

## THE $F_N$ METHOD FOR RADIATIVE TRANSFER MODELS THAT INCLUDE POLARIZATION EFFECTS

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**Abstract**—The  $F_N$  method is used to construct an accurate solution for all components ( $m \geq 0$ ) in a Fourier representation of the Stokes vector basic to the scattering of polarized light. Computational aspects of the solution are reported in detail, and the established solution is used to obtain numerical results, accurate to six significant figures, for the four Stokes parameters for each of two considered test problems.

### 1. INTRODUCTION

In a recent work,<sup>1</sup> we reported a solution, based on the generalized spherical harmonics method, for the Stokes vector basic to the scattering of polarized light. Here, we consider the same class of problems, and we use the  $F_N$  method<sup>2-4</sup> to develop a highly accurate solution.

We let  $\mathbf{I}(\tau, \mu, \phi)$  denote the density vector with the four Stokes parameters  $I(\tau, \mu, \phi)$ ,  $Q(\tau, \mu, \phi)$ ,  $U(\tau, \mu, \phi)$  and  $V(\tau, \mu, \phi)$  as components and consider the equation of transfer<sup>5-7</sup>

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu, \phi) + \mathbf{I}(\tau, \mu, \phi) = \frac{\omega}{4\pi} \int_0^{2\pi} \int_{-1}^1 \mathbf{P}(\mu, \mu', \phi - \phi') \mathbf{I}(\tau, \mu', \phi') d\mu' d\phi', \quad (1)$$

where we use the analytical representation<sup>8</sup>

$$\mathbf{P}(\mu, \mu', \phi - \phi') = \frac{1}{2} \sum_{m=0}^L (2 - \delta_{0,m}) [C^m(\mu, \mu') \cos m(\phi - \phi') + S^m(\mu, \mu') \sin m(\phi - \phi')] \quad (2)$$

for the phase matrix. Here,

$$C^m(\mu, \mu') = A^m(\mu, \mu') + \mathbf{D}A^m(\mu, \mu')\mathbf{D} \quad (3a)$$

and

$$S^m(\mu, \mu') = A^m(\mu, \mu')\mathbf{D} - \mathbf{D}A^m(\mu, \mu'), \quad (3b)$$

where

$$A^m(\mu, \mu') = \sum_{l=m}^L \Pi_l^m(\mu) \mathbf{B}_l \Pi_l^m(\mu') \quad (4a)$$

and

$$\mathbf{D} = \text{diag} \{1, 1, -1, -1\}. \quad (4b)$$

In addition,

$$\Pi_l^m(\mu) = \left[ \frac{(l-m)!}{(l+m)!} \right]^{1/2} \begin{pmatrix} P_l^m(\mu) & 0 & 0 & 0 \\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0 \\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0 \\ 0 & 0 & 0 & P_l^m(\mu) \end{pmatrix}, \quad (5)$$

where

$$P_l^m(\mu) = (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu) \quad (6)$$

is used to denote the associated Legendre functions. The functions  $R_l^m(\mu)$  and  $T_l^m(\mu)$  are as defined and used in Refs. 6 and 8. In contrast to Ref. 8, we have as a matter of computational convenience included the factor  $[(l-m)!/(l+m)!]^{1/2}$  in our current definitions of the matrices  $\Pi_l^m(\mu)$ . Continuing to follow Ref. 1, we note that  $\tau \in [0, \tau_0]$  is the optical variable,  $\mu$  is the direction cosine of the propagating radiation,  $w$  is the single-scattering albedo and the scattering law is defined by the *Greek constants*  $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \epsilon_l, \zeta_l\}$  so that

$$\mathbf{B}_l = \begin{pmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\epsilon_l \\ 0 & 0 & \epsilon_l & \delta_l \end{pmatrix}. \quad (7)$$

We seek a solution of Eq. (1) subject to the boundary conditions

$$\mathbf{I}(0, \mu, \phi) = \pi \delta(\mu - \mu_0) \delta(\phi - \phi_0) \mathbf{F} \quad (8a)$$

and

$$\mathbf{I}(\tau_0, -\mu, \phi) = \frac{\lambda_0}{\pi} \mathbf{L} \int_0^{2\pi} \int_0^1 \mathbf{I}(\tau_0, \mu', \phi') \mu' d\mu' d\phi' \quad (8b)$$

for  $\mu \in [0, 1]$  and  $\phi \in [0, 2\pi]$ . Here,  $\lambda_0$  is the coefficient for Lambert reflection,  $\mathbf{L} = \text{diag}\{1, 0, 0, 0\}$  and the flux vector  $\mathbf{F}$  has entries  $F_I, F_Q, F_U$  and  $F_V$  that are considered given.

## 2. FOURIER DECOMPOSITION

As Eqs. (1) and (8) define the radiation field we seek, we now utilize the Fourier decomposition of the Stokes vector that was developed in Ref. 6 and subsequently used by McCormick and Sanchez in a work<sup>9</sup> concerning the inverse problem in radiative transfer with polarization. First of all, we define

$$\Phi_1^m(\chi) = (2 - \delta_{0,m}) \text{diag} \{ \cos m\chi, \cos m\chi, \sin m\chi, \sin m\chi \} \quad (9a)$$

and

$$\Phi_2^m(\chi) = (2 - \delta_{0,m}) \text{diag} \{ -\sin m\chi, -\sin m\chi, \cos m\chi, \cos m\chi \} \quad (9b)$$

