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ON THE USE OF THE F_N METHOD WITH SPLINES FOR RADIATIVE TRANSFER PROBLEMS

R. C. Y. Chin, G. W. Hedstrom, and C. E. Siewert†
Lawrence Livermore National Laboratory, University of California, Livermore, CA 94550, U.S.A.

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Abstract—A variant of the F_N method that is based on splines is used to develop a solution to a class of basic problems in the theory of radiation transport that is computationally attractive in a parallel-processing environment. In addition to selected numerical results, some observations regarding the merits and the accuracy of the method are reported.

1. INTRODUCTION

The F_N method[1,2] has enjoyed considerable success in solving radiation transport problems relevant to, for example, one-speed applications that require many-term phase functions[3] and multigroup shielding calculations in multilayered media.[4] Polynomial basis functions were used in these previous versions of the F_N method, and so the required matrix elements were computed efficiently and accurately from various two- or three-term recursion formulas. Now to have a variant of the F_N method that utilizes finite-element techniques and one that also can effectively take advantage of the newly emerging parallel-processing machines, we develop a solution that leads to an algorithm where each of the required matrix elements can be evaluated accurately, efficiently and independently.

We consider the case of isotropic scattering and plane geometry, and thus we seek a solution of the transport equation[5]

$$\mu \frac{\partial}{\partial x}\psi(x,\mu) + \psi(x,\mu) = \frac{c}{2}\int_{-1}^{1}\psi(x,\mu)\,\mathrm{d}\mu \tag{1}$$

subject to boundary conditions

$$\psi(L,\mu) = F_{\mathrm{L}}(\mu) \quad \text{for} \quad \mu \in [0,1] \tag{2a}$$

and

$$\psi(R,-\mu) = F_{\mathrm{R}}(\mu) \quad \text{for} \quad \mu \in [0,1] \tag{2b}$$

where $F_{\mathrm{L}}(\mu)$ and $F_{\mathrm{R}}(\mu)$ are given. Here $\psi(x,\mu)$ is the angular flux, $x \in (L,R)$ is the position variable measured in optical units, μ is the direction cosine of the propagating radiation and $c \in (0,1)$ is the mean number of secondary particles per collision.

2. SINGULAR INTEGRAL EQUATIONS AND INTEGRAL CONDITIONS

As a review of the general aspects of the F_N method and to provide an introduction to our current computations, we follow Siewert[6] and use an integral transformation technique to deduce a set of singular integral equations and integral conditions from which we develop our F_N solution to Eqs. (1) and (2). We begin by changing μ to $-\mu$ in Eq. (1). We then multiply the resulting equation by $\exp(-x/s)$ and integrate over x from a to b to find

$$s\mu B(\mu,s) - (\mu - s)\int_a^b \exp(-x/s)\psi(x,-\mu)\,\mathrm{d}x = \frac{cs}{2}\rho^*(s) \tag{3}$$

†Permanent address: Mathematics Department, North Carolina State University, Raleigh, NC 27695-8205, U.S.A.

where

$$\rho^*(s) = \int_a^b \exp(-x/s) \int_{-1}^{1} \psi(x, \mu)\, d\mu\, dx \tag{4}$$

and

$$B(\mu, s) = \psi(a, -\mu)\exp(-a/s) - \psi(b, -\mu)\exp(-b/s). \tag{5}$$

Here $a, b \in [L, R]$ and $a < b$. For $s \notin [-1, 1]$ we divide Eq. (3) by $\mu - s$ and integrate over μ to find

$$\Lambda(s)\rho^*(s) = s \int_{-1}^{1} \mu B(\mu, s) \frac{d\mu}{\mu - s} \tag{6}$$

where

$$\Lambda(s) = 1 + \frac{cs}{2} \int_{-1}^{1} \frac{d\mu}{\mu - s}. \tag{7}$$

It is clear that the dispersion function $\Lambda(s)$, as defined by Eq. (7), is analytic in the complex s plane cut from -1 to 1 along the real axis.

We find it convenient to multiply Eq. (6) by $\exp(a/s)$ and to consider the resulting equation only for $\mathscr{R}s > 0$. Then in order to take into account the left half of the s plane, we change s to $-s$ in Eq. (6) and, for convenience, multiply the resulting equation by $\exp(-b/s)$; we subsequently consider the resulting equation only for $\mathscr{R}s > 0$. In this way we obtain for $\mathscr{R}s > 0$, the pair of equations

$$\Lambda(s)I(s) = \int_{-1}^{1} \mu C(\mu, s) \frac{d\mu}{\mu - s} \tag{8a}$$

and

$$\Lambda(s)J(s) = \int_{-1}^{1} \mu D(\mu, s) \frac{d\mu}{\mu - s} \tag{8b}$$

where

$$I(s) = \frac{1}{s} \int_a^b \exp[-(x - a)/s] \int_{-1}^{1} \psi(x, \mu)\, d\mu\, dx, \tag{9a}$$

$$J(s) = \frac{1}{s} \int_a^b \exp[-(b - x)/s] \int_{-1}^{1} \psi(x, \mu)\, d\mu\, dx, \tag{9b}$$

$$C(\mu, s) = \psi(a, -\mu) - \psi(b, -\mu)\exp[-(b - a)/s] \tag{10a}$$

and

$$D(\mu, s) = \psi(b, \mu) - \psi(a, \mu)\exp[-(b - a)/s]. \tag{10b}$$

We note[7] that the dispersion function $\Lambda(s)$ has only two zeros $\pm\nu_0$ with $|\nu_0| > 1$. Thus on evaluating Eqs. (8) at $s = \nu_0$, we find

$$\frac{c\nu_0}{2} \int_{-1}^{1} \mu C(\mu, \nu_0) \frac{d\mu}{\mu - \nu_0} = 0 \tag{11a}$$

and

$$\frac{c\nu_0}{2} \int_{-1}^{1} \mu D(\mu, \nu_0) \frac{d\mu}{\mu - \nu_0} = 0. \tag{11b}$$

We can also let s approach $\nu \in [0, 1)$ from above and below the cut and use the Plemelj formulae[8] to deduce from Eqs. (8) that

$$\left[\lambda(\nu) \pm i\pi \frac{c\nu}{2}\right] I(\nu) = ⨍_{-1}^{1} \mu C(\mu, \nu) \frac{d\mu}{\mu - \nu} \pm i\pi\nu C(\nu, \nu) \tag{12a}$$

and

$$\left[\lambda(\nu) \pm i\pi\frac{c\nu}{2}\right]J(\nu) = -\!\!\!\!\!\!\int_{-1}^{1} \mu D(\mu, \nu)\frac{d\mu}{\mu - \nu} \pm i\pi\nu D(\nu, \nu) \tag{12b}$$

where $-\!\!\!\!\!\!\int$ denotes the principal-value integral and

$$\lambda(\nu) = 1 + \frac{c\nu}{2}\log\left(\frac{1-\nu}{1+\nu}\right). \tag{12c}$$

