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On Two-Group Critical Problems in Neutron-Transport Theory

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Abstract—*The F_N method is used, in regard to two-group neutron-transport theory, to compute accurate critical dimensions for slabs and spheres.*

Sur des problèmes critiques à deux groupes dans la théorie de transport des neutrons

Résumé—*Dans la théorie de transport des neutrons à deux groupes la méthode F_N est employée pour calculer les dimensions précises critiques de plaques et de sphères.*

Über kritische Zweigruppen-Probleme der Neutronentransporttheorie

Zusammenfassung—*In der Zweigruppen-Neutronentransporttheorie wird die F_N-Methode zur Berechnung kritischer Abmessungen von Platten und Kugeln benutzt.*

INTRODUCTION

The F_N method[1] has enjoyed considerable success in producing accurate numerical results for multigroup problems relevant to radiation shielding calculations.[2-5] In these previous works, only downscattering was allowed, however, so now we consider a fully coupled two-group model to demonstrate the viability of the F_N method for a new class of problems.

We follow Siewert and Shieh[6] and write the two-group transport equation in plane geometry as

$$\mu \frac{\partial}{\partial x} \boldsymbol{\Psi}(x,\mu) + \boldsymbol{\Sigma}\boldsymbol{\Psi}(x,\mu) = \boldsymbol{C}\int_{-1}^{1} \boldsymbol{\Psi}(x,\mu')\,d\mu' \ . \quad (1)$$

Here the elements of $\boldsymbol{\Psi}(x,\mu)$ are the group angular fluxes, x is the position variable measured (in optical units) in terms of σ_2, the smaller of the group total cross sections σ_1 and σ_2, and μ is the direction cosine of the propagating neutrons. In addition,

$$\boldsymbol{\Sigma} = \text{diag}\{\sigma,1\} \ , \quad (2)$$

where $\sigma = \sigma_1/\sigma_2 > 1$, and the transfer matrix $\boldsymbol{C}$ has nonnegative elements c_{ij}. For our critical slab problem we seek the half-thickness for a multiplying medium,[7] i.e., $k_{BMS} > 1$ where

$$k_{BMS} = \frac{1}{\sigma} c_{11} + c_{22} + \left[\left(\frac{1}{\sigma} c_{11} + c_{22}\right)^2 - \frac{4}{\sigma} \det \boldsymbol{C}\right]^{1/2} , \quad (3)$$

such that the elements of $\boldsymbol{\Psi}(x,\mu)$ are nonnegative and $\boldsymbol{\Psi}(x,\mu)$ satisfies Eq. (1) and the boundary conditions

$$\boldsymbol{\Psi}(-a,\mu) = \boldsymbol{\Psi}(a,-\mu) = \mathbf{0} \ , \quad \mu \in [0,1] \ . \quad (4)$$

The vector $\boldsymbol{\Psi}(x,\mu)$ is clearly symmetric, i.e., $\boldsymbol{\Psi}(x,\mu) = \boldsymbol{\Psi}(-x,-\mu)$.

ANALYSIS

To start, we follow Ref. 8 and change μ to $-\mu$ in Eq. (1), multiply the resulting equation by $\exp(-x/s)$ and integrate over x from $-a$ to a to obtain, after an integration by parts,

$$(s\boldsymbol{\Sigma} - \mu\boldsymbol{I})\int_{-a}^{a} \boldsymbol{\Psi}(x,-\mu)\exp(-x/s)\,dx = s\boldsymbol{C}\boldsymbol{\rho}^*(s) - s\mu\boldsymbol{B}(\mu,s) \quad , \tag{5}$$

where

$$\boldsymbol{B}(\mu,s) = \boldsymbol{\Psi}(-a,-\mu)\exp(a/s) - \boldsymbol{\Psi}(a,-\mu)\exp(-a/s) \tag{6}$$

and

$$\boldsymbol{\rho}^*(s) = \int_{-a}^{a} \exp(-x/s)\int_{-1}^{1} \boldsymbol{\Psi}(x,\mu)\,d\mu\,dx \quad . \tag{7}$$