and note that we can write the phase matrix as

$$\mathbf{P}(\mu, \mu', \phi - \phi') = \sum_{m=0}^L [\Phi_1^m(\phi - \phi') \mathbf{A}^m(\mu, \mu') \mathbf{D}_1 + \Phi_2^m(\phi - \phi') \mathbf{A}^m(\mu, \mu') \mathbf{D}_2], \quad (10)$$

where  $\mathbf{A}^m(\mu, \mu')$  is given by Eq. (4a) and where

$$\mathbf{D}_1 = \text{diag} \{ 1, 1, 0, 0 \} \quad (11a)$$

and

$$\mathbf{D}_2 = \text{diag} \{ 0, 0, 1, 1 \}. \quad (11b)$$

To accomplish our Fourier decomposition, we first let

$$\mathbf{I}_c(\tau, \mu, \phi) = \pi \delta(\mu - \mu_0) \delta(\phi - \phi_0) \mathbf{F} \exp\{-\tau/\mu\} \quad (12)$$

and then express the Stokes vector as

$$\mathbf{I}(\tau, \mu, \phi) = \mathbf{I}_c(\tau, \mu, \phi) + \sum_{m=0}^L \sum_{k=1}^2 \Phi_k^m(\phi - \phi_0) [\mathbf{I}_k^m(\tau, \mu) - \mathbf{I}_k^m(0, \mu) \exp\{-\tau/\mu\}] \quad (13a)$$

and

$$\mathbf{I}(\tau, -\mu, \phi) = \sum_{m=0}^L \sum_{k=1}^2 \Phi_k^m(\phi - \phi_0) \mathbf{I}_k^m(\tau, -\mu) \quad (13b)$$

for  $\mu \in [0, 1]$  and  $\phi \in [0, 2\pi]$ . We can now substitute Eqs. (13) into Eq. (1) and use, for  $k = 1$  and 2,

$$\int_0^{2\pi} \mathbf{P}(\mu, \mu', \phi - \phi') \Phi_k^m(\phi' - \phi_0) d\phi' = 2\pi \Phi_k^m(\phi - \phi_0) \mathbf{K}^m(\mu' \rightarrow \mu), \quad (14)$$

where

$$\mathbf{K}^m(\mu' \rightarrow \mu) = \sum_{l=m}^L \Pi_l^m(\mu) \mathbf{B}_l \Pi_l^m(\mu') \quad (15)$$

to obtain

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}_k^m(\tau, \mu) + \mathbf{I}_k^m(\tau, \mu) = \frac{\varpi}{2} \int_{-1}^1 \mathbf{K}^m(\mu' \rightarrow \mu) \mathbf{I}_k^m(\tau, \mu') d\mu' \quad (16)$$

and the boundary conditions

$$\mathbf{I}_k^m(0, \mu) = \frac{1}{2} \delta(\mu - \mu_0) \mathbf{D}_k \mathbf{F} \quad (17a)$$

and

$$\mathbf{I}_k^m(\tau_0, -\mu) = 2\lambda_0 \delta_{0,m} \delta_{1,k} \mathbf{L} \int_0^1 \mathbf{I}_k^m(\tau_0, \mu') \mu' d\mu' \quad (17b)$$

for  $\mu \in [0, 1]$ .

We thus must solve Eq. (16) subject to Eqs. (17) and obtain the desired results from Eqs. (12) and (13).

### 3. INTEGRAL EQUATIONS AND CONSTRAINTS

To make our notation more concise, we let  $\mathbf{I}(\tau, \mu)$  denote  $\mathbf{I}_k^m(\tau, \mu)$ . We then let

$$\mathbf{I}_l(\tau) = \int_{-1}^1 \Pi_l^m(\mu) \mathbf{I}(\tau, \mu) d\mu \quad (18)$$

and write Eq. (16), after changing  $\mu$  to  $-\mu$ , as

$$-\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, -\mu) + \mathbf{I}(\tau, -\mu) = \frac{\varpi}{2} \sum_{l=m}^L (-1)^{l-m} \mathbf{D} \Pi_l^m(\mu) \mathbf{D} \mathbf{B}_l \mathbf{I}_l(\tau), \quad (19)$$

where we have used the fact that

$$\Pi_l^m(-\mu) = (-1)^{l-m} \mathbf{D} \Pi_l^m(\mu) \mathbf{D}. \quad (20)$$

We now multiply Eq. (19) by  $\exp\{-\tau/s\}$  and integrate over  $\tau$  from  $\tau = a$  to  $\tau = b$  and obtain, after an integration by parts,

$$s\mu \mathbf{B}(\mu, s) - (\mu - s) \int_a^b \exp\{-\tau/s\} \mathbf{I}(\tau, -\mu) d\tau = \frac{\varpi s}{2} \sum_{l=m}^L (-1)^{l-m} \mathbf{D} \Pi_l^m(\mu) \mathbf{D} \mathbf{B}_l \mathbf{I}_l^*(s), \quad (21)$$

where

$$\mathbf{B}(\mu, s) = \mathbf{I}(a, -\mu) \exp\{-a/s\} - \mathbf{I}(b, -\mu) \exp\{-b/s\} \quad (22)$$

and

$$\mathbf{I}_l^*(s) = \int_a^b \exp\{-\tau/s\} \mathbf{I}_l(\tau) d\tau. \quad (23)$$