For $\nu \in [0, 1)$ we may eliminate $I(\nu)$ from the two versions of Eq. (12a) to obtain

$$\nu\lambda(\nu)C(\nu, \nu) = \frac{c\nu}{2} -\!\!\!\!\!\!\int_{-1}^{1} \mu C(\mu, \nu)\frac{d\mu}{\mu - \nu}, \tag{13a}$$

and similarly it follows from Eq. (12b) that

$$\nu\lambda(\nu)D(\nu, \nu) = \frac{c\nu}{2} -\!\!\!\!\!\!\int_{-1}^{1} \mu D(\mu, \nu)\frac{d\mu}{\mu - \nu}. \tag{13b}$$

Equations (11) and (13) define a system of singular integral equations and integral conditions which relates $\psi(a, \mu)$ and $\psi(b, \mu)$ for $\mu \in [-1, 1]$.

3. THE BOUNDARY INTEGRAL EQUATIONS AND CONSTRAINTS

Seeking first to establish the boundary results $\psi(L, -\mu)$ and $\psi(R, \mu)$ for $\mu \in [0, 1]$, we let $a = L$ and $b = R$ and use Eq. (2) to find from Eqs. (11) and (13)

$$-\frac{c\nu_0}{2}\int_0^1 \mu\psi(L, -\mu)\frac{d\mu}{\mu - \nu_0} + \frac{c\nu_0}{2}\exp(-\Delta/\nu_0)\int_0^1 \mu\psi(R, \mu)\frac{d\mu}{\mu + \nu_0} = K_1(\nu_0) \tag{14a}$$

and

$$-\frac{c\nu_0}{2}\int_0^1 \mu\psi(R, \mu)\frac{d\mu}{\mu - \nu_0} + \frac{c\nu_0}{2}\exp(-\Delta/\nu_0)\int_0^1 \mu\psi(L, -\mu)\frac{d\mu}{\mu + \nu_0} = K_2(\nu_0) \tag{14b}$$

and, for $\nu \in [0, 1)$,

$$\nu\lambda(\nu)\psi(L, -\nu) - \frac{c\nu}{2} -\!\!\!\!\!\!\int_0^1 \mu\psi(L, -\mu)\frac{d\mu}{\mu - \nu} + \frac{c\nu}{2}\exp(-\Delta/\nu)\int_0^1 \mu\psi(R, \mu)\frac{d\mu}{\mu + \nu} = K_1(\nu) \tag{14c}$$

and

$$\nu\lambda(\nu)\psi(R, \nu) - \frac{c\nu}{2} -\!\!\!\!\!\!\int_0^1 \mu\psi(R, \mu)\frac{d\mu}{\mu - \nu} + \frac{c\nu}{2}\exp(-\Delta/\nu)\int_0^1 \mu\psi(L, -\mu)\frac{d\mu}{\mu + \nu} = K_2(\nu) \tag{14d}$$

where $\Delta = R - L$,

$$K_1(\nu_0) = \frac{c\nu_0}{2}\int_0^1 \mu F_L(\mu)\frac{d\mu}{\mu + \nu_0} - \frac{c\nu_0}{2}\exp(-\Delta/\nu_0)\int_0^1 \mu F_R(\mu)\frac{d\mu}{\mu - \nu_0}, \tag{15a}$$

$$K_2(\nu_0) = \frac{c\nu_0}{2}\int_0^1 \mu F_R(\mu)\frac{d\mu}{\mu + \nu_0} - \frac{c\nu_0}{2}\exp(-\Delta/\nu_0)\int_0^1 \mu F_L(\mu)\frac{d\mu}{\mu - \nu_0}, \tag{15b}$$

$$K_1(\nu) = \frac{c\nu}{2}\int_0^1 \mu F_L(\mu)\frac{d\mu}{\mu + \nu} + \exp(-\Delta/\nu)\left[\nu\lambda(\nu)F_R(\nu) - \frac{c\nu}{2} -\!\!\!\!\!\!\int_0^1 \mu F_R(\mu)\frac{d\mu}{\mu - \nu}\right] \tag{15c}$$

and

$$K_2(\nu) = \frac{c\nu}{2}\int_0^1 \mu F_R(\mu)\frac{d\mu}{\mu + \nu} + \exp(-\Delta/\nu)\left[\nu\lambda(\nu)F_L(\nu) - \frac{c\nu}{2} -\!\!\!\!\!\!\int_0^1 \mu F_L(\mu)\frac{d\mu}{\mu - \nu}\right]. \tag{15d}$$

We note that Mullikin[9] has discussed the questions of existence and uniqueness of the solution of a system of singular integral equations and constraints similar to Eqs. (14).

4. AN APPROXIMATION BASED ON COLLOCATION

Equations (14) have been derived from Eqs. (1) and (2) without approximation; however, we want to use the F_N method to solve these equations in an approximate, but accurate, manner. We choose to approximate the scattered component of the solution by a function in a finite-dimensional subspace, and thus we let $\{\phi_\alpha(\mu)\}$ be a set of $N+1$ basis functions and write the desired boundary fluxes as

$$\psi(L, -\mu) = F_{\mathrm{R}}(\mu)\exp(-\Delta/\mu) + \frac{c}{2}\sum_{\alpha=0}^{N} a_\alpha \phi_\alpha(\mu) \tag{16a}$$

and

$$\psi(R, \mu) = F_{\mathrm{L}}(\mu)\exp(-\Delta/\mu) + \frac{c}{2}\sum_{\alpha=0}^{N} b_\alpha \phi_\alpha(\mu) \tag{16b}$$

for $\mu \in [0, 1]$. We can substitute Eqs. (16) into Eqs. (14) to obtain

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi) + b_\alpha c A_\alpha(\xi)\exp(-\Delta/\xi)] = 2R_1(\xi) \tag{17a}$$

and

$$\sum_{\alpha=0}^{N} [a_\alpha c A_\alpha(\xi)\exp(-\Delta/\xi) + b_\alpha B_\alpha(\xi)] = 2R_2(\xi) \tag{17b}$$

