tag{A28}$$

can be used to show that the $N+1$ zeros of $P_{m,2}^{M+1}(\xi)$ are the eigenvalues of the tridiagonal matrix

$$\Gamma = \begin{vmatrix} B_m^m & C_m^m & & & \\ A_m^{m+1} & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & C_m^{M-1} \\ & & & A_m^M & B_m^M \end{vmatrix}. \tag{A29}$$

Here, for $l \geqslant m$ and $m \geqslant 2$,

$$A_m^l = \frac{[(l^2-m^2)(l^2-4)]^{1/2}}{l(2l+1)} \tag{A30a}$$

$$B_m^l = \frac{2m}{l(l+1)} \tag{A30b}$$

and

$$C_m^l = \frac{[(l+1)^2-m^2]^{1/2}[(l+1)^2-4]^{1/2}}{(l+1)(2l+1)}. \tag{A30c}$$

Finally, we note that to include the case $m = 1$ in the foregoing discussion we simply do not use $\xi = 1$ that is an eigenvalue of Γ, for $m = 1$, but that clearly is not a zero of det $\Pi_{N+2}^1(\mu)$ in the interval (0, 1).