In order to keep our development general, we do not, at this point, specify  $a$  and  $b$  more precisely than  $a, b \in [0, \tau_0]$  and  $a < b$ . Considering that  $s \in [-1, 1]$ , we next multiply Eq. (21) by  $(-1)^m \mathbf{D} \Pi_a^m(\mu) \mathbf{D} (\mu - s)^{-1}$  and integrate over  $\mu$  from  $-1$  to  $1$  to obtain

$$\begin{aligned} (-1)^a \mathbf{D} \mathbf{I}_a^*(s) + \frac{\varpi s}{2} \int_{-1}^1 \Pi_a^m(\mu) \sum_{l=m}^L (-1)^l \Pi_l^m(\mu) \mathbf{B}_l \mathbf{D} \mathbf{I}_l^*(s) \frac{d\mu}{\mu - s} \\ = (-1)^m s \int_{-1}^1 \mu \Pi_a^m(\mu) \mathbf{D} \mathbf{B}(\mu, s) \frac{d\mu}{\mu - s}. \end{aligned} \quad (24)$$

We now require the polynomial matrices  $\mathbf{G}_l^m(\xi)$  introduced in Ref. 1 and redefined in Ref. 10. Thus,

$$\mathbf{G}_{l+1}^m(\xi) = (\mathbf{U}_{l+1}^m)^{-1}[(\xi \mathbf{h}_l + \mathbf{V}_l^m)\mathbf{G}_l^m(\xi) - \mathbf{U}_l^m \mathbf{G}_{l-1}^m(\xi)], \quad (25)$$

where  $l = 2, 3, \dots$ , for  $m = 0$  and  $1$  and  $l = m, m+1, \dots$ , for  $m \geq 2$ . Here,

$$\mathbf{h}_l = (2l+1)\mathbf{I} - \varpi \mathbf{B}_l, \quad (26)$$

$$\mathbf{U}_l^m = (l^2 - m^2)^{1/2} \text{diag}\{1, (l^2 - 4)^{1/2}/l, (l^2 - 4)^{1/2}/l, 1\} \quad (27)$$

and

$$\mathbf{V}_l^m = \frac{2m(2l+1)}{l(l+1)} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (28)$$

As in Ref. 10, we use the starting values

$$\mathbf{G}_0^0(\xi) = \text{diag}\{1, 0, 0, 1\}, \quad (29a)$$

$$\mathbf{G}_1^0(\xi) = \text{diag}\{(1 - \varpi)\xi, 0, 0, (1 - \varpi\delta_0)\xi\}, \quad (29b)$$

$$\mathbf{G}_2^0(\xi) = \text{diag}\left\{\frac{1}{2}[(1 - \varpi)(3 - \varpi\beta_1)\xi^2 - 1], 1, 1, \frac{1}{2}[(1 - \varpi\delta_0)(3 - \varpi\delta_1)\xi^2 - 1]\right\}, \quad (29c)$$

$$\mathbf{G}_1^1(\xi) = 2^{-1/2} \text{diag}\{1, 0, 0, 1\}, \quad (29d)$$

$$\mathbf{G}_2^1(\xi) = \text{diag}\{6^{-1/2}(3 - \varpi\beta_1)\xi, 1, 1, 6^{-1/2}(3 - \varpi\delta_1)\xi\} \quad (29e)$$

and, for  $m \geq 2$ ,

$$\mathbf{G}_m^m(\xi) = \Delta_m = S_m \text{diag}\{1, R_m, R_m, 1\}, \quad (29f)$$

where

$$S_m = (2m-1)!![(2m)!]^{-1/2} \quad (30)$$

and

$$R_m = \left[ \frac{m(m-1)}{(m+2)(m+1)} \right]^{1/2}. \quad (31)$$

We note that, in general,

$$\mathbf{G}_l^m(-\xi) = (-1)^{l-m} \mathbf{D} \mathbf{G}_l^m(\xi) \mathbf{D} \mathbf{K}^m, \quad (32)$$

where

$$\mathbf{K}^m = \text{diag}\{1, 1 - 2\delta_{m,1}, 1 - 2\delta_{m,1}, 1\}. \quad (33)$$

Introducing  $\mathbf{J} = \text{diag}\{1, 1, 1, -1\}$ , we now multiply Eq. (24) by  $[\mathbf{G}_\alpha^m(s)]^T \mathbf{J} \mathbf{B}_\alpha$  and sum the resulting equation over  $\alpha$  from  $\alpha = m$  to  $\alpha = L$  to obtain

$$\sum_{l=m}^L (-1)^{l-m} [\mathbf{W}_l^m(s)]^T \mathbf{J} \mathbf{B}_l \mathbf{D} \mathbf{I}_l^*(s) = s \int_{-1}^1 \mu \mathbf{G}^T(s, \mu) \mathbf{J} \mathbf{D} \mathbf{B}(\mu, s) \frac{d\mu}{\mu - s}, \quad (34)$$

where, since  $\mathbf{J} \mathbf{B}_l^T = \mathbf{B}_l \mathbf{J}$ ,

$$\mathbf{G}(s, \mu) = \sum_{l=m}^L \mathbf{\Pi}_l^m(\mu) \mathbf{B}_l \mathbf{G}_l^m(s) \quad (35)$$

and

$$\mathbf{W}_l^m(s) = \mathbf{G}_l^m(s) + \frac{\varpi s}{2} \int_{-1}^1 \mathbf{\Pi}_l^m(\mu) \mathbf{G}(s, \mu) \frac{d\mu}{\mu - s}. \quad (36)$$