for $\xi \in \nu_0 \cup [0, 1]$. Here we have defined

$$A_\alpha(\xi) = \int_0^1 \mu\phi_\alpha(\mu)\frac{\mathrm{d}\mu}{\mu + \xi} \tag{18a}$$

for $\xi \in \nu_0 \cup [0, 1]$,

$$B_\alpha(\nu_0) = -c\int_0^1 \mu\phi_\alpha(\mu)\frac{\mathrm{d}\mu}{\mu - \nu_0} \tag{18b}$$

and

$$B_\alpha(\nu) = 2\lambda(\nu)\phi_\alpha(\nu) - c\,⨍_0^1 \mu\phi_\alpha(\mu)\frac{\mathrm{d}\mu}{\mu - \nu} \tag{18c}$$

for $\nu \in [0, 1]$. We also have used

$$R_1(\xi) = \int_0^1 [F_{\mathrm{L}}(\mu)S(\Delta; \mu, \xi) + F_{\mathrm{R}}(\mu)C(\Delta; \mu, \xi)]\mu\, \mathrm{d}\mu \tag{19a}$$

and

$$R_2(\xi) = \int_0^1 [F_{\mathrm{R}}(\mu)S(\Delta; \mu, \xi) + F_{\mathrm{L}}(\mu)C(\Delta; \mu, \xi)]\mu\, \mathrm{d}\mu \tag{19b}$$

where, in general,

$$S(a; x, y) = \frac{1 - \exp(-a/x)\exp(-a/y)}{x + y} \tag{20a}$$

and

$$C(a; x, y) = \frac{\exp(-a/x) - \exp(-a/y)}{x - y}. \tag{20b}$$

To define our collocation scheme, we select ξ_β, for $\beta = 0, 1, \ldots, N-1$, from the interval [0, 1] and take $\xi_N = \nu_0$. We thus can solve the linear system

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi_\beta) + b_\alpha c A_\alpha(\xi_\beta)\exp(-\Delta/\xi_\beta)] = 2R_1(\xi_\beta) \tag{21a}$$

and

$$\sum_{\alpha=0}^{N} [a_\alpha c A_\alpha(\xi_\beta)\exp(-\Delta/\xi_\beta) + b_\alpha B_\alpha(\xi_\beta)] = 2R_2(\xi_\beta), \tag{21b}$$

with $\beta = 0, 1, \ldots, N$, to find the required coefficients $\{a_\alpha, b_\alpha\}$.

In other applications of the F_N method,[3,4,6] it has been found useful to perform a *post processing* iteration based on Eqs. (14) as follows. We rewrite Eq. (18c) as

$$B_\alpha(\mu) = 2\phi_\alpha(\mu) - cB_\alpha^*(\mu) \tag{22}$$

with

$$B_\alpha^*(\mu) = \int_0^1 \tau\phi_\alpha(\tau)\frac{d\tau}{\tau-\mu} - \mu\log\left(\frac{1-\mu}{1+\mu}\right)\phi_\alpha(\mu), \tag{23}$$

and we deduce from Eqs. (17) for $\xi = \mu \in [0, 1]$ that we can write Eqs. (16) as

$$\psi(L, -\mu) = F_R(\mu)\exp(-\Delta/\mu) + \frac{c}{2}R_1(\mu) + \frac{c^2}{4}\sum_{\alpha=0}^{N}[a_\alpha B_\alpha^*(\mu) - b_\alpha A_\alpha(\mu)\exp(-\Delta/\mu)] \tag{24a}$$

and

$$\psi(R, \mu) = F_L(\mu)\exp(-\Delta/\mu) + \frac{c}{2}R_2(\mu) + \frac{c^2}{4}\sum_{\alpha=0}^{N}[b_\alpha B_\alpha^*(\mu) - a_\alpha A_\alpha(\mu)\exp(-\Delta/\mu)] \tag{24b}$$

for $\mu \in [0, 1]$.

5. THE INTERIOR SOLUTION

Generalizing the foregoing development, we can use, for $x \in (L, R)$ and $\mu \in [0, 1]$, the approximations

$$\psi(x, -\mu) = F_R(\mu)\exp[-(R-x)/\mu] + \frac{c}{2}\sum_{\alpha=0}^{N} c_\alpha(x)\phi_\alpha(\mu) \tag{25a}$$

and

$$\psi(x, \mu) = F_L(\mu)\exp[-(x-L)/\mu] + \frac{c}{2}\sum_{\alpha=0}^{N} d_\alpha(x)\phi_\alpha(\mu) \tag{25b}$$

in Eqs. (11a) and (13a) with $a = x$ and $b = R$ and in Eqs. (11b) and (13b) with $a = L$ and $b = x$. Thus upon using $N + 1$ collocation points ξ_β, we find that

$$\sum_{\alpha=0}^{N}[c_\alpha(x)B_\alpha(\xi_\beta) - d_\alpha(x)cA_\alpha(\xi_\beta)] = 2R_1(x, \xi_\beta) - c\exp[-(R-x)/\xi_\beta]\sum_{\alpha=0}^{N} b_\alpha A_\alpha(\xi_\beta) \tag{26a}$$

and

$$\sum_{\alpha=0}^{N}[-c_\alpha(x)cA_\alpha(\xi_\beta) + d_\alpha(x)B_\alpha(\xi_\beta)] = 2R_2(x, \xi_\beta) - c\exp[-(x-L)/\xi_\beta]\sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi_\beta) \tag{26b}$$

where

$$R_1(x, \xi) = \int_0^1 \{F_L(\mu)\exp[-(x-L)/\mu]S(R-x; \mu, \xi) + F_R(\mu)C(R-x; \mu, \xi)\}\mu\, d\mu \tag{27a}$$

and

$$R_2(x, \xi) = \int_0^1 \{F_R(\mu)\exp[-(R-x)/\mu]S(x-L; \mu, \xi) + F_L(\mu)C(x-L; \mu, \xi)\}\mu\, d\mu. \tag{27b}$$

It is clear that the coefficient matrix in the linear system defined by Eqs. (26) does not depend on x, and so we can solve Eqs. (26) for various right-hand sides, namely, for different values of x, to obtain the expansion coefficients $\{c_\alpha(x), d_\alpha(x)\}$. We can again use Eq. (22) and post processing to obtain

$$\psi(x, -\mu) = F_R(\mu)\exp[-(R-x)/\mu] + \frac{c}{2}R_1(x,\mu) + \frac{c^2}{4}\sum_{\alpha=0}^{N}\langle c_\alpha(x)B_\alpha^*(\mu) + \{d_\alpha(x) - b_\alpha\exp[-(R-x)/\mu]\}A_\alpha(\mu)\rangle \tag{28a}$$

and

$$\psi(x, \mu) = F_L(\mu)\exp[-(x-L)/\mu] + \frac{c}{2}R_2(x,\mu) + \frac{c^2}{4}\sum_{\alpha=0}^{N}\langle d_\alpha(x)B_\alpha^*(\mu) + \{c_\alpha(x) - a_\alpha\exp[-(x-L)/\mu]\}A_\alpha(\mu)\rangle \tag{28b}$$

for $\mu \in [0, 1]$.