Multiplying Eq. (5) by $(s\boldsymbol{\Sigma} - \mu\boldsymbol{I})^{-1}$, for $s \notin [-1,1]$, and integrating over μ, we find

$$\boldsymbol{\Lambda}(s)\boldsymbol{\rho}^*(s) = s\int_0^1 \mu(\mu\boldsymbol{I} - s\boldsymbol{\Sigma})^{-1}\boldsymbol{B}(\mu,s)\,d\mu + s\int_0^1 \mu(\mu\boldsymbol{I} + s\boldsymbol{\Sigma})^{-1}\boldsymbol{B}(-\mu,s)\,d\mu \quad , \tag{8}$$

where

$$\boldsymbol{\Lambda}(s) = \boldsymbol{I} + s\int_{-1}^{1} (\mu\boldsymbol{I} - s\boldsymbol{\Sigma})^{-1}\,d\mu\,\boldsymbol{C} \tag{9}$$

or[9]

$$\boldsymbol{\Lambda}(s) = \boldsymbol{I} + s\int_{-1}^{1} \boldsymbol{\Psi}(\mu)\,\frac{d\mu}{\mu - s} \quad , \tag{10}$$

where

$$\boldsymbol{\Psi}(\mu) = \boldsymbol{\Theta}(\mu)\boldsymbol{C} = \mathrm{diag}\{\theta(\mu),1\}\boldsymbol{C} \quad , \tag{11}$$

with

$$\theta(\mu) = 1 \quad , \quad \mu \in [-1/\sigma, 1/\sigma] \quad ,$$
$$= 0 \quad , \quad \text{otherwise} \quad . \tag{12}$$

Since for the considered critical problem we have $\boldsymbol{B}(-\mu,s) = -\boldsymbol{B}(\mu,-s)$, we rewrite Eq. (8) as

$$\boldsymbol{\Lambda}(s)\boldsymbol{\rho}^*(s) = s\int_0^1 \mu(\mu\boldsymbol{I} - s\boldsymbol{\Sigma})^{-1}\boldsymbol{B}(\mu,s)\,d\mu - s\int_0^1 \mu(\mu\boldsymbol{I} + s\boldsymbol{\Sigma})^{-1}\boldsymbol{B}(\mu,-s)\,d\mu \tag{13}$$

and consider only $\mathrm{Re}\,s > 0$. Now introducing the notation

$$\boldsymbol{B}^\dagger(\mu,s) = \begin{bmatrix} B_1(\sigma\mu,s) \\ B_2(\mu,s) \end{bmatrix} \quad , \tag{14}$$

we write Eq. (13) as

$$\boldsymbol{\Lambda}(s)\boldsymbol{\rho}^*(s) = s\boldsymbol{\Sigma}\int_0^1 \mu\boldsymbol{\Theta}(\mu)\boldsymbol{B}^\dagger(\mu,s)\,\frac{d\mu}{\mu - s} - s\boldsymbol{\Sigma}\int_0^1 \mu\boldsymbol{\Theta}(\mu)\boldsymbol{B}^\dagger(\mu,-s)\,\frac{d\mu}{\mu + s} \quad . \tag{15}$$

We note from Ref. 6 that $\boldsymbol{\Lambda}(s)$ can be singular, for $\mathrm{Re}\,s > 0$ and $s \notin [0,1]$, for at most two values of s, and so we express this fact as

$$\boldsymbol{\Lambda}^T(\nu_\beta)\boldsymbol{M}(\nu_\beta) = \boldsymbol{0} \quad , \quad \beta = 1,2,\ldots,\aleph \quad , \tag{16}$$

where $\boldsymbol{M}(\nu_\beta)$ is a null vector of $\boldsymbol{\Lambda}^T(\nu_\beta)$ and $\aleph$ can be 1 or 2 depending[6] on the transfer matrix $\boldsymbol{C}$ and σ. We now use Eq. (4) and deduce from Eq. (15) that

$$\boldsymbol{M}^T(\nu_\beta)\boldsymbol{\Sigma}\int_0^1 \mu\boldsymbol{\Theta}(\mu)\boldsymbol{\Psi}^\dagger(a,\mu) \times \left[\frac{1}{\nu_\beta - \mu} + \frac{1}{\nu_\beta + \mu}\exp(-2a/\nu_\beta)\right] d\mu = 0 \tag{17}$$

for $\beta = 1,2,\ldots,\aleph$. Here $\boldsymbol{\Psi}^\dagger(a,\mu)$ has elements $\psi_1(a,\sigma\mu)$ and $\psi_2(a,\mu)$. Continuing to follow Ref. 8, we let $s \to \nu \in (0,1)$ and find from Eq. (15), after using Eq. (4), that

$$\boldsymbol{\Lambda}^\pm(\nu)\boldsymbol{\rho}^*(\nu)\exp(-a/\nu) = \nu\boldsymbol{\Sigma}\left\{\int_0^1 \mu\boldsymbol{\Theta}(\mu)\boldsymbol{\Psi}^\dagger(a,\mu) \times \left[\frac{P}{\mu - \nu} - \frac{1}{\mu + \nu}\exp(-2a/\nu)\right] d\mu \pm \pi i\nu\boldsymbol{\Theta}(\nu)\boldsymbol{\Psi}^\dagger(a,\nu)\right\} \tag{18}$$

or, after we eliminate $\boldsymbol{\rho}^*(\nu)$,

$$\boldsymbol{\Theta}(\nu)\boldsymbol{C}\boldsymbol{\lambda}^{-1}(\nu) \times \left\{\boldsymbol{\lambda}(\nu)\boldsymbol{C}^{-1}\boldsymbol{\Sigma}\boldsymbol{\Psi}^\dagger(a,\nu) + \boldsymbol{\Sigma}\int_0^1 \mu\boldsymbol{\Theta}(\mu)\boldsymbol{\Psi}^\dagger(a,\mu) \times \left[\frac{P}{\nu - \mu} + \frac{1}{\nu + \mu}\exp(-2a/\nu)\right] d\mu\right\} = \boldsymbol{0} \quad , \tag{19}$$

where

$$\boldsymbol{\lambda}(\nu) = \boldsymbol{I} + \nu P\int_{-1}^{1} \boldsymbol{\Psi}(\mu)\,\frac{d\mu}{\mu - \nu} \quad . \tag{20}$$

At this point we introduce $\boldsymbol{\Omega}(z) = \boldsymbol{C}\boldsymbol{\Lambda}(z)\boldsymbol{C}^{-1}$, i.e.,

$$\boldsymbol{\Omega}(z) = \boldsymbol{I} + z\boldsymbol{C}\int_{-1}^{1} \boldsymbol{\Theta}(\mu)\,\frac{d\mu}{\mu - z} \quad , \tag{21}$$

and rewrite Eqs. (17) and (19) as

$$\boldsymbol{N}^T(\nu_\beta)\boldsymbol{C\Sigma}\int_0^1 \mu\boldsymbol{\Theta}(\mu)\boldsymbol{\Psi}^\dagger(a,\mu) \times\left[\frac{1}{\nu_\beta-\mu}+\frac{1}{\nu_\beta+\mu}\exp(-2a/\nu_\beta)\right]d\mu=0 \quad (22a)$$

and

$$\boldsymbol{\Theta}(\nu)\boldsymbol{\omega}^{-1}(\nu)\left\{\boldsymbol{\omega}(\nu)\boldsymbol{\Sigma\Psi}^\dagger(a,\nu)+\boldsymbol{C\Sigma}\int_0^1 \mu\boldsymbol{\Theta}(\mu)\boldsymbol{\Psi}^\dagger(a,\mu) \times\left[\frac{P}{\nu-\mu}+\frac{1}{\nu+\mu}\exp(-2a/\nu)\right]d\mu\right\}=\mathbf{0}\ , \quad (22b)$$

where

$$\boldsymbol{\Omega}^T(\nu_\beta)\boldsymbol{N}(\nu_\beta)=\mathbf{0} \quad (23)$$

and

$$\boldsymbol{\omega}(\nu)=\boldsymbol{I}+\nu\boldsymbol{C}P\int_{-1}^1 \boldsymbol{\Theta}(\mu)\frac{d\mu}{\mu-\nu}\ . \quad (24)$$

Equation (22a), for $\beta=1,2,\ldots,\aleph$, and Eq. (22b) for $\nu\in(0,1)$ are the basic equations on which we base our F_N solution.

THE F_N APPROXIMATION

We write our approximate solution for the exiting fluxes as

$$\boldsymbol{\Psi}(a,\mu)=\sum_{\alpha=0}^N \boldsymbol{a}_\alpha S_\alpha(\mu)\ , \quad (25)$$

where the $S_\alpha(\mu)$ denote a set of basis functions to be selected and the expansion vectors $\boldsymbol{a}_\alpha$ are to be determined, and we then substitute Eq. (25) into Eqs. (22) to obtain

$$\boldsymbol{N}^T(\nu_\beta)\sum_{\alpha=0}^N[\boldsymbol{B}_\alpha(\nu_\beta)+\boldsymbol{CA}_\alpha(\nu_\beta)\exp(-2a/\nu_\beta)]\boldsymbol{a}_\alpha=0 \quad (26a)$$

and

$$\boldsymbol{\Theta}(\nu)\boldsymbol{\omega}^{-1}(\nu)\sum_{\alpha=0}^N[\boldsymbol{B}_\alpha(\nu)+\boldsymbol{CA}_\alpha(\nu)\exp(-2a/\nu)]\boldsymbol{a}_\alpha=\mathbf{0} \quad (26b)$$

for $\nu\in(0,1)$. Here,

$$\boldsymbol{A}_\alpha(\xi)=\boldsymbol{\Sigma}\int_0^1 \mu\boldsymbol{\Theta}(\mu)\boldsymbol{\Pi}_\alpha(\mu)\frac{d\mu}{\mu+\xi}\ ,\quad \xi\notin[-1,0)\ , \quad (27)$$

$$\boldsymbol{B}_\alpha(\nu_\beta)=\boldsymbol{C\Sigma}\int_0^1 \mu\boldsymbol{\Theta}(\mu)\boldsymbol{\Pi}_\alpha(\mu)\frac{d\mu}{\nu_\beta-\mu}\ , \quad (28a)$$

and

$$\boldsymbol{B}_\alpha(\nu)=\boldsymbol{\omega}(\nu)\boldsymbol{\Sigma\Pi}_\alpha(\nu)+\boldsymbol{C\Sigma}P\int_0^1 \mu\boldsymbol{\Theta}(\mu)\boldsymbol{\Pi}_\alpha(\mu)\frac{d\mu}{\nu-\mu}\ ,\quad \nu\in(0,1)\ , \quad (28b)$$

where

$$\boldsymbol{\Pi}_\alpha(\mu)=\text{diag}\{S_\alpha(\sigma\mu),S_\alpha(\mu)\}\ . \quad (29)$$

To be specific we now express the null vector defined by Eq. (23) as

$$\boldsymbol{N}(\nu_\beta)=\begin{bmatrix}\Omega_{22}(\nu_\beta)-\Omega_{21}(\nu_\beta)\\ \Omega_{11}(\nu_\beta)-\Omega_{12}(\nu_\beta)\end{bmatrix}\ , \quad (30)$$

and we next multiply Eq. (26b) by $\omega(\nu)=\det\boldsymbol{\omega}(\nu)$ and rewrite Eqs. (26) as the three scalar equations

$$\sum_{\alpha=0}^N[\boldsymbol{X}_\alpha(\nu_\beta)+\boldsymbol{Y}_\alpha(\nu_\beta)\exp(-2a/\nu_\beta)]\boldsymbol{a}_\alpha=0\ , \quad (31a)$$

$$\sum_{\alpha=0}^N[\boldsymbol{X}_\alpha^{(1)}(\nu)+\boldsymbol{Y}_\alpha^{(1)}(\nu)\exp(-2a/\nu)]\boldsymbol{a}_\alpha=0 \quad (31b)$$

for $\nu\in(0,1/\sigma)$, and

$$\sum_{\alpha=0}^N[\boldsymbol{X}_\alpha^{(2)}(\nu)+\boldsymbol{Y}_\alpha^{(2)}(\nu)\exp(-2a/\nu)]\boldsymbol{a}_\alpha=0 \quad (31c)$$

for $\nu\in(0,1)$. Here,

$$\boldsymbol{Y}_\alpha(\nu_\beta)=\begin{bmatrix}c_{21}+c_{11}+\det\boldsymbol{C}L(\nu_\beta)\\ c_{12}+c_{22}+\dfrac{1}{\sigma}\det\boldsymbol{C}L(\sigma\nu_\beta)\end{bmatrix}^T \times\text{diag}\{F_\alpha(\sigma\nu_\beta),F_\alpha(\nu_\beta)\}\ , \quad (32a)$$

$$\boldsymbol{Y}_\alpha^{(1)}(\nu)=\begin{bmatrix}c_{11}+\det\boldsymbol{C}L(\nu)\\ c_{12}\end{bmatrix}^T \times\text{diag}\{F_\alpha(\sigma\nu),F_\alpha(\nu)\}\ , \quad (32b)$$

and

$$\boldsymbol{Y}_\alpha^{(2)}(\nu)=\begin{bmatrix}c_{21}\\ c_{22}+\dfrac{1}{\sigma}\det\boldsymbol{C}L(\sigma\nu)\end{bmatrix}^T \times\text{diag}\{F_\alpha(\sigma\nu),F_\alpha(\nu)\}\ , \quad (32c)$$

where

$$L(\xi)=\xi\ln\left|\frac{1-\xi}{1+\xi}\right|\ ,\quad \xi\ \text{real}\ , \quad (33)$$

$$L(i|\nu_\beta|)=-2|\nu_\beta|\tan^{-1}\left(\frac{1}{|\nu_\beta|}\right)\ , \quad (34)$$

and

$$F_\alpha(\xi) = \int_0^1 \mu S_\alpha(\mu)\, \frac{d\mu}{\mu+\xi} \quad , \quad \xi \notin [-1,0) \ . \qquad (35)$$

In addition,

$$X_\alpha(\nu_\beta) = \begin{bmatrix} c_{21} + c_{11} + \det CL(\nu_\beta) \\ c_{12} + c_{22} + \dfrac{1}{\sigma}\det CL(\sigma\nu_\beta) \end{bmatrix}^T \times \mathrm{diag}\{-F_\alpha(-\sigma\nu_\beta), -F_\alpha(-\nu_\beta)\} \ , \qquad (36a)$$

$$X_\alpha^{(1)}(\nu) = \omega(\nu)\begin{bmatrix} \sigma S_\alpha(\sigma\nu) \\ 0 \end{bmatrix}^T + \begin{bmatrix} c_{11} + \det CL(\nu) \\ c_{12} \end{bmatrix}^T \times \mathrm{diag}\{G_\alpha(\sigma\nu), G_\alpha(\nu)\} \ , \qquad (36b)$$

and

$$X_\alpha^{(2)}(\nu) = \omega(\nu)\begin{bmatrix} 0 \\ S_\alpha(\nu) \end{bmatrix}^T + \begin{bmatrix} c_{21} \\ c_{22} + \dfrac{1}{\sigma}\det CL(\sigma\nu) \end{bmatrix}^T \times \mathrm{diag}\{G_\alpha(\sigma\nu), G_\alpha(\nu)\} \ , \qquad (36c)$$

where

$$G_\alpha(\xi) = P\int_0^1 \mu S_\alpha(\mu)\, \frac{d\mu}{\xi-\mu} \ . \qquad (37)$$

Here we use the shifted Legendre polynomials as our basis functions, and thus in the accompanying Appendix we discuss our procedure for evaluating the required $F_\alpha(\xi)$ and $G_\alpha(\xi)$ for the case $S_\alpha(\mu) = P_\alpha(2\mu - 1)$.

Considering that $F_\alpha(\xi)$ and $G_\alpha(\xi)$ are available, we now wish to define a collocation strategy and to solve Eqs. (31) to find the critical half-thickness a. It is clear that Eq. (31a) provides one equation for $\aleph = 1$ and two equations for $\aleph = 2$. We therefore use

$$\xi_\beta = \frac{1}{2\sigma}\left[1 + \cos\left(\frac{\beta\pi}{N+1}\right)\right] \quad , \quad \beta = 1,2,\ldots,N \quad (38a)$$

and

$$\zeta_\beta = \frac{1}{2}\left[1 + \cos\left(\frac{\beta+2-\aleph}{N+3-\aleph}\,\pi\right)\right] \ , \quad \beta = \aleph-1, \aleph, \ldots, N \ , \qquad (38b)$$

and consider

$$\sum_{\alpha=0}^{N} [X_\alpha(\nu_\beta) + Y_\alpha(\nu_\beta)\exp(-2a/\nu_\beta)]\,a_\alpha = 0 \ , \quad \beta = 1,2,\ldots,\aleph \ , \qquad (39a)$$

$$\sum_{\alpha=0}^{N} [X_\alpha^{(1)}(\xi_\beta) + Y_\alpha^{(1)}(\xi_\beta)\exp(-2a/\xi_\beta)]\,a_\alpha = 0 \ , \quad \beta = 1,2,\ldots,N \ , \qquad (39b)$$

and

$$\sum_{\alpha=0}^{N} [X_\alpha^{(2)}(\zeta_\beta) + Y_\alpha^{(2)}(\zeta_\beta)\exp(-2a/\zeta_\beta)]\,a_\alpha = 0 \ , \quad \beta = \aleph-1, \aleph, \ldots, N \ . \qquad (39c)$$

Equations (39) provide exactly $2(N+1)$ scalar equations. It is clear that we must not use a value of N in Eq. (38b) that would yield $\zeta_\beta = 1/\sigma$ for any appropriate value of β.

We note that since $k_{BMS} > 1$ we have at least one purely imaginary eigenvalue $\nu_1 = i|\nu_1|$ and that if $\aleph = 2$ the second eigenvalue can be real, infinite, or purely imaginary.[6] At this point we normalize our solution by taking $a_{0,1} = -1$ and writing Eq. (25) as

$$\Psi(a,\mu) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + \sum_{\alpha=0}^{N} A_\alpha P_\alpha(2\mu - 1) \ , \qquad (40)$$

where

$$A_\alpha = \begin{bmatrix} (1-\delta_{\alpha,0})a_{\alpha,1} \\ a_{\alpha,2} \end{bmatrix} \ . \qquad (41)$$

We now assume an initial value of a and solve the following rearrangement of Eqs. (39) to find A_α, $\alpha = 0,1,\ldots,N$:

$$\sum_{\alpha=0}^{N} [X_\alpha(\nu_2) + Y_\alpha(\nu_2)\exp(-2a/\nu_2)]A_\alpha = [X_0(\nu_2) + Y_0(\nu_2)\exp(-2a/\nu_2)]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \qquad (42a)$$

if $\aleph = 2$,

$$\sum_{\alpha=0}^{N} [X_\alpha^{(1)}(\xi_\beta) + Y_\alpha^{(1)}(\xi_\beta)\exp(-2a/\xi_\beta)]A_\alpha = [X_0^{(1)}(\xi_\beta) + Y_0^{(1)}(\xi_\beta)\exp(-2a/\xi_\beta)]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad (42b)$$

for $\beta = 1,2,\ldots,N$, and

$$\sum_{\alpha=0}^{N} [X_\alpha^{(2)}(\zeta_\beta) + Y_\alpha^{(2)}(\zeta_\beta)\exp(-2a/\zeta_\beta)]A_\alpha = [X_0^{(2)}(\zeta_\beta) + Y_0^{(2)}(\zeta_\beta)\exp(-2a/\zeta_\beta)]\begin{bmatrix} 1 \\ 0 \end{bmatrix} \ , \qquad (42c)$$

for $\beta = \aleph - 1, \aleph, \ldots, N$. We next find an improved result for a from

$$\exp(-2a/\nu_1) = -\left[\sum_{\alpha=0}^{N} \boldsymbol{Y}_\alpha(\nu_1)\boldsymbol{A}_\alpha\right]^{-1} \sum_{\alpha=0}^{N} \boldsymbol{X}_\alpha(\nu_1)\boldsymbol{A}_\alpha \ , \tag{43}$$

and we repeat the process until a converged result for a is found.

NUMERICAL RESULTS

For our numerical work we consider the seven data cases that were defined by Forster[10,11] and Boffi and Premuda[12] and used by Kriese et al.[7] Thus we write $\sigma = \sigma_1/\sigma_2$ and $c_{ij} = \sigma_{ij}/(2\sigma_2)$ and use

$$\sigma_{11} = \sigma_{11s} + (1-\chi)\bar{\nu}_1\sigma_{1f} \ , \tag{44a}$$

$$\sigma_{12} = \sigma_{12s} + (1-\chi)\bar{\nu}_2\sigma_{2f} \ , \tag{44b}$$

$$\sigma_{21} = \chi\bar{\nu}_1\sigma_{1f} \ , \tag{44c}$$

and

$$\sigma_{22} = \sigma_{22s} + \chi\bar{\nu}_2\sigma_{2f} \tag{44d}$$

for the data of Forster[10,11] given in Table I and $\sigma_1 = a_2/b, \sigma_2 = a_1/b$ and

$$\sigma_{11} = \sigma_1 C_A(2 \to 2) \ , \tag{45a}$$

$$\sigma_{12} = \sigma_1 C_A(1 \to 2) \ , \tag{45b}$$

$$\sigma_{21} = \sigma_2 C_A(2 \to 1) \ , \tag{45c}$$

and

$$\sigma_{22} = \sigma_2 C_A(1 \to 1) \tag{45d}$$

for the data sets of Boffi and Premuda[12] given in Table II to define our basic data σ and $\boldsymbol{C}$. It is important to note that we assume the data in Tables I and II to be exact.

Given the data from Table I or II, we first compute the discrete eigenvalue (n.b., $\aleph = 1$) by using Newton's method to solve $\det \boldsymbol{\Lambda}(\nu_1) = 0$. Then we solve Eqs. (42) and (43) in the manner mentioned to find the critical half-thickness a for a given order of the approximation. In Table III we report our results for critical half-thicknesses along with reference values thought to be correct to six significant figures (based on the calculations of Ref. 7 and our F_N results for large N). Since, as discussed for example in Ref. 7, the critical sphere problem requires only that Eq. (4) be changed to read

$$\boldsymbol{\Psi}(-a,\mu) = -\boldsymbol{\Psi}(a,-\mu) \ , \quad \mu \in [0,1] \ , \tag{46}$$

and that we interpret a as the critical radius, we have incorporated the relevant minus sign in our developed equations, and thus we list in Table IV our results for critical radii for spheres.

CONCLUSIONS

It is thought important to notice that, for the first time, the F_N method has been used to solve a class of two-group problems in neutron-transport theory that is based on a full scattering matrix (both up- and

TABLE I

Two-Group Macroscopic Cross Sections (in cm^{-1})*

Case	σ_1	σ_{11s}	$\bar{\nu}_1\sigma_{1f}$	σ_{12s}	σ_2	σ_{22s}	$\bar{\nu}_2\sigma_{2f}$	χ
I	0.54628	0.4241	0.2425	0.0045552	0.33588	0.3198	0.0070425	1.0
II	2.52025	2.44383	0.12658	0.029227	0.65696	0.62568	0.002621	1.0
III	0.3456	0.26304	0.1728	0.072	0.216	0.07824	0.167184	0.575
IV	0.336	0.23616	0.2503392	0.0432	0.2208	0.0792	0.29016	0.575

*From Refs. 10 and 11.

TABLE II

Two-Group Parameters (Ref. 12)

Case	b (cm)	a_1	a_2	$C_A(1\to1)$	$C_A(1\to2)$	$C_A(2\to2)$	$C_A(2\to1)$
V	7.831	2.1	10	0.923	0.016	0.95	0.640648
VI	0.7831	0.21	1	0.923	0.016	0.95	31.5311
VII	0.07831	0.021	0.1	0.923	0.016	0.95	1917.83

TABLE III

Critical Half-Thicknesses for Slabs

Case	$N=2$	$N=5$	$N=10$	$N=30$	Reference
I	2.84366(2)[a]	2.84367(2)	2.84367(2)	2.84367(2)	2.84367(2)
II	4.97076	4.97111	4.97111	4.97112	4.97112
III	6.47548(−1)	6.49382(−1)	6.49377(−1)	6.49377(−1)	6.49377(−1)
IV	3.94159(−1)	3.96460(−1)	3.96468(−1)	3.96469(−1)	3.96469(−1)
V	2.09919	2.10008	2.09994	2.09994	2.09994
VI	2.11583(−1)	2.09635(−1)	2.10011(−1)	2.10000(−1)	2.10000(−1)
VII	1.93485(−2)	2.18713(−2)	2.09781(−2)	2.10000(−2)	2.10000(−2)

[a]Read as 2.84366×10^2.

TABLE IV

Critical Radii for Spheres

Case	$N=2$	$N=5$	$N=10$	$N=30$	Reference
I	5.69429(2)[a]	5.69430(2)	5.69430(2)	5.69430(2)	5.69430(2)
II	1.05437(1)	1.05441(1)	1.05441(1)	1.05441(1)	1.05441(1)
III	1.70658	1.70844	1.70844	1.70844	1.70844
IV	1.15235	1.15513	1.15513	1.15513	1.15513
V	4.73730	4.73798	4.73785	4.73786	4.73786
VI	6.22797(−1)	6.19421(−1)	6.19651(−1)	6.19651(−1)	6.19651(−1)
VII	9.78106(−2)	9.95800(−2)	9.93264(−2)	9.92464(−2)	9.92464(−2)

[a]Read as 5.69429×10^2.

downscattering). In our numerical work we have, for the sake of having a comparison with known high-quality results,[7,10,12] considered here only the critical problems for slabs and spheres; however, it is clear that subcritical problems (with or without inhomogeneous source terms) can readily be solved with the developed formalism. We note that Malvagi and Pomraning[13] have generalized the given development of the F_N method for fully coupled systems to solve, with the F_N method, a basic problem relevant to transport in evacuated ducts.

With the exception of the calculation of the discrete spectrum, the development reported here does not rely on the fact that only two groups were considered, and so we expect that the analysis required for the extension to the case of a fully coupled multigroup model can be carried out without fundamental difficulties. A work that defines an efficient scheme for computing the discrete spectrum for multigroup transport theory is being completed and will be reported.[14]

As a matter of interest, we have observed that our F_N solution of the considered critical problems for slabs and spheres is considerably easier to implement and requires much less computer time than the exact analysis reported in Ref. 7. We are hopeful that soon we will have a generalization to a general multigroup model so that, in the spirit of Westfall,[15] we will be able to provide large-code developers a set of high-quality results that can be used as test cases.

APPENDIX

As the Legendre polynomials can be computed accurately by forward recursion, i.e., $P_0(\xi) = 1$ and

$$P_{l+1}(\xi) = \frac{1}{l+1}\left[(2l+1)\xi P_l(\xi) - lP_{l-1}(\xi)\right] \quad \text{(A.1)}$$

for $l = 0, 1, \ldots$, we focus our attention here on recording our way of computing accurately the functions

$$F_\alpha(\xi) = \int_0^1 \mu P_\alpha(2\mu - 1)\,\frac{d\mu}{\mu+\xi}\ , \quad \xi \notin [-1,0)\ , \quad \text{(A.2)}$$

and

$$G_\alpha(\xi) = P\int_0^1 \mu P_\alpha(2\mu - 1)\,\frac{d\mu}{\xi-\mu}\ . \quad \text{(A.3)}$$

Since $G_\alpha(\xi) = -F_\alpha(-\xi)$, $\xi \notin (0,1]$, we require only $F_\alpha(\xi)$, $\xi \notin [-1,0)$, and

$$G_\alpha(\nu) = P\int_0^1 \mu P_\alpha(2\mu - 1)\frac{d\mu}{\nu - \mu}\ , \quad \nu \in (0,1)\ . \tag{A.4}$$

For real $\xi \notin [-1,0)$ we use the schemes reported in Refs. 4 and 16 to compute $F_\alpha(\xi)$, viz., for $\xi \in [-1.001,-1) \cup [0,0.001]$ we use

$$F_0(\xi) = 1 - \xi\ln(1 + 1/\xi) \tag{A.5}$$

and forward recursion, i.e.,

$$F_{\alpha+1}(\xi) = \frac{1}{\alpha+1}[-(2\alpha+1)(2\xi+1)F_\alpha(\xi) - \alpha F_{\alpha-1}(\xi) + \delta_{\alpha,0} + \delta_{\alpha,1}] \tag{A.6}$$

for $\alpha = 0,1,\ldots,N-1$. For real $\xi \notin [-1.001,0.001]$ we use backward recursion, i.e., for some $L > N$ we take

$$R_L(\xi) = 0 \tag{A.7}$$

and, for $\alpha = L, L-1, \ldots, 2$,

$$R_{\alpha-1}(\xi) = -\left(\frac{\alpha}{\alpha+1}\right) \times \left[\left(\frac{2\alpha+1}{\alpha+1}\right)(2\xi+1) + R_\alpha(\xi)\right]^{-1}\ . \tag{A.8}$$

We then use

$$F_1(\xi) = \xi[(2\xi+1)\ln(1+1/\xi) - 2] \tag{A.9}$$

and

$$F_{\alpha+1}(\xi) = R_\alpha(\xi)F_\alpha(\xi) \tag{A.10}$$

for $\alpha = 1,2,\ldots,N-1$. Finally we increase L and repeat the calculation until the $F_\alpha(\xi)$, for $\alpha = 2,3,\ldots,N$, have converged to the desired degree of accuracy. For imaginary values of ξ we use backward recursion for $|\xi| > 0.001$ and forward recursion otherwise. We note that

$$F_0(i|x|) = 1 - |x|\tan^{-1}\frac{1}{|x|} - i\frac{1}{2}|x|\ln(1 + 1/|x|^2) \tag{A.11}$$

and

$$F_1(i|x|) = i|x|\left\{-2 + \frac{1}{2}\ln(1 + 1/|x|^2) + 2|x|\tan^{-1}\frac{1}{|x|} + i\left[|x|\ln(1+1/|x|^2) - \tan^{-1}\frac{1}{|x|}\right]\right\}\ . \tag{A.12}$$

We do not require $F_\alpha(\xi)$ for values of ξ that are neither real nor purely imaginary.

We find that we can use forward recursion to compute $G_\alpha(\nu)$ for $\nu \in (0,1)$. Thus we start with

$$G_0(\nu) = -1 - \nu\ln\left(\frac{1-\nu}{\nu}\right) \tag{A.13}$$

and then use

$$G_{\alpha+1}(\nu) = \frac{1}{\alpha+1}\left[(2\alpha+1)(2\nu-1)G_\alpha(\nu) - \alpha G_{\alpha-1}(\nu) - \delta_{\alpha,0} - \delta_{\alpha,1}\right] \tag{A.14}$$

for $\alpha = 0,1,\ldots,N-1$.

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