In a related paper<sup>10</sup> devoted to a study of the discrete spectrum for these polarization problems, we found we could express  $W_l^m(s)$ ,  $s \in [-1, 1]$ , as

$$W_l^m(s) = Y_l^m(s)A^m(s), \quad (37)$$

where

$$Y_l^0(s) = \Pi_l^0(s)[\Pi_2^0(s)]^{-1} \text{diag}\{P_2(s), 1, 1, P_2(s)\}, \quad (38a)$$

$$Y_l^1(s) = \Pi_l^1(s)[\Pi_2^1(s)]^{-1} \text{diag}\{\frac{1}{2}6^{1/2}s, 1, 1, \frac{1}{2}6^{1/2}s\} \quad (38b)$$

and, for  $m \geq 2$ ,

$$Y_l^m(s) = \Pi_l^m(s)[\Pi_m^m(s)]^{-1} \Delta_m. \quad (38c)$$

In addition

$$A^m(s) = \mathbf{I} + \frac{\omega s}{2} \int_{-1}^1 \Xi^m(\mu) \mathbf{G}(s, \mu) \frac{d\mu}{\mu - s}, \quad (39)$$

where

$$\Xi^0(\mu) = \text{diag}\{1, R_2^0(\mu), R_2^0(\mu), 1\}, \quad (40a)$$

$$\Xi^1(\mu) = (1 - \mu^2)^{1/2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{\mu}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{\mu}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (40b)$$

and, for  $m \geq 2$ ,

$$\Xi^m(\mu) = (1 - \mu^2)^{m/2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1 + \mu^2}{1 - \mu^2} & -\frac{2\mu}{1 - \mu^2} & 0 \\ 0 & -\frac{2\mu}{1 - \mu^2} & \frac{1 + \mu^2}{1 - \mu^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (40c)$$

Using Eq. (37), we now can write Eq. (34) as

$$[A^m(s)]^T \mathbf{X}(s) = s \int_{-1}^1 \mu \mathbf{G}^T(s, \mu) \mathbf{JDB}(\mu, s) \frac{d\mu}{\mu - s}, \quad (41)$$

where

$$\mathbf{X}(s) = \sum_{i=m}^L (-1)^{l-m} [Y_l^m(s)]^T \mathbf{JDB}_l \mathbf{I}_l^*(s). \quad (42)$$

We note that the transforms  $\mathbf{I}_l^*(s)$  are analytic everywhere in the complex  $s$  plane except for the origin where there are essential singularities. Rather than consider Eq. (41) for all  $s$ , we first multiply Eq. (41) by  $\exp\{a/s\}$  and consider the resulting equation with  $a = \tau$ ,  $b = \tau_0$  and  $\Re s > 0$ . Next we multiply Eq. (41), after changing  $s$  to  $-s$ , by  $\exp\{-b/s\}$  and consider the resulting equation with  $a = 0$ ,  $b = \tau$  and  $\Re s > 0$ . We therefore write, for  $\Re s > 0$ ,

$$[A^m(s)]^T \mathbf{I}(s) = \int_{-1}^1 \mu \mathbf{G}^T(s, \mu) \mathbf{JDC}(\mu, s) \frac{d\mu}{\mu - s} \quad (43a)$$

and

$$[A^m(s)]^T J(s) = \int_{-1}^1 \mu G^T(s, \mu) J D(\mu, s) \frac{d\mu}{\mu - s}, \quad (43b)$$

where

$$I(s) = \frac{1}{s} \sum_{l=m}^L (-1)^{l-m} [Y_l^m(s)]^T J D B_l \int_{\tau}^{\tau_0} \exp\{-(\tau' - \tau)/s\} I_l(\tau') d\tau', \quad (44a)$$

$$J(s) = \frac{1}{s} \sum_{l=m}^L [Y_l^m(s)]^T J B_l \int_0^{\tau} \exp\{-(\tau - \tau')/s\} I_l(\tau') d\tau', \quad (44b)$$

$$C(\mu, s) = I(\tau, -\mu) - I(\tau_0, -\mu) \exp\{-(\tau_0 - \tau)/s\} \quad (45a)$$

and

$$D(\mu, s) = I(\tau, \mu) - I(0, \mu) \exp\{-\tau/s\}. \quad (45b)$$

If we now consider that the discrete spectrum (in the right half-plane) is defined, as in Ref. 10, by

$$A^m(v_\beta) M(v_\beta) = 0, \quad \beta = 1, 2, \dots, N, \quad v_\beta \in [0, 1], \quad (46)$$

then we can readily deduce from Eqs. (43) the integral constraints

$$M^T(v_\beta) \int_{-1}^1 \mu G^T(v_\beta, \mu) J D C(\mu, v_\beta) \frac{d\mu}{\mu - v_\beta} = 0 \quad (47a)$$

and

$$M^T(v_\beta) \int_{-1}^1 \mu G^T(v_\beta, \mu) J D(\mu, v_\beta) \frac{d\mu}{\mu - v_\beta} = 0 \quad (47b)$$

for  $\beta = 1, 2, \dots, N$ .

We can also let  $s \rightarrow v \in [0, 1)$  from above (+) and below (-) the branch cut of  $A^m(s)$  and use the Plemelj formulae<sup>11</sup> to deduce from Eqs. (43) that

$$\begin{aligned} & \left\{ [\lambda^m(v)]^T \pm \frac{\pi i \varpi v}{2} G^T(v, v) \Xi^m(v) \right\} I(v) \\ & = \int_{-1}^1 \mu G^T(v, \mu) J D C(\mu, v) \frac{d\mu}{\mu - v} \pm \pi i v G^T(v, v) J D C(v, v) \end{aligned} \quad (48a)$$

and

$$\begin{aligned} & \left\{ [\lambda^m(v)]^T \pm \frac{\pi i \varpi v}{2} G^T(v, v) \Xi^m(v) \right\} J(v) \\ & = \int_{-1}^1 \mu G^T(v, \mu) J D(\mu, v) \frac{d\mu}{\mu - v} \pm \pi i v G^T(v, v) J D(v, v). \end{aligned} \quad (48b)$$