6. NUMERICAL RESULTS BASED ON LINEAR SPLINES

We note that Gerasoulis and Srivastav[10] and Jen and Srivastav,[11] for example, have used collocation methods with linear and cubic splines to establish a procedure for solving numerically a class of singular integral equations derived from elasticity theory. Now, in order to begin an evaluation of the merits of our F_N solution developed here in terms of general basis functions, we consider the case of linear splines,[12] and we report numerical results for some slab problems.

The splines we use are specified as follows. For the *knots* for our splines we take $\zeta_0 = 0$ and $\zeta_N = 1$ and leave the remaining ζ_α, $\alpha = 1, 2, \dots, N-1$, arbitrary at this point. We may therefore write

$$\phi_0(\mu) = \begin{cases} (\zeta_1 - \mu)/h_1, & \text{for } \mu \in [0, \zeta_1], \\ 0, & \text{otherwise,} \end{cases} \tag{29a}$$

$$\phi_\alpha(\mu) = \begin{cases} (\mu - \zeta_{\alpha-1})/h_\alpha, & \text{for } \mu \in [\zeta_{\alpha-1}, \zeta_\alpha], \\ (\zeta_{\alpha+1} - \mu)/h_{\alpha+1}, & \text{for } \mu \in [\zeta_\alpha, \zeta_{\alpha+1}], \\ 0, & \text{otherwise,} \end{cases} \tag{29b}$$

for $\alpha = 1, 2, \dots, N-1$, and

$$\phi_N(\mu) = \begin{cases} (\mu - \zeta_{N-1})/h_N, & \text{for } \mu \in [\zeta_{N-1}, 1], \\ 0, & \text{otherwise,} \end{cases} \tag{29c}$$

with $h_\alpha = \zeta_\alpha - \zeta_{\alpha-1}$.

Considering first Eqs. (18b) and (18c), we can evaluate the integrals to obtain

$$B_\alpha(\nu_0) = -cD_\alpha(\nu_0) \tag{30a}$$

and

$$B_\alpha(\nu) = -cD_\alpha(\nu) + 2\lambda(\nu)\phi_\alpha(\nu) \quad \text{for} \quad \nu \in [0, 1] \tag{30b}$$

where, in general,

$$D_\alpha(\xi) = \int_0^1 \mu\phi_\alpha(\mu)\frac{d\mu}{\mu - \xi} \tag{31}$$

and, in particular,

$$D_0(\xi) = \frac{\zeta_1}{2} - \xi - \xi\left(\frac{\xi - \zeta_1}{h_1}\right)\log\left|\frac{\xi - \zeta_1}{\xi}\right|, \tag{32a}$$

$$D_\alpha(\xi) = \frac{h_\alpha + h_{\alpha+1}}{2} - \xi\left[\left(\frac{\xi - \zeta_{\alpha+1}}{h_{\alpha+1}}\right)\log|\xi - \zeta_{\alpha+1}| - \left(\frac{1}{h_\alpha} + \frac{1}{h_{\alpha+1}}\right)(\xi - \zeta_\alpha)\log|\xi - \zeta_\alpha| + \left(\frac{\xi - \zeta_{\alpha-1}}{h_\alpha}\right)\log|\xi - \zeta_{\alpha-1}|\right], \tag{32b}$$

for $\alpha = 1, 2, \ldots, N-1$, and

$$D_N(\xi) = \frac{h_N}{2} + \xi - \xi\left(\frac{\xi - \zeta_{N-1}}{h_N}\right)\log\left|\frac{\xi - \zeta_{N-1}}{\xi - 1}\right|. \tag{32c}$$

We note that some terms in Eq. (30b) must be combined before the limit $\nu \to 1$ can be observed. Note also that if $|\xi - \zeta_{\alpha-1}|/h_\alpha$ or $|\zeta_{\alpha+1} - \xi|/h_{\alpha+1}$ is large compared to 1, there may be loss of accuracy (due to the cancellation of large numbers in finite-precision arithmetic) if the functions $D_\alpha(\xi)$ are always evaluated by the explicit expressions given by Eqs. (32). We have avoided this potential loss of accuracy by making use of appropriate binomial expansions in Eqs. (32) under these conditions.

Having defined $D_\alpha(\xi)$ for $\alpha = 0, 1, 2, \ldots, N$, we can obtain the other basic F_N functions from

$$A_\alpha(\xi) = D_\alpha(-\xi) \tag{33}$$

for $\xi \in \nu_0 \cup [0, 1]$. Finally for $\mu \in [0, 1]$ we can write

$$B_\alpha^*(\mu) = D_\alpha(\mu) - \mu\phi_\alpha(\mu)\log\left(\frac{1-\mu}{1+\mu}\right) \tag{34}$$

where again some care must be exercised to obtain the desired result for $\mu \to 1$.

As an application we consider the finite-slab albedo problem. Thus we seek a solution to Eq. (1) subject to the boundary conditions

$$\psi(L, \mu) = \tfrac{1}{2}\delta(\mu - \mu_0) \tag{35a}$$

and

$$\psi(R, -\mu) = 0 \tag{35b}$$

for $\mu \in [0, 1]$, where μ_0 is the direction cosine of the incident radiation.

After experimenting with various knot distributions for the purpose of computing the albedo and the transmission factor with linear splines, we have concluded that the accuracy depends very weakly on the knot distribution and that a linear distribution of knots is adequate here, and so we write

$$\zeta_\alpha = \frac{\alpha}{N} \tag{36}$$

for $\alpha = 0, 1, \ldots, N$. For the collocation points we use the N midpoints of the intervals joining the knots

$$\xi_\beta = \tfrac{1}{2}(\zeta_\beta + \zeta_{\beta+1}) \tag{37a}$$

for $\beta = 0, 1, \ldots, N-1$ and, for the consistency condition,

$$\xi_N = \nu_0. \tag{37b}$$

Having defined the knots and the collocation scheme, we solve Eqs. (21) to obtain the expansion coefficients $\{a_\alpha, b_\alpha\}$.

As a comparison of the F_N method based on linear splines with the F_N method based on Legendre polynomials $\{P_\alpha(2\mu - 1)\}$, we display in Table 1 the values of the albedo

$$A^* = \frac{2}{\mu_0}\int_0^1 \psi(L, -\mu)\mu \, d\mu \tag{38a}$$

and the transmission factor

$$B^* = \frac{2}{\mu_0}\int_0^1 \psi(R, \mu)\mu \, d\mu \tag{38b}$$

Table 1. Values of A^* and B^* for $\mu_0 = 1$ and $\Delta = 1$.

c	N	Polynomials A^*	Polynomials B^*	Linear Splines A^*	Linear Splines B^*	Cubic Splines A^*	Cubic Splines B^*
0.5	3	9.91 (-2)	4.460 (-1)	9.90 (-2)	4.462 (-1)	9.90 (-2)	4.462 (-1)
	7	9.9119 (-2)	4.46058(-1)	9.91 (-2)	4.461 (-1)	9.912 (-2)	4.4606 (-1)
	15	9.91192(-2)	4.46058(-1)	9.911 (-2)	4.4607 (-1)	9.91191(-2)	4.46058(-1)
	31	9.91192(-2)	4.46058(-1)	9.9118 (-2)	4.4606 (-1)	9.91192(-2)	4.46058(-1)
0.9	3	2.6740 (-1)	5.916 (-1)	2.673 (-1)	5.917 (-1)	2.67 (-1)	5.92 (-1)
	7	2.67410(-1)	5.91625(-1)	2.6740 (-1)	5.917 (-1)	2.6741 (-1)	5.91625(-1)
	15	2.67410(-1)	5.91625(-1)	2.67409(-1)	5.9163 (-1)	2.67410(-1)	5.91625(-1)
	31	2.67410(-1)	5.91625(-1)	2.67410(-1)	5.9163 (-1)	2.67410(-1)	5.91625(-1)
0.99	3	3.3293 (-1)	6.510 (-1)	3.329 (-1)	6.510 (-1)	3.328 (-1)	6.511 (-1)
	7	3.32922(-1)	6.50977(-1)	3.3291 (-1)	6.510 (-1)	3.3292 (-1)	6.50979(-1)
	15	3.32922(-1)	6.50978(-1)	3.3292 (-1)	6.5098 (-1)	3.32922(-1)	6.50978(-1)
	31	3.32922(-1)	6.50978(-1)	3.32921(-1)	6.50978(-1)	3.32922(-1)	6.50978(-1)

computed by both methods with three different values of c and four values of N for the case of $\mu_0 = 1$ and $\Delta = 1$. We believe that the results based on polynomials are, for $N = 31$, correct to 6 significant figures.

For the sake of completeness, we display the formulas for A^* and B^* from Eqs. (38) when linear splines are used as the basis functions in Eq. (16). Thus we have

$$A^* = \frac{c}{\mu_0} \sum_{\alpha=0}^{N} a_\alpha J_\alpha \tag{39a}$$

and

$$B^* = \exp(-\Delta/\mu_0) + \frac{c}{\mu_0} \sum_{\alpha=0}^{N} b_\alpha J_\alpha \tag{39b}$$

where

$$J_\alpha = \int_0^1 \mu \phi_\alpha(\mu)\, d\mu \tag{40}$$

or more explicitly

$$J_0 = \tfrac{1}{6}\zeta_1^2, \tag{41a}$$

$$J_\alpha = \tfrac{1}{6}(\zeta_{\alpha+1} - \zeta_{\alpha-1})(\zeta_{\alpha+1} + \zeta_\alpha + \zeta_{\alpha-1}), \tag{41b}$$

for $\alpha = 1, 2, \ldots, N-1$, and

$$J_N = \tfrac{1}{6}(2 + \zeta_{N-1})(1 - \zeta_{N-1}). \tag{41c}$$

7. NUMERICAL RESULTS BASED ON HERMITE CUBIC SPLINES

Although the computations discussed in the previous section indicate that the radiation problem in a slab can be solved by using linear splines with the F_N method, we also made some calculations with cubic splines in a search for a more efficient method. We report on our experience with Hermite cubic splines in this section.

For the Hermite cubics there are two basis functions associated with each knot, so that the value of N in Eqs. (16) must be odd and > 1. Let the knots be denoted by ζ_α with $\alpha = 0, 1, \ldots, M$ and $M = (N-1)/2$. We choose $\zeta_0 = 0$ and $\zeta_M = 1$ and leave the rest of the knots arbitrary for the moment. To define the Hermite basis functions $\Phi_\alpha(\mu)$ and $\Psi_\alpha(\mu)$ we make use of the representations given by Schultz[12] and write

$$\Phi_0(\mu) = \begin{cases} 3(\zeta_1 - \mu)^2/h_1^2 - 2(\zeta_1 - \mu)^3/h_1^3, & \text{for} \quad \mu \in [0, \zeta_1], \\ 0, & \text{otherwise,} \end{cases} \tag{42a}$$

$$\Phi_\alpha(\mu) = \begin{cases} 3(\mu - \zeta_{\alpha-1})^2/h_\alpha^2 - 2(\mu - \zeta_{\alpha-1})^3/h_\alpha^3, & \text{for} \quad \mu \in [\zeta_{\alpha-1}, \zeta_\alpha], \\ 3(\zeta_{\alpha+1} - \mu)^2/h_{\alpha+1}^2 - 2(\zeta_{\alpha+1} - \mu)^3/h_{\alpha+1}^3, & \text{for} \quad \mu \in [\zeta_\alpha, \zeta_{\alpha+1}], \\ 0, & \text{otherwise,} \end{cases} \tag{42b}$$

for $\alpha = 1, 2, \ldots, M-1$,

$$\Phi_M(\mu) = \begin{cases} 3(\mu - \zeta_{M-1})^2/h_M^2 - 2(\mu - \zeta_{M-1})^3/h_M^3, & \text{for} \quad \mu \in [\zeta_{M-1}, 1], \\ 0, & \text{otherwise,} \end{cases} \tag{42c}$$

$$\Psi_0(\mu) = \begin{cases} \mu(\zeta_1 - \mu)^2/h_1^2, & \text{for} \quad \mu \in [0, \zeta_1], \\ 0, & \text{otherwise,} \end{cases} \tag{43a}$$

$$\Psi_\alpha(\mu) = \begin{cases} (\mu - \zeta_\alpha)(\mu - \zeta_{\alpha-1})^2/h_\alpha^2, & \text{for} \quad \mu \in [\zeta_{\alpha-1}, \zeta_\alpha], \\ (\mu - \zeta_\alpha)(\zeta_{\alpha+1} - \mu)^2/h_{\alpha+1}^2, & \text{for} \quad \mu \in [\zeta_\alpha, \zeta_{\alpha+1}], \\ 0, & \text{otherwise,} \end{cases} \tag{43b}$$

for $\alpha = 1, 2, \ldots, M-1$, and

$$\Psi_M(\mu) = \begin{cases} (\mu - 1)(\mu - \zeta_{M-1})^2/h_M^2, & \text{for} \quad \mu \in [\zeta_{M-1}, 1], \\ 0, & \text{otherwise.} \end{cases} \tag{43c}$$

Now to establish a correspondence with our general development given in Secs. 4 and 5, we let

$$\phi_{2\beta}(\mu) = \Phi_\beta(\mu) \quad \text{and} \quad \phi_{2\beta+1}(\mu) = \Psi_\beta(\mu) \tag{44a and b}$$

for $\beta = 0, 1, \ldots, M$. Thus with the definitions

$$D_{2\beta}(\xi) = \int_0^1 \mu \Phi_\beta(\mu) \frac{d\mu}{\mu - \xi} \tag{45a}$$

and

$$D_{2\beta+1}(\xi) = \int_0^1 \mu \Psi_\beta(\mu) \frac{d\mu}{\mu - \xi} \tag{45b}$$

for all ξ except for $\xi = 1$ in regard to $D_{2M}(\xi)$, we can again write

$$B_\alpha(\nu_0) = -cD_\alpha(\nu_0), \tag{46a}$$

$$B_\alpha(\nu) = -cD_\alpha(\nu) + 2\lambda(\nu)\phi_\alpha(\nu) \tag{46b}$$

and

$$B_\alpha^*(\nu) = D_\alpha(\nu) - \nu\phi_\alpha(\nu)\log\left(\frac{1-\nu}{1+\nu}\right) \tag{47}$$

for $\nu \in [0, 1]$ and

$$A_\alpha(\xi) = D_\alpha(-\xi), \tag{48}$$

for $\xi \in \nu_0 \cup [0, 1]$ and make available all of the general formalism developed in Secs. 4 and 5. To be specific, we substitute Eqs. (42) and Eqs. (43) into Eqs. (45) to find that

$$D_0(\xi) = P_0(\xi) + \xi\left(1 - \frac{\xi}{\zeta_1}\right)^2\left(1 + \frac{2\xi}{\zeta_1}\right)\log\left|\frac{\xi - \zeta_1}{\xi}\right|, \tag{49a}$$

$$D_1(\xi) = P_1(\xi) + \xi^2\left(1 - \frac{\xi}{\zeta_1}\right)^2\log\left|\frac{\xi - \zeta_1}{\xi}\right|, \tag{49b}$$

$$D_{2\beta}(\xi) = P_{2\beta}(\xi) + \frac{\xi(\zeta_{\beta+1} - \xi)^2}{h_{\beta+1}^2}\left[3 - \frac{2(\zeta_{\beta+1} - \xi)}{h_{\beta+1}}\right]\log|\xi - \zeta_{\beta+1}|$$

$$- \frac{\xi(\xi - \zeta_{\beta-1})^2}{h_\beta^2}\left[3 - \frac{2(\xi - \zeta_{\beta-1})}{h_\beta}\right]\log|\xi - \zeta_{\beta-1}|$$

$$+ \xi\left\{\frac{(\xi - \zeta_{\beta-1})^2}{h_\beta^2}\left[3 - \frac{2(\xi - \zeta_{\beta-1})}{h_\beta}\right] - \frac{(\zeta_{\beta+1} - \xi)^2}{h_{\beta+1}^2}\left[3 - \frac{2(\zeta_{\beta+1} - \xi)}{h_{\beta+1}}\right]\right\}\log|\xi - \zeta_\beta| \tag{49c}$$

and

$$D_{2\beta+1}(\xi) = P_{2\beta+1}(\xi) + \frac{\xi(\xi - \zeta_\beta)(\zeta_{\beta+1} - \xi)^2}{h_{\beta+1}^2}\log|\xi - \zeta_{\beta+1}| - \frac{\xi(\xi - \zeta_\beta)(\xi - \zeta_{\beta-1})^2}{h_\beta^2}$$

$$\times \log|\xi - \zeta_{\beta-1}| + \xi(\xi - \zeta_\beta)\left[\frac{(\xi - \zeta_{\beta-1})^2}{h_\beta^2} - \frac{(\zeta_{\beta+1} - \xi)^2}{h_{\beta+1}^2}\right]\log|\xi - \zeta_\beta|, \tag{49d}$$

and $\beta = 1, 2, \ldots, M-1$,

$$D_{2M}(\xi) = P_{2M}(\xi) + \frac{\xi}{h_M^2}(\xi - \zeta_{M-1})^2\left[3 - \frac{2(\xi - \zeta_{M-1})}{h_M}\right]\log\left|\frac{\xi - 1}{\xi - \zeta_{M-1}}\right| \tag{49e}$$

and

$$D_{2M+1}(\xi) = P_{2M+1}(\xi) + \frac{\xi}{h_M^2}(\xi - 1)(\xi - \zeta_{M-1})^2\log\left|\frac{\xi - 1}{\xi - \zeta_{M-1}}\right|, \tag{49f}$$

where

$$P_0(\xi) = \zeta_1\left[\frac{1}{2} - \frac{5}{6}\frac{\xi}{\zeta_1} - 2\left(\frac{\xi}{\zeta_1}\right)^2 + 2\left(\frac{\xi}{\zeta_1}\right)^3\right], \tag{50a}$$

$$P_1(\xi) = \zeta_1^2\left[\frac{1}{12} + \frac{1}{3}\frac{\xi}{\zeta_1} - \frac{3}{2}\left(\frac{\xi}{\zeta_1}\right)^2 + \left(\frac{\xi}{\zeta_1}\right)^3\right], \tag{50b}$$

$$P_{2\beta}(\xi) = \frac{1}{2}(h_\beta + h_{\beta+1}) - 2\left(\frac{\zeta_{\beta-1}}{h_\beta} + \frac{\zeta_{\beta+1}}{h_{\beta+1}} + \frac{\zeta_{\beta-1}^2}{h_\beta^2} - \frac{\zeta_{\beta+1}^2}{h_{\beta+1}^2}\right)\xi$$

$$+ 2\left(\frac{1}{h_\beta} + \frac{1}{h_{\beta+1}} + 2\frac{\zeta_{\beta-1}}{h_\beta^2} - 2\frac{\zeta_{\beta+1}}{h_{\beta+1}^2}\right)\xi^2 + 2\left(\frac{1}{h_{\beta+1}^2} - \frac{1}{h_\beta^2}\right)\xi^3 \tag{50c}$$

and

$$P_{2\beta+1}(\xi) = \frac{1}{12}(h_{\beta+1}^2 - h_\beta^2) + \frac{1}{3}\left(3\frac{\zeta_{\beta-1}^2}{h_\beta} + 3\frac{\zeta_{\beta+1}^2}{h_{\beta+1}} - 2h_\beta - 2h_{\beta+1}\right)\xi$$

$$- 2\left(\frac{\zeta_{\beta-1}}{h_\beta} + \frac{\zeta_{\beta+1}}{h_{\beta+1}}\right)\xi^2 + \left(\frac{1}{h_\beta} + \frac{1}{h_{\beta+1}}\right)\xi^3, \tag{50d}$$

for $\beta = 1, 2, \ldots, M-1$,

$$P_{2M}(\xi) = \frac{1}{2}h_M + \frac{1}{6}\left(5 + \frac{12}{h_M} - \frac{12}{h_M^2}\right)\xi + 2(2 - h_M)\frac{\xi^2}{h_M^2} - \frac{2\xi^3}{h_M^2} \tag{50e}$$

and

$$P_{2M+1}(\xi) = -\frac{1}{12}h_M^2 + \frac{1}{6}\left(\frac{6}{h_M} - 9 + 2h_M\right)\xi - \frac{1}{2}\left(\frac{4}{h_M} - 3\right)\xi^2 + \frac{\xi^3}{h_M}. \tag{50f}$$

We note that, as for the case of linear splines, there may be loss of accuracy (caused by the cancellation of large numbers in finite-precision arithmetic) if the explicit expressions Eqs. (49) and (50) are used to evaluate $D_{2\beta}(\xi)$ or $D_{2\beta+1}(\xi)$ if $|\xi - \zeta_{\beta-1}|/h_\beta$ or $|\zeta_{\beta+1} - \xi|/h_{\beta+1}$ is large compared to 1. Under these conditions we have evaluated $D_{2\beta}(\xi)$ and $D_{2\beta+1}(\xi)$ by making appropriate binomial expansions in Eqs. (49).

Proceeding with the computation of the albedo A^* and transmission factor B^* as given by Eqs. (39), we substitute Eqs. (42) and (43) into

$$J_{2\beta} = \int_0^1 \mu \Phi_\beta(\mu)\,\mathrm{d}\mu \quad \text{and} \quad J_{2\beta+1} = \int_0^1 \mu \Psi_\beta(\mu)\,\mathrm{d}\mu \tag{51a, b}$$

to obtain

$$J_0 = \tfrac{3}{20}\zeta_1^2, \quad J_1 = \tfrac{1}{30}\zeta_1^3, \tag{52a, b}$$

$$J_{2\beta} = \tfrac{1}{20}(\zeta_{\alpha+1} - \zeta_{\alpha-1})(3\zeta_{\alpha+1} + 4\zeta_\alpha + 3\zeta_{\alpha-1}) \tag{52c}$$

and

$$J_{2\beta+1} = \tfrac{1}{60}[(\zeta_{\alpha+1} - \zeta_\alpha)^2(2\zeta_{\alpha+1} + 3\zeta_\alpha) - (\zeta_\alpha - \zeta_{\alpha-1})^2(3\zeta_\alpha + 2\zeta_{\alpha-1})], \tag{52d}$$

for $\beta = 1, 2, \ldots, M-1$,

$$J_{2M} = \tfrac{1}{20}(7 + 3\zeta_{M-1})(1 - \zeta_{M-1}) \tag{52e}$$

and

$$J_{2M+1} = -\tfrac{1}{60}(3 + 2\zeta_{M-1})(1 - \zeta_{M-1})^2. \tag{52f}$$

Finally, to complete the specification of the F_N method based on Hermite cubic splines, we must define a knot and collocation-point strategy. We have implemented several different strategies and have found that one good choice is to distribute the knots quadratically, which is in keeping with the spirit of de Boor's analysis,[13] and to use the knots and the midpoints of the subintervals as collocation points. Thus we take the knots as

$$\zeta_\alpha = \left(\frac{\alpha}{M}\right)^2 \tag{53}$$

for $\alpha = 0, 1, \ldots, M$, and we take the collocation points to be

$$\xi_{2\beta} = \zeta_\beta \quad \text{for} \quad \beta = 0, 1, \ldots, M, \tag{54a}$$

$$\xi_{2\beta-1} = \tfrac{1}{2}(\zeta_{\beta-1} + \zeta_\beta) \quad \text{for} \quad \beta = 1, 2, \ldots, M \tag{54b}$$

and, for the consistency condition,

$$\xi_N = \nu_0. \tag{54c}$$

The results of our computations with the Hermite cubic spline formulation of the F_N method are summarized in Tables 1 and 2. In the last two columns of Table 1 are listed the albedos and the transmission factors for the albedo problem of Sec. 6. We note that for each of the three methods summarized in Table 1 the dimension of the approximating subspace is $N + 1$. Further for the linear splines the interval [0, 1] has been partitioned into N subintervals, and for the cubic splines the same interval has been partitioned into $M = (N-1)/2$ subintervals.

For this same albedo problem we have computed the angular distribution of the diffuse radiation field, i.e.,

$$\psi_*(x, \mu) = \psi(x, \mu) - \tfrac{1}{2}\delta(\mu - \mu_0)\exp[-(x - L)/\mu], \tag{55}$$

at the boundaries, $x = L$ and $x = R$, and at selected interior points in the slab. In presenting our results in Table 2 we take $L = 0$ and report, for the case $\mu_0 = 1$, $\Delta = 1$ and $c = 0.9$, the converged results we found for $\psi_*(x, \mu)$ from the solution based on Hermite cubic splines. In order to obtain the results reported in Table 2 (and thought to be correct to within ± 1 in the last digits given) we used $N = 127$.

Table 2. Values of $\psi_*(x, \mu)$ for $\mu_0 = 1$, $\Delta = 1$ and $c = 0.9$.

μ	$x = 0$	$x = 0.05$	$x = 0.1$	$x = 0.2$	$x = 0.5$	$x = 0.75$	$x = 1$
-1.0	2.10001(-1)	2.01847(-1)	1.92839(-1)	1.73234(-1)	1.08052(-1)	5.20545(-2)	
-0.9	2.23885(-1)	2.15593(-1)	2.06342(-1)	1.86021(-1)	1.17325(-1)	5.71217(-2)	
-0.8	2.39472(-1)	2.31116(-1)	2.21674(-1)	2.00691(-1)	1.28289(-1)	6.32737(-2)	
-0.7	2.56974(-1)	2.48678(-1)	2.39138(-1)	2.17620(-1)	1.41428(-1)	7.08963(-2)	
-0.6	2.76545(-1)	2.68510(-1)	2.59037(-1)	2.37232(-1)	1.57408(-1)	8.05794(-2)	
-0.5	2.98151(-1)	2.90702(-1)	2.81572(-1)	2.59941(-1)	1.77150(-1)	9.32667(-2)	
-0.4	3.21267(-1)	3.14917(-1)	3.06584(-1)	2.85937(-1)	2.01882(-1)	1.10552(-1)	
-0.3	3.44241(-1)	3.39767(-1)	3.32927(-1)	3.14583(-1)	2.32986(-1)	1.35283(-1)	
-0.2	3.63306(-1)	3.61744(-1)	3.57282(-1)	3.42984(-1)	2.70683(-1)	1.72635(-1)	
-0.1	3.72669(-1)	3.75369(-1)	3.74075(-1)	3.64932(-1)	3.08157(-1)	2.28599(-1)	
0.0	3.59371(-1)	3.74856(-1)	3.79338(-1)	3.77346(-1)	3.32750(-1)	2.72344(-1)	
0.0		3.74856(-1)	3.79338(-1)	3.77346(-1)	3.32750(-1)	2.72344(-1)	1.86138(-1)
0.1		1.45459(-1)	2.36850(-1)	3.26737(-1)	3.47040(-1)	2.96869(-1)	2.24033(-1)
0.2		8.17099(-2)	1.47169(-1)	2.38445(-1)	3.27320(-1)	3.06245(-1)	2.48328(-1)
0.3		5.66940(-2)	1.05960(-1)	1.83409(-1)	2.91657(-1)	2.96255(-1)	2.57582(-1)
0.4		4.33879(-2)	8.26581(-2)	1.48251(-1)	2.57648(-1)	2.77711(-1)	2.55202(-1)
0.5		3.51359(-2)	6.77242(-2)	1.24185(-1)	2.28845(-1)	2.57409(-1)	2.46679(-1)
0.6		2.95196(-2)	5.73489(-2)	1.06760(-1)	2.05035(-1)	2.38028(-1)	2.35536(-1)
0.7		2.54507(-2)	4.97251(-2)	9.35871(-2)	1.85331(-1)	2.20417(-1)	2.23633(-1)
0.8		2.23672(-2)	4.38879(-2)	8.32899(-2)	1.68882(-1)	2.04711(-1)	2.11883(-1)
0.9		1.99500(-2)	3.92759(-2)	7.50244(-2)	1.55000(-1)	1.90787(-1)	2.00704(-1)
1.0		1.80042(-2)	3.55402(-2)	6.82457(-2)	1.43159(-1)	1.78448(-1)	1.90265(-1)

We comment on several numerical considerations for this problem. We note that as $c \to 0$, we have $\nu_0 \to 1$, so that Eqs. (17) for $\xi = 1$ and $\xi = \nu_0$ become nearly linearly dependent; this clearly leads to ill conditioning of the linear system corresponding to Eqs. (21). In this case we remove $\xi = 1$ from the set of collocation points and in either the first or the last interval, instead of using the midpoint Eq. (37a), we use two interior collocation points. We have observed, however, that if this strategy is used when ν_0 is far from 1, say, $\nu_0 > 1.5$, then the spline may be very much in error at the point $\mu = 1$.

It is known that $\psi_*(x, \mu)$ has singularities at $x = 0$ for $\mu = 0$ and at $x = \Delta$ for $\mu = 0$. We note that Kaper and Kellogg[14] have reported an analysis of the asymptotic behavior of $\psi_*(x, \mu)$ near the singularities. One advantage of the F_N method is that on the boundries of the slab the discontinuities at $\mu = 0$ in $\psi_*(0, \mu)$ and $\psi_*(\Delta, \mu)$ are automatically resolved. Another advantage of the method is that the $|\mu|\log|\mu|$ singularities of the exit angular fluxes are resolved in the postprocessing.

The most difficult values to compute accurately in Table 2 are the entries at $x = 0.05$ for $\mu > 0$. In fact, we have found that as $x \to 0$, it is necessary to use ever increasing values of N in order to maintain the same degree of accuracy in the computation of $\psi_*(x, \mu)$. It is also difficult to compute accurately the interior values of $\psi_*(x, \mu)$ for x near Δ and $\mu < 0$. We have chosen to use the same value of N for all values of x because we then need to compose and factor the matrix in Eqs. (26) only once. Alternatively, if one is interested in computing the diffuse radiation field very close to the boundaries, one might consider letting N depend on x or incorporating the appropriate singular functions in the spline.

8. CONCLUSIONS

We have found that both linear and Hermite cubic splines are convenient to use as basis functions for the F_N method. Furthermore the Hermite cubic splines yield, when the approximating subspaces have the same dimensionality, an accuracy almost as good as that offered by the more traditional

Legendre polynomial basis functions. Linear splines are considerably less effective, however, probably because the radiation field is quite smooth, while the first derivative of linear splines is discontinuous at each knot.

We have found that the accuracy of the solution with either linear or Hermite cubic splines is influenced only weakly by the choice of the location of the knots. A bad choice of the collocation points can, however, lead to disastrous results.

To comment on the parallel processing aspects of different implementations of the F_N method, we note that in the traditional version based on Legendre polynomials the coefficients $A_\alpha(\xi_\beta)$ and $B_\alpha(\xi_\beta)$ in Eqs. (26) are computed recursively. To compute the matrix elements in this manner is clearly a sequential operation with respect to α. In the spline version of the method each of these matrix elements may be computed independently, and so a significantly greater degree of parallelism can be used in the calculation.

Finally we note that the spline version of the F_N method has additional features in that basis functions that include more of the basic structure of the solution in a limited domain can be included in the approximating subspace, and the distribution of knots can be selected, perhaps adaptively, to improve the calculation.

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