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THE DISCRETE SPECTRUM FOR RADIATIVE TRANSFER WITH POLARIZATION

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Abstract—Elementary considerations are used to define and analyze the discrete spectrum for a general radiative transfer model that includes polarization effects

INTRODUCTION

In regard to radiative transfer models that include polarization effects, we consider the vector equation of transfer[1]

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau, \mu, \phi) + \boldsymbol{I}(\tau, \mu, \phi) = \frac{\varpi}{4\pi} \int_0^{2\pi} \int_{-1}^{1} \boldsymbol{P}(\mu, \mu', \phi - \phi') \boldsymbol{I}(\tau, \mu', \phi') \, d\mu' \, d\phi' \tag{1}$$

where the Stokes vector $\boldsymbol{I}(\tau, \mu, \phi)$ has the four Stokes parameters $I(\tau, \mu, \phi)$, $Q(\tau, \mu, \phi)$, $U(\tau, \mu, \phi)$ and $V(\tau, \mu, \phi)$ as components Here $\varpi \in (0, 1]$ is the albedo for single scattering and $\boldsymbol{P}(\mu, \mu', \phi - \phi')$ is the phase matrix As noted in previous works[2-5] we can use the analytical representation

$$\boldsymbol{P}(\mu, \mu', \phi - \phi') = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) [\boldsymbol{C}^m(\mu, \mu') \cos m(\phi - \phi') + \boldsymbol{S}^m(\mu, \mu') \sin m(\phi - \phi')] \tag{2}$$

to carry out an analytical Fourier decomposition of the Stokes vector $\boldsymbol{I}(\tau, \mu, \phi)$ Here

$$\boldsymbol{C}^m(\mu, \mu') = \boldsymbol{A}^m(\mu, \mu') + \boldsymbol{D}\boldsymbol{A}^m(\mu, \mu')\boldsymbol{D} \tag{3a}$$

and

$$\boldsymbol{S}^m(\mu, \mu') = \boldsymbol{A}^m(\mu, \mu')\boldsymbol{D} - \boldsymbol{D}\boldsymbol{A}^m(\mu, \mu') \tag{3b}$$

where

$$\boldsymbol{A}^m(\mu, \mu') = \sum_{l=m}^{L} \boldsymbol{\Pi}_l^m(\mu) \boldsymbol{B}_l \boldsymbol{\Pi}_l^m(\mu') \tag{4}$$

and $\boldsymbol{D} = \mathrm{diag}\{1, 1, -1, -1\}$ In addition

$$\boldsymbol{\Pi}_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} \begin{bmatrix} P_l^m(\mu) & 0 & 0 & 0 \\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0 \\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0 \\ 0 & 0 & 0 & P_l^m(\mu) \end{bmatrix} \tag{5}$$

where

$$P_l^m(\mu) = (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu) \tag{6}$$

is used to denote the associated Legendre functions and the functions $R_l^m(\mu)$ and $T_l^m(\mu)$ are as defined and used in Refs [2] and [3] We note that $\tau \in [0, \tau_0]$ is the optical variable, μ is the direction

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cosine of the propagating radiation and that the scattering law is defined by the *Greek constants* $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \epsilon_l, \zeta_l\}$ so that

$$B_l = \begin{bmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\epsilon_l \\ 0 & 0 & \epsilon_l & \delta_l \end{bmatrix} \tag{7}$$

Referring now to some of our previous work,[2,3] we note that in carrying out the Fourier decomposition of the Stokes vector $\boldsymbol{I}(\tau, \mu, \phi)$ we find that the Fourier components can be expressed in terms of solutions to the vector equations

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau, \mu) + \boldsymbol{I}(\tau, \mu) = \frac{\varpi}{2} \int_{-1}^{1} \boldsymbol{K}^m(\mu' \to \mu) \boldsymbol{I}(\tau, \mu') \, d\mu' \tag{8}$$

for $m = 0, 1, \ldots, L$. Here the scattering kernel is

$$\boldsymbol{K}^m(\mu' \to \mu) = \sum_{l=m}^{L} \boldsymbol{\Pi}_l^m(\mu) \boldsymbol{B}_l \boldsymbol{\Pi}_l^m(\mu') \tag{9}$$

We consider here only $L \geqslant 2$ since equation (1) for $L < 2$ reduces to uncoupled equations that have been considered previously.[6,7]

In this work we focus our attention on the Fourier components of $\boldsymbol{I}(\tau, \mu, \phi)$ and report some observations concerning the discrete spectrum basic to equation (8). Since one of the fundamental aspects of developing exact or approximate solutions of equation (8), say by the method of elementary solutions[8] or by the F_N method,[9] is a computation of the discrete spectrum, we develop here the dispersion function appropriate to studies of equation (8), and we prove that the zeros of this dispersion function can be approximated by the zeros of certain polynomials. Having in mind a completeness proof[8] for the elementary solutions of equation (8), we also prove that the boundary values of the dispersion matrix cannot be singular on a certain subset of the real axis.

ELEMENTARY SOLUTIONS

In order to define the discrete spectrum for the Fourier-component problems, we first substitute the proposed solution

$$\boldsymbol{I}_\xi(\tau, \mu) = \boldsymbol{\Phi}(\xi, \mu) \exp\{-\tau/\xi\} \tag{10}$$

into

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau, \mu) + \boldsymbol{I}(\tau, \mu) = \frac{\varpi}{2} \int_{-1}^{1} \boldsymbol{K}^m(\mu' \to \mu) \boldsymbol{I}(\tau, \mu') \, d\mu' \tag{11}$$

to find

$$(\xi - \mu) \boldsymbol{\Phi}(\xi, \mu) = \frac{\varpi \xi}{2} \sum_{k=m}^{L} \boldsymbol{\Pi}_k^m(\mu) \boldsymbol{B}_k \boldsymbol{G}_k^m(\xi) \boldsymbol{M}(\xi) \tag{12}$$

where we have defined

$$\boldsymbol{G}_k^m(\xi) \boldsymbol{M}(\xi) = \int_{-1}^{1} \boldsymbol{\Pi}_k^m(\mu) \boldsymbol{\Phi}(\xi, \mu) \, d\mu \tag{13}$$

Now upon multiplying equation (12) by $\boldsymbol{\Pi}_l^m(\mu)$, integrating over μ from -1 to 1 and using the fact[3] that

$$[(2l+1)\xi \boldsymbol{I} + \boldsymbol{V}_l^m] \boldsymbol{\Pi}_l^m(\xi) = \boldsymbol{U}_{l+1}^m \boldsymbol{\Pi}_{l+1}^m(\xi) + \boldsymbol{U}_l^m \boldsymbol{\Pi}_{l-1}^m(\xi) \tag{14}$$

where

$$\boldsymbol{U}_l^m = (l^2 - m^2)^{1/2} \operatorname{diag}\{1, l^{-1}(l^2-4)^{1/2}, l^{-1}(l^2-4)^{1/2}, 1\} \tag{15a}$$

and

$$V_l^m = \frac{2m(2l+1)}{l(l+1)} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{15b}$$

we find

$$[\boldsymbol{U}_l^m \boldsymbol{G}_{l-1}^m(\xi) - (\xi \boldsymbol{h}_l + \boldsymbol{V}_l^m)\boldsymbol{G}_l^m(\xi) + \boldsymbol{U}_{l+1}^m \boldsymbol{G}_{l+1}^m(\xi)]\boldsymbol{M}(\xi) = \boldsymbol{0} \tag{16}$$

where

$$\boldsymbol{h}_l = (2l+1)\boldsymbol{I} - \varpi \boldsymbol{B}_l \tag{17}$$

Considering that $\xi \notin [-1, 1]$, we solve equation (12) to find

$$\boldsymbol{\Phi}(\xi, \mu) = \frac{\varpi\xi}{2}\left(\frac{1}{\xi - \mu}\right)\sum_{k=m}^{L} \boldsymbol{\Pi}_k^m(\mu)\boldsymbol{B}_k \boldsymbol{G}_k^m(\xi)\boldsymbol{M}(\xi), \tag{18}$$

and after multiplying equation (18) by $\boldsymbol{\Pi}_l^m(\mu)$ and integrating over μ, we find

$$\left[\boldsymbol{G}_l^m(\xi) + \frac{\varpi\xi}{2}\int_{-1}^{1} \boldsymbol{\Pi}_l^m(\mu)\boldsymbol{G}(\xi, \mu)\frac{\mathrm{d}\mu}{\mu - \xi}\right]\boldsymbol{M}(\xi) = \boldsymbol{0} \tag{19}$$

where

$$\boldsymbol{G}(\xi, \mu) = \sum_{l=m}^{L} \boldsymbol{\Pi}_l^m(\mu)\boldsymbol{B}_l \boldsymbol{G}_l^m(\xi) \tag{20}$$

As we would like to use the matrices $\boldsymbol{G}_l^m(\xi)$ without restrictions on ξ, we follow Ref [5] and consider equation (16) without the factor $\boldsymbol{M}(\xi)$ We therefore take the matrices $\boldsymbol{G}_l^m(\xi)$ to be defined by

$$\boldsymbol{G}_{l+1}^m(\xi) = (\boldsymbol{U}_{l+1}^m)^{-1}[(\xi\boldsymbol{h}_l + \boldsymbol{V}_l^m)\boldsymbol{G}_l^m(\xi) - \boldsymbol{U}_l^m \boldsymbol{G}_{l-1}^m(\xi)] \tag{21}$$

where $l = 2, 3, \ldots$, for $m = 0$ and 1, and $l = m, m+1, \ldots$, for $m \geqslant 2$ Here, in contrast to Ref [5], we use the starting values

$$\boldsymbol{G}_0^0(\xi) = \mathrm{diag}\{1, 0, 0, 1\} \tag{22a}$$

$$\boldsymbol{G}_1^0(\xi) = \mathrm{diag}\{(1-\varpi)\xi, 0, 0, (1-\varpi\delta_0)\xi\} \tag{22b}$$

$$\boldsymbol{G}_2^0(\xi) = \mathrm{diag}\{\tfrac{1}{2}[(1-\varpi)(3-\varpi\beta_1)\xi^2 - 1], 1, 1, \tfrac{1}{2}[(1-\varpi\delta_0)(3-\varpi\delta_1)\xi^2 - 1]\}, \tag{22c}$$

$$\boldsymbol{G}_1^1(\xi) = 2^{-1/2}\,\mathrm{diag}\{1, 0, 0, 1\} \tag{22d}$$

$$\boldsymbol{G}_2^1(\xi) = \mathrm{diag}\{6^{-1/2}(3-\varpi\beta_1)\xi, 1, 1, 6^{-1/2}(3-\varpi\delta_1)\xi\} \tag{22e}$$

and, for $m \geqslant 2$,

$$\boldsymbol{G}_m^m(\xi) = \boldsymbol{\Delta}_m = S_m\,\mathrm{diag}\{1, R_m, R_m, 1\} \tag{22f}$$

where

$$S_m = (2m-1)!![(2m)!]^{-1/2} \tag{23}$$

and

$$R_m = \left[\frac{m(m-1)}{(m+2)(m+1)}\right]^{1/2} \tag{24}$$

We can now use equations (14) and (21) and the fact that

$$\int_{-1}^{1} \boldsymbol{\Pi}_l^m(\mu)\boldsymbol{\Pi}_l^m(\mu)\,\mathrm{d}\mu = \left(\frac{2}{2l+1}\right)\boldsymbol{\Delta}_{l\,l} \tag{25}$$

where

$$\boldsymbol{\Delta}_{l\,l} = \delta_{l\,l}\,\mathrm{diag}\{1, (1-\delta_{0\,l})(1-\delta_{1\,l}), (1-\delta_{0\,l})(1-\delta_{1\,l}), 1\} \tag{26}$$

to show that

$$\boldsymbol{W}_l^m(\xi) = \boldsymbol{G}_l^m(\xi) + \frac{\varpi\xi}{2}\int_{-1}^{1} \boldsymbol{\Pi}_l^m(\mu)\boldsymbol{G}(\xi,\mu)\frac{d\mu}{\mu-\xi} \tag{27}$$

satisfies

$$[(2l+1)\xi\boldsymbol{I} + \boldsymbol{V}_l^m]\boldsymbol{W}_l^m(\xi) = \boldsymbol{U}_{l+1}^m\boldsymbol{W}_{l+1}^m(\xi) + \boldsymbol{U}_l^m\boldsymbol{W}_{l-1}^m(\xi) \tag{28}$$

Considering first of all the cases $m \geqslant 2$, we can write

$$\boldsymbol{W}_l^m(\xi) = \boldsymbol{\Pi}_l^m(\xi)[\boldsymbol{\Pi}_m^m(\xi)]^{-1}\boldsymbol{W}_m^m(\xi), \quad \xi \notin [-1, 1], \tag{29}$$

$l = m, m+1, \ldots$. Using equations (22f) and (29) and the fact[3] that for $m \geqslant 2$,

$$\boldsymbol{\Pi}_m^m(\mu) = \boldsymbol{\Delta}_m\boldsymbol{\Xi}^m(\mu) \tag{30}$$

where

$$\boldsymbol{\Xi}^m(\mu) = (1-\mu^2)^{m/2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{1+\mu^2}{1-\mu^2} & -\dfrac{2\mu}{1-\mu^2} & 0 \\ 0 & -\dfrac{2\mu}{1-\mu^2} & \dfrac{1+\mu^2}{1-\mu^2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{31}$$

we can rewrite equation (19) as

$$\boldsymbol{\Lambda}^m(\xi)\boldsymbol{M}(\xi) = \boldsymbol{0} \tag{32}$$

where, in general,

$$\boldsymbol{\Lambda}^m(\xi) = \boldsymbol{I} + \frac{\varpi\xi}{2}\int_{-1}^{1}\boldsymbol{\Xi}^m(\mu)\boldsymbol{G}(\xi,\mu)\frac{d\mu}{\mu-\xi} \tag{33}$$

The cases $m = 0$ and $m = 1$ require separate considerations since the matrices $\boldsymbol{\Pi}_0^0(\xi)$, $\boldsymbol{\Pi}_1^0(\xi)$ and $\boldsymbol{\Pi}_1^1(\xi)$ are not invertible. For $m = 0$ we find we can write, for $\xi \notin [-1, 1]$ and $l = 0, 1, \ldots$,

$$\boldsymbol{W}_l^0(\xi) = \boldsymbol{\Pi}_l^0(\xi)[\boldsymbol{\Pi}_2^0(\xi)]^{-1}\mathrm{diag}\{P_2(\xi), 1, 1, P_2(\xi)\}\boldsymbol{\Lambda}^0(\xi) \tag{34}$$

where we have defined

$$\boldsymbol{\Lambda}^0(\xi) = \boldsymbol{I} + \frac{\varpi\xi}{2}\int_{-1}^{1}\boldsymbol{\Xi}^0(\mu)\boldsymbol{G}(\xi,\mu)\frac{d\mu}{\mu-\xi} \tag{35}$$

and

$$\boldsymbol{\Xi}^0(\mu) = \mathrm{diag}\{1, R_2^0(\mu), R_2^0(\mu), 1\} \tag{36}$$

In a similar manner we find for $m = 1$ that we can write, for $\xi \notin [-1, 1]$ and $l = 1, 2, \ldots$,

$$\boldsymbol{W}_l^1(\xi) = \boldsymbol{\Pi}_l^1(\xi)[\boldsymbol{\Pi}_2^1(\xi)]^{-1}\mathrm{diag}\{\tfrac{1}{2}\xi 6^{1/2}, 1, 1, \tfrac{1}{2}\xi 6^{1/2}\}\boldsymbol{\Lambda}^1(\xi) \tag{37}$$

where

$$\boldsymbol{\Lambda}^1(\xi) = \boldsymbol{I} + \frac{\varpi\xi}{2}\int_{-1}^{1}\boldsymbol{\Xi}^1(\mu)\boldsymbol{G}(\xi,\mu)\frac{d\mu}{\mu-\xi} \tag{38}$$

with

$$\boldsymbol{\Xi}^1(\mu) = (1-\mu^2)^{1/2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\dfrac{\mu}{2} & \dfrac{1}{2} & 0 \\ 0 & \dfrac{1}{2} & -\dfrac{\mu}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

It now follows that we can define the discrete spectrum for each Fourier-component problem as those values of $\xi \notin [-1, 1]$ such that

$$\Lambda^m(\xi)\boldsymbol{M}(\xi) = \boldsymbol{0} \tag{40}$$

Clearly ξ must be a zero of $\det \Lambda^m(\xi)$, and $\boldsymbol{M}(\xi)$ must be a resulting null vector of $\Lambda^m(\xi)$

BASIC IDENTITIES

We now follow a procedure used by Inonu,[6] Garcia and Siewert[7] and Siewert and Thomas[10] and develop a set of identities that will prove helpful in our analysis of the discrete spectrum for this polarization problem First of all we introduce, for $l \geqslant m$

$$\boldsymbol{Q}_l^m(\xi) = \frac{1}{2}\int_{-1}^{1} \boldsymbol{\Xi}^m(\mu)\boldsymbol{\Pi}_l^m(\mu)\frac{d\mu}{\xi - \mu} \tag{41}$$

and rewrite equations (33), (35) and (38) as

$$\Lambda^m(\xi) = \boldsymbol{I} - \varpi\xi \sum_{l=m}^{L} \boldsymbol{Q}_l^m(\xi)\boldsymbol{B}_l\boldsymbol{G}_l^m(\xi) \tag{42}$$

We can multiply equation (41) by $(2l + 1)\xi$ and use equation (14) to obtain

$$[(2l + 1)\xi\boldsymbol{I} + \boldsymbol{V}_l^m]\boldsymbol{Q}_l^m(\xi) = \boldsymbol{U}_{l+1}^m\boldsymbol{Q}_{l+1}^m(\xi) + \boldsymbol{U}_l^m\boldsymbol{Q}_{l-1}^m(\xi) + \boldsymbol{K}_l^m \tag{43}$$

where

$$\boldsymbol{K}_l^0 = \operatorname{diag}\{\delta_{0,l}, \delta_{2,l}, \delta_{2,l}, \delta_{0,l}\}, \tag{44a}$$

$$\boldsymbol{K}_l^1 = \operatorname{diag}\{2^{1/2}\delta_{1,l}, \delta_{2,l}, \delta_{2,l}, 2^{1/2}\delta_{1,l}\} \tag{44b}$$

and

$$\boldsymbol{K}_l^m = \delta_{l,m}\boldsymbol{\Delta}_m^{-1}, \quad m \geqslant 2 \tag{44c}$$

Noting that $\boldsymbol{Q}_l^m(\xi)$ is symmetric, we first multiply the transpose of equation (43) on the right by $\boldsymbol{G}_l^m(\xi)$, we then multiply

$$(\xi\boldsymbol{h}_l + \boldsymbol{V}_l^m)\boldsymbol{G}_l^m(\xi) = \boldsymbol{U}_{l+1}^m\boldsymbol{G}_{l+1}^m(\xi) + \boldsymbol{U}_l^m\boldsymbol{G}_{l-1}^m(\xi) \tag{45}$$

by $\boldsymbol{Q}_l^m(\xi)$ on the left, subtract the two equations, one from the other, and then sum the resulting equation from $l = m$ to $l = L$ to obtain, after we note equations (42) and (44),

$$\Lambda^m(\xi) = \boldsymbol{U}_{L+1}^m[\boldsymbol{Q}_L^m(\xi)\boldsymbol{G}_{L+1}^m(\xi) - \boldsymbol{Q}_{L+1}^m(\xi)\boldsymbol{G}_L^m(\xi)] \tag{46}$$

for all $m \geqslant 0$ We carry out a similar elimination between equation (45) and

$$[(2l + 1)\xi\boldsymbol{I} + \boldsymbol{V}_l^m]\boldsymbol{\Pi}_l^m(\xi) = \boldsymbol{U}_{l+1}^m\boldsymbol{\Pi}_{l+1}^m(\xi) + \boldsymbol{U}_l^m\boldsymbol{\Pi}_{l-1}^m(\xi) \tag{47}$$

to find

$$\xi\boldsymbol{\Psi}^m(\xi) = \tfrac{1}{2}\boldsymbol{\Xi}^m(\xi)\boldsymbol{U}_{L+1}^m[\boldsymbol{\Pi}_{L+1}^m(\xi)\boldsymbol{G}_L^m(\xi) - \boldsymbol{\Pi}_L^m(\xi)\boldsymbol{G}_{L+1}^m(\xi)] \tag{48}$$

where

$$\boldsymbol{\Psi}^m(\xi) = \frac{\varpi}{2}\boldsymbol{\Xi}^m(\xi)\boldsymbol{G}(\xi, \xi) \tag{49}$$

We also find, from equations (43) and (47), that

$$\boldsymbol{\Xi}^m(\xi) = \boldsymbol{U}_{L+1}^m[\boldsymbol{\Pi}_{L+1}^m(\xi)\boldsymbol{Q}_L^m(\xi) - \boldsymbol{\Pi}_L^m(\xi)\boldsymbol{Q}_{L+1}^m(\xi)] \tag{50}$$

Finally we use equation (50) to eliminate $\boldsymbol{Q}_L^m(\xi)$ from equation (46) and equation (48) to eliminate $\boldsymbol{G}_L^m(\xi)$ from equation (46) so that we can obtain

$$\boldsymbol{\Xi}^m(\xi)\boldsymbol{\Pi}_{L+1}^m(\xi)\Lambda^m(\xi) = [\boldsymbol{\Xi}^m(\xi)]^2\boldsymbol{G}_{L+1}^m(\xi) - 2\xi\boldsymbol{Q}_{L+1}^m(\xi)\boldsymbol{\Psi}^m(\xi) \tag{51}$$

BASIC RESULTS

Having established a set of identities, we now can prove two basic results concerning the Λ matrix We first wish to show that the boundary values $[\Lambda^m(\eta)]^\pm$ cannot be singular for $\eta \in (-1, 1)$ This proof is by contradiction and follows the one given in Ref [7], and so for a given m we assume there is a nonzero null vector $\boldsymbol{M}(\eta_0)$ so that

$$[\Lambda^m(\eta_0)]^\pm \boldsymbol{M}(\eta_0) = \boldsymbol{0} \tag{52}$$

for some $\eta_0 \in (-1, 1)$ We can use the Plemelj formulas[11] to deduce from equations (33), (35) and (38) that

$$[\Lambda^m(\eta)]^\pm = \boldsymbol{\lambda}^m(\eta) \pm \pi i \eta \boldsymbol{\Psi}^m(\eta) \tag{53}$$

for $\eta \in (-1, 1)$, here

$$\boldsymbol{\lambda}^m(\eta) = \boldsymbol{I} + \frac{\varpi\eta}{2} \fint_{-1}^{1} \Xi^m(\mu) \boldsymbol{G}(\eta\ \mu) \frac{\mathrm{d}\mu}{\mu - \eta} \tag{54}$$

and the symbol $\fint$ is used to indicate that the integral is to be evaluated in the Cauchy principal-value sense Considering equation (52) to be valid we conclude from equation (53) that

$$\boldsymbol{\lambda}^m(\eta_0) \boldsymbol{M}(\eta_0) = \boldsymbol{0} \tag{55}$$

and

$$\eta_0 \boldsymbol{\Psi}^m(\eta_0) \boldsymbol{M}(\eta_0) = \boldsymbol{0} \tag{56}$$

We can again use the Plemelj formulas to deduce from equation (51) that

$$\Xi^m(\eta) \boldsymbol{\Pi}^m_{L+1}(\eta) \boldsymbol{\lambda}^m(\eta) = [\Xi^m(\eta)]^2 \boldsymbol{G}^m_{L+1}(\eta) - 2\eta \boldsymbol{q}^m_{L+1}(\eta) \boldsymbol{\Psi}^m(\eta) \tag{57}$$

where, in general,

$$\boldsymbol{q}^m_l(\eta) = \frac{1}{2} \fint_{-1}^{1} \Xi^m(\mu) \boldsymbol{\Pi}^m_l(\mu) \frac{\mathrm{d}\mu}{\eta - \mu} \tag{58}$$

We can multiply equation (57), for $\eta = \eta_0$ by $\boldsymbol{M}(\eta_0)$ and use equations (55) and (56) to obtain

$$\boldsymbol{G}^m_{L+1}(\eta_0) \boldsymbol{M}(\eta_0) = \boldsymbol{0} \tag{59}$$

since $\Xi^m(\eta)$ is nonsingular for $\eta \in (-1, 1)$ We can also set $\xi = \eta_0$ in equation (48) and multiply the resulting equation by $\boldsymbol{M}(\eta_0)$ to obtain, after we use equation (59),

$$\boldsymbol{\Pi}^m_{L+1}(\eta_0) \boldsymbol{G}^m_L(\eta_0) \boldsymbol{M}(\eta_0) = \boldsymbol{0} \tag{60}$$

It is clear from equation (45) that $\boldsymbol{G}^m_L(\eta_0)\boldsymbol{M}(\eta_0) \neq \boldsymbol{0}$ if $\boldsymbol{G}^m_{L+1}(\eta_0)\boldsymbol{M}(\eta_0) = \boldsymbol{0}$, and so we let

$$\boldsymbol{N}(\eta_0) = \boldsymbol{G}^m_L(\eta_0) \boldsymbol{M}(\eta_0) \tag{61}$$

be the nonzero null vector of $\boldsymbol{\Pi}^m_{L+1}(\eta_0)$ and write equation (60) as

$$\boldsymbol{\Pi}^m_{L+1}(\eta_0) \boldsymbol{N}(\eta_0) = \boldsymbol{0} \tag{62}$$

Equation (46) yields

$$\boldsymbol{\lambda}^m(\eta) = \boldsymbol{U}^m_{L+1}[\boldsymbol{q}^m_L(\eta)\boldsymbol{G}^m_{L+1}(\eta) - \boldsymbol{q}^m_{L+1}(\eta)\boldsymbol{G}^m_L(\eta)] \tag{63}$$

for $\eta \in (-1, 1)$, and so we set $\eta = \eta_0$ and multiply by $\boldsymbol{M}(\eta_0)$ to find from equation (63), after noting equations (55), (59) and (61),

$$\boldsymbol{q}^m_{L+1}(\eta_0) \boldsymbol{N}(\eta_0) = \boldsymbol{0} \tag{64}$$

The Plemelj formulas allow us to conclude from equation (50) that

$$\Xi^m(\eta) = \boldsymbol{U}^m_{L+1}[\boldsymbol{\Pi}^m_{L+1}(\eta)\boldsymbol{q}^m_L(\eta) - \boldsymbol{\Pi}^m_L(\eta)\boldsymbol{q}^m_{L+1}(\eta)] \tag{65}$$

Setting $\eta = \eta_0$ in equation (65) and multiplying by $\boldsymbol{N}(\eta_0)$, we find

$$\Xi^m(\eta_0)\boldsymbol{N}(\eta_0) = \boldsymbol{U}^m_{L+1}\boldsymbol{\Pi}^m_{L+1}(\eta_0)\boldsymbol{q}^m_L(\eta_0)\boldsymbol{N}(\eta_0) \tag{66}$$

after we note equation (64) Finally we observe that $\boldsymbol{\Pi}^m_{L+1}(\eta_0)$ and $\boldsymbol{q}^m_L(\eta_0)$ commute, so that we can use equation (62) to conclude from equation (66) that

$$\boldsymbol{\Xi}^m(\eta_0)\boldsymbol{N}(\eta_0) = \boldsymbol{0} \tag{67}$$

which clearly is impossible since $\boldsymbol{\Xi}^m(\eta_0)$ is nonsingular for $\eta_0 \in (-1, 1)$ and $\boldsymbol{N}(\eta_0)$ is nonzero It therefore follows that our premise of equation (52) is false

We consider now the discrete spectrum defined by

$$\boldsymbol{\Lambda}^m(\xi)\boldsymbol{M}(\xi) = \boldsymbol{0} \tag{68}$$

where $\xi \notin [-1, 1]$ By using the asymptotic formulas for the Jacobi polynomials and Jacobi functions of the second kind given by Szego,[12] we have concluded that

$$\lim_{L \to \infty} [\boldsymbol{\Pi}^m_{L+1}(\xi)]^{-1}\boldsymbol{Q}^m_{L+1}(\xi) = \boldsymbol{0} \tag{69}$$

for all $\xi \notin [-1, 1]$, and so we deduce from equation (51) that the discrete spectrum defined by equation (68) can be approximated by

$$\boldsymbol{G}^m_{N+1}(\xi)\boldsymbol{M}_{N+1}(\xi) = \boldsymbol{0} \tag{70}$$

with better and better accuracy as $N > L$ tends to infinity In Ref [5] we used the generalized spherical harmonics method to solve equation (1) subject to appropriate boundary conditions, and in the process of constructing our solution we were required to develop an algorithm to find all the zeros of $\det \boldsymbol{G}^m_{N+1}(\xi)$, for N odd It follows then that this same algorithm can be used to compute an approximation to the discrete spectrum defined here by equation (68)

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