Here, we use  $\int$  to denote that the integration is to be carried out in the Cauchy principal-value sense. Also,

$$\lambda^m(v) = I + \frac{\varpi v}{2} \int_{-1}^1 \Xi^m(\mu) G(v, \mu) \frac{d\mu}{\mu - v}. \quad (49)$$

We can also eliminate  $I(v)$  and  $J(v)$  respectively between the two versions of Eqs. (48a) and (48b) to find

$$[\lambda^m(v)]^T [\Xi^m(v)]^{-1} J D C(v, v) + \frac{\varpi}{2} \int_{-1}^1 \mu G^T(v, \mu) J D C(\mu, v) \frac{d\mu}{v - \mu} = 0 \quad (50a)$$

and

$$[\lambda^m(v)]^T [\Xi^m(v)]^{-1} \mathbf{J}\mathbf{D}(v, v) + \frac{\varpi}{2} \int_{-1}^1 \mu \mathbf{G}^T(v, \mu) \mathbf{J}\mathbf{D}(\mu, v) \frac{d\mu}{v - \mu} = \mathbf{0} \quad (50b)$$

for  $v \in [0, 1]$ .

Intending first to establish  $\mathbf{I}(0, -\mu)$  and  $\mathbf{I}(\tau_0, \mu)$  for  $\mu \in [0, 1]$ , we use Eq. (17b) and deduce from Eqs. (47a) and (50a), with  $\tau = 0$ , and Eqs. (47b) and (50b), with  $\tau = \tau_0$ , the system of equations

$$\mathbf{X}[\mathbf{DI}(0, -\mu)](v_\beta) + \exp\{-\tau_0/v_\beta\} \{\mathbf{Y}[\mathbf{I}(\tau_0, \mu)](v_\beta) - 2\lambda_0 \delta_{0,m} \delta_{1,k} \mathbf{X}[\mathbf{LE}](v_\beta)\} = \mathbf{Y}[\mathbf{I}(0, \mu)](v_\beta) \quad (51a)$$

and

$$\begin{aligned} \mathbf{X}[\mathbf{I}(\tau_0, \mu)](v_\beta) - 2\lambda_0 \delta_{0,m} \delta_{1,k} \mathbf{Y}[\mathbf{LE}](v_\beta) \\ + \exp\{-\tau_0/v_\beta\} \mathbf{Y}[\mathbf{DI}(0, -\mu)](v_\beta) = \exp\{-\tau_0/v_\beta\} \mathbf{X}[\mathbf{I}(0, \mu)](v_\beta) \end{aligned} \quad (51b)$$

for  $\beta = 1, 2, \dots, \aleph$  and

$$\mathbf{X}[\mathbf{DI}(0, -\mu)](v) + \exp\{-\tau_0/v\} \{\mathbf{Y}[\mathbf{I}(\tau_0, \mu)](v) - 2\lambda_0 \delta_{0,m} \delta_{1,k} \mathbf{X}[\mathbf{LE}](v)\} = \mathbf{Y}[\mathbf{I}(0, \mu)](v) \quad (52a)$$

and

$$\begin{aligned} \mathbf{X}[\mathbf{I}(\tau_0, \mu)](v) - 2\lambda_0 \delta_{0,m} \delta_{1,k} \mathbf{Y}[\mathbf{LE}](v) \\ + \exp\{-\tau_0/v\} \mathbf{Y}[\mathbf{DI}(0, -\mu)](v) = \exp\{-\tau_0/v\} \mathbf{X}[\mathbf{I}(0, \mu)](v) \end{aligned} \quad (52b)$$

for  $v \in [0, 1]$ . In writing Eqs. (51) and (52), we have used

$$\mathbf{E} = \int_0^1 \mu \mathbf{I}(\tau_0, \mu) d\mu \quad (53)$$

and the notation

$$\mathbf{X}[\mathbf{F}(\mu)](v_\beta) = \mathbf{M}^T(v_\beta) \int_0^1 \mu \mathbf{G}^T(v_\beta, \mu) \mathbf{J}\mathbf{F}(\mu) \frac{d\mu}{v_\beta - \mu}, \quad (54a)$$

$$\mathbf{Y}[\mathbf{F}(\mu)](v_\beta) = \mathbf{M}^T(v_\beta) \mathbf{D}\mathbf{K}^m \int_0^1 \mu \mathbf{G}^T(-v_\beta, \mu) \mathbf{J}\mathbf{F}(\mu) \frac{d\mu}{v_\beta + \mu}, \quad (54b)$$

$$\mathbf{X}[\mathbf{F}(\mu)](v) = [\lambda^m(v)]^T [\Xi^m(v)]^{-1} \mathbf{J}\mathbf{F}(v) + \frac{\varpi}{2} \int_0^1 \mu \mathbf{G}^T(v, \mu) \mathbf{J}\mathbf{F}(\mu) \frac{d\mu}{v - \mu} \quad (54c)$$

and

$$\mathbf{Y}[\mathbf{F}(\mu)](v) = \frac{\varpi}{2} \mathbf{D}\mathbf{K}^m \int_0^1 \mu \mathbf{G}^T(-v, \mu) \mathbf{J}\mathbf{F}(\mu) \frac{d\mu}{v + \mu}. \quad (54d)$$

The integral constraints and the singular integral equations given by Eqs. (51) and (52) are to be solved to yield the boundary solutions  $\mathbf{I}(0, -\mu)$  and  $\mathbf{I}(\tau_0, \mu)$  for  $\mu \in [0, 1]$ .

If we consider that  $\mathbf{I}(0, -\mu)$  and  $\mathbf{I}(\tau_0, \mu)$ , for  $\mu \in [0, 1]$ , have been established we can, for  $\tau \in (0, \tau_0)$ , rewrite Eqs. (47) and (50) as

$$\mathbf{X}[\mathbf{DI}(\tau, -\mu)](v_\beta) - \mathbf{Y}[\mathbf{I}(\tau, \mu)](v_\beta) = \exp\{-(\tau_0 - \tau)/v_\beta\} \{\mathbf{X}[\mathbf{DI}(\tau_0, -\mu)](v_\beta) - \mathbf{Y}[\mathbf{I}(\tau_0, \mu)](v_\beta)\} \quad (55a)$$

and

$$\mathbf{X}[\mathbf{I}(\tau, \mu)](v_\beta) - \mathbf{Y}[\mathbf{DI}(\tau, -\mu)](v_\beta) = \exp\{-\tau/v_\beta\} \{\mathbf{X}[\mathbf{I}(0, \mu)](v_\beta) - \mathbf{Y}[\mathbf{DI}(0, -\mu)](v_\beta)\} \quad (55b)$$

for  $\beta = 1, 2, \dots, \aleph$  and

$$\mathbf{X}[\mathbf{DI}(\tau, -\mu)](v) - \mathbf{Y}[\mathbf{I}(\tau, \mu)](v) = \exp\{-(\tau_0 - \tau)/v\} \{\mathbf{X}[\mathbf{DI}(\tau_0, -\mu)](v) - \mathbf{Y}[\mathbf{I}(\tau_0, \mu)](v)\} \quad (56a)$$

and

$$\mathbf{X}[\mathbf{I}(\tau, \mu)](v) - \mathbf{Y}[\mathbf{DI}(\tau, -\mu)](v) = \exp\{-\tau/v\} \{\mathbf{X}[\mathbf{I}(0, \mu)](v) - \mathbf{Y}[\mathbf{DI}(0, -\mu)](v)\} \quad (56b)$$

for  $v \in [0, 1]$ . Clearly, once  $\mathbf{I}(0, -\mu)$  and  $\mathbf{I}(\tau_0, \mu)$ , for  $\mu \in [0, 1]$ , have been established, the right-hand sides of Eqs. (55) and (56) can be considered known, and so the system given by Eqs. (55) and (56) can, for any  $\tau \in (0, \tau_0)$ , be solved to yield  $\mathbf{I}(\tau, \pm\mu)$  for  $\mu \in [0, 1]$ .

#### 4. AN $F_N$ SOLUTION

In this section, we proceed to use the  $F_N$  method<sup>2-4</sup> to establish approximate but accurate solutions to Eqs. (51) and (52) for the boundary results and Eqs. (55) and (56) for the interior results. We first let  $\mathbf{I}_0(\tau, \mu)$  denote the exact solution for the case  $\varpi = 0$ . It follows from Eqs. (16) and (17) that

$$\mathbf{I}_0(\tau, \mu) = \frac{1}{2}\delta(\mu - \mu_0)\mathbf{D}_k\mathbf{F}\exp\{-\tau/\mu\} \quad (57a)$$

and

$$\mathbf{I}_0(\tau, -\mu) = \lambda_0\mu_0\delta_{0,m}\delta_{1,k}\exp\{-\tau_0/\mu_0\}\mathbf{L}\mathbf{F}\exp\{-(\tau_0 - \tau)/\mu\} \quad (57b)$$

for  $\mu \in [0, 1]$  and  $\tau \in [0, \tau_0]$ . We next let  $P_\alpha(\mu)$  denote the Legendre polynomial of order  $\alpha$  and express our  $F_N$  solutions in the forms

$$\mathbf{I}(0, -\mu) = \mathbf{I}_0(0, -\mu) + \varpi\mathbf{D}\mathbf{E}^m(\mu) \sum_{\alpha=0}^N \mathbf{a}_\alpha P_\alpha(2\mu - 1) \quad (58a)$$

and

$$\mathbf{I}(\tau_0, \mu) = \mathbf{I}_0(\tau_0, \mu) + \varpi\mathbf{E}^m(\mu) \sum_{\alpha=0}^N \mathbf{b}_\alpha P_\alpha(2\mu - 1) \quad (58b)$$

for  $\mu \in [0, 1]$  and

$$\mathbf{I}(\tau, -\mu) = \mathbf{I}_0(\tau, -\mu) + \varpi\mathbf{D}\mathbf{E}^m(\mu) \sum_{\alpha=0}^N \mathbf{c}_\alpha(\tau) P_\alpha(2\mu - 1) \quad (59a)$$

and

$$\mathbf{I}(\tau, \mu) = \mathbf{I}_0(\tau, \mu) + \varpi\mathbf{E}^m(\mu) \sum_{\alpha=0}^N \mathbf{d}_\alpha(\tau) P_\alpha(2\mu - 1) \quad (59b)$$

for  $\mu \in [0, 1]$  and  $\tau \in (0, \tau_0)$ . Here, the vectors  $\mathbf{a}_\alpha$ ,  $\mathbf{b}_\alpha$ ,  $\mathbf{c}_\alpha(\tau)$  and  $\mathbf{d}_\alpha(\tau)$  are to be found such that Eqs. (58) satisfy Eqs. (51) and (52) and Eqs. (59) satisfy Eqs. (55) and (56) at selected values of  $\xi \in \{v_\beta\} \cup [0, 1]$ .

We now substitute Eqs. (58) into Eqs. (51) and (52) to find

$$\sum_{\alpha=0}^N \{\mathbf{B}_\alpha(\xi)\mathbf{a}_\alpha + \exp\{-\tau_0/\xi\}[\mathbf{A}_\alpha(\xi) - \lambda_0\delta_{0,m}\delta_{1,k}(\delta_{0,\alpha} + \frac{1}{3}\delta_{1,\alpha})\mathbf{B}_0(\xi)\mathbf{L}]\mathbf{b}_\alpha\} = \mathbf{T}_1(0, \xi) \quad (60a)$$

and

$$\sum_{\alpha=0}^N \{[\mathbf{B}_\alpha(\xi) - \lambda_0\delta_{0,m}\delta_{1,k}(\delta_{0,\alpha} + \frac{1}{3}\delta_{1,\alpha})\mathbf{A}_0(\xi)\mathbf{L}]\mathbf{b}_\alpha + \exp\{-\tau_0/\xi\}\mathbf{A}_\alpha(\xi)\mathbf{a}_\alpha\} = \mathbf{T}_2(\tau_0, \xi) \quad (60b)$$

for  $\xi = v_\beta$  or  $\xi = v \in [0, 1]$ . Here, we have defined

$$\mathbf{B}_\alpha(v_\beta) = \varpi\mathbf{M}^T(v_\beta) \int_0^1 \mu P_\alpha(2\mu - 1)\mathbf{G}^T(v_\beta, \mu)\mathbf{J}\mathbf{E}^m(\mu) \frac{d\mu}{v_\beta - \mu} \quad (61a)$$

and

$$\mathbf{A}_\alpha(v_\beta) = \varpi\mathbf{M}^T(v_\beta)\mathbf{D}\mathbf{K}^m \int_0^1 \mu P_\alpha(2\mu - 1)\mathbf{G}^T(-v_\beta, \mu)\mathbf{J}\mathbf{E}^m(\mu) \frac{d\mu}{v_\beta + \mu} \quad (61b)$$

for  $\beta = 1, 2, \dots, \aleph$  and

$$\mathbf{B}_\alpha(v) = 2[\lambda^m(v)]^T \mathbf{J}P_\alpha(2v - 1) + \varpi \int_0^1 \mu P_\alpha(2\mu - 1)\mathbf{G}^T(v, \mu)\mathbf{J}\mathbf{E}^m(\mu) \frac{d\mu}{v - \mu} \quad (62a)$$



and

$$\mathbf{A}_\alpha(v) = \varpi \mathbf{DK}^m \int_0^1 \mu P_\alpha(2\mu - 1) \mathbf{G}^T(-v, \mu) \mathbf{J} \mathbf{E}^m(\mu) \frac{d\mu}{v + \mu} \quad (62b)$$

for  $v \in [0, 1]$ . In writing Eqs. (60), we have also used the general definitions

$$\begin{aligned} \mathbf{T}_1(\tau, \xi) = \mathbf{E}(\xi) \mathbf{DK}^m \int_0^1 \mu \mathbf{G}^T(-\xi, \mu) \mathbf{J} \mathbf{I}_0(0, \mu) \exp\{-\tau/\mu\} S(\tau_0 - \tau; \mu, \xi) d\mu \\ + \mathbf{E}(\xi) \int_0^1 \mu \mathbf{G}^T(\xi, \mu) \mathbf{J} \mathbf{D} \mathbf{I}_0(\tau_0, -\mu) C(\tau_0 - \tau; \mu, \xi) d\mu \end{aligned} \quad (63a)$$

and

$$\begin{aligned} \mathbf{T}_2(\tau, \xi) = \mathbf{E}(\xi) \mathbf{DK}^m \int_0^1 \mu \mathbf{G}^T(-\xi, \mu) \mathbf{J} \mathbf{D} \mathbf{I}_0(\tau_0, -\mu) \exp\{-(\tau_0 - \tau)/\mu\} S(\tau; \mu, \xi) d\mu \\ + \mathbf{E}(\xi) \int_0^1 \mu \mathbf{G}^T(\xi, \mu) \mathbf{J} \mathbf{I}_0(0, \mu) C(\tau; \mu, \xi) d\mu, \end{aligned} \quad (63b)$$

where

$$\mathbf{E}(v_\beta) = \mathbf{M}^T(v_\beta) \quad (64a)$$

and

$$\mathbf{E}(v) = \text{diag}\{1, 1, 1, 1\}, \quad v \in [0, 1]. \quad (64b)$$

In addition, we have used

$$S(a; x, y) = \frac{1 - \exp\{-a/x\} \exp\{-a/y\}}{x + y} \quad (65a)$$

and

$$C(a; x, y) = \frac{\exp\{-a/x\} - \exp\{-a/y\}}{x - y}. \quad (65b)$$

It is clear that Eqs. (60) yield, for  $\xi = v_\beta$ ,  $\beta = 1, 2, \dots, \aleph$ , exactly  $2\aleph$  linear equations for the  $8(N + 1)$  unknowns—the  $4(N + 1)$  elements of  $\mathbf{a}_\alpha$ ,  $\alpha = 0, 1, \dots, N$ , and the  $4(N + 1)$  elements of  $\mathbf{b}_\alpha$ ,  $\alpha = 0, 1, \dots, N$ . On the other hand, for  $\xi = \eta_\alpha \in (0, 1)$ ,  $\alpha = 1, 2, \dots, M$ , Eqs. (60) yield exactly  $8M$  linear equations. It follows that we can have a *square* system only if we take  $M = N + 1 - \frac{1}{4}\aleph$ , which clearly is possible only for  $\aleph = 4k$ ,  $k = 0, 1, \dots$ . Continuing we let  $\aleph = 4k + j$ , for  $k = 0, 1, \dots$ , and  $j = 0, 1, 2$  or 3. Of course we could take  $M = N + 1 - k$  and compute a *least squares* solution for those values of  $j$  that yield an overdetermined system; however we intend, in general, to use Gaussian elimination to compute a solution to our linear system, and so we prefer to define a collocation strategy that yields a square system for all  $j$ . To start, we define

$$\xi_\beta = v_\beta, \quad \beta = 1, 2, \dots, \aleph, \quad (66a)$$

and

$$\xi_\beta = \eta_\beta \in (0, 1), \quad \beta = \aleph + 1, \aleph + 2, \dots, \aleph + M, \quad (66b)$$

where  $M = N + 1 - k$ . Next we let  $\eta_{\aleph+1}$  denote the smallest of the  $\eta_\beta$ ,  $\beta = \aleph + 1, \aleph + 2, \dots, \aleph + M$ , and then define

$$\mathbf{P}_\beta = 1, \quad \beta = 1, 2, \dots, \aleph, \quad (67a)$$

$$\mathbf{P}_{\aleph+1} = \text{diag}\{1, 1, 1, 1\}, \quad \aleph = 4k, \quad (67b)$$

$$\mathbf{P}_{\aleph+1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \aleph = 4k + 1, \quad (67c)$$

$$\mathbf{P}_{\aleph+1} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad \aleph = 4k + 2, \quad (67d)$$

$$\mathbf{P}_{\aleph+1} = (1 \quad 1 \quad 1 \quad 1), \quad \aleph = 4k + 3, \quad (67e)$$

and

$$\mathbf{P}_\beta = \text{diag}\{1, 1, 1, 1\}, \quad \beta = \aleph + 2, \aleph + 3, \dots, \aleph + M. \quad (67f)$$

With these definitions, we can now solve the square linear system defined by

$$\mathbf{P}_\beta \sum_{\alpha=0}^N \{\mathbf{B}_\alpha(\xi_\beta) \mathbf{a}_\alpha + \exp\{-\tau_0/\xi_\beta\} [\mathbf{A}_\alpha(\xi_\beta) - \rho_\alpha \mathbf{B}_0(\xi_\beta) \mathbf{L}] \mathbf{b}_\alpha\} = \mathbf{P}_\beta \mathbf{T}_1(0, \xi_\beta) \quad (68a)$$

and

$$\mathbf{P}_\beta \sum_{\alpha=0}^N \{[\mathbf{B}_\alpha(\xi_\beta) - \rho_\alpha \mathbf{A}_0(\xi_\beta) \mathbf{L}] \mathbf{b}_\alpha + \exp\{-\tau_0/\xi_\beta\} \mathbf{A}_\alpha(\xi_\beta) \mathbf{a}_\alpha\} = \mathbf{P}_\beta \mathbf{T}_2(\tau_0, \xi_\beta) \quad (68b)$$

for  $\beta = 1, 2, \dots, \aleph + M$  to find the required  $\mathbf{a}_\alpha$  and  $\mathbf{b}_\alpha$  for  $\alpha = 0, 1, \dots, N$ . Here, we have let

$$\rho_\alpha = \lambda_0 \delta_{0,m} \delta_{1,k} (\delta_{0,\alpha} + \frac{1}{3} \delta_{1,\alpha}). \quad (69)$$

We consider now that we have specified a definition of the continuum collocation points  $\eta_\beta$  and that we have solved Eqs. (68) to establish the constants  $\{\mathbf{a}_\alpha\}$  and  $\{\mathbf{b}_\alpha\}$ . Thus to find the vectors  $\{\mathbf{c}_\alpha(\tau)\}$  and  $\{\mathbf{d}_\alpha(\tau)\}$  required in Eqs. (59) to complete the desired solution at interior points in the layer,  $\tau \in (0, \tau_0)$ , we substitute Eqs. (59) into Eqs. (55) and (56) to find

$$\sum_{\alpha=0}^N [\mathbf{B}_\alpha(\xi) \mathbf{c}_\alpha(\tau) - \mathbf{A}_\alpha(\xi) \mathbf{d}_\alpha(\tau)] = \mathbf{R}_1(\tau, \xi) \quad (70a)$$

and

$$\sum_{\alpha=0}^N [\mathbf{B}_\alpha(\xi) \mathbf{d}_\alpha(\tau) - \mathbf{A}_\alpha(\xi) \mathbf{c}_\alpha(\tau)] = \mathbf{R}_2(\tau, \xi), \quad (70b)$$

where

$$\mathbf{R}_1(\tau, \xi) = \mathbf{T}_1(\tau, \xi) - \exp\{-\tau_0 - \tau\}/\xi \sum_{\alpha=0}^N [\mathbf{A}_\alpha(\xi) - \rho_\alpha \mathbf{B}_0(\xi) \mathbf{L}] \mathbf{b}_\alpha \quad (71a)$$

and

$$\mathbf{R}_2(\tau, \xi) = \mathbf{T}_2(\tau, \xi) - \exp\{-\tau/\xi\} \sum_{\alpha=0}^N \mathbf{A}_\alpha(\xi) \mathbf{a}_\alpha \quad (71b)$$

for  $\xi = v_\beta$  or  $\xi = v \in [0, 1]$ . We elect to use the same collocation strategy we used to find the constants  $\{\mathbf{a}_\alpha\}$  and  $\{\mathbf{b}_\alpha\}$ , and so now we can solve the square linear system

$$\mathbf{P}_\beta \sum_{\alpha=0}^N [\mathbf{B}_\alpha(\xi_\beta) \mathbf{c}_\alpha(\tau) - \mathbf{A}_\alpha(\xi_\beta) \mathbf{d}_\alpha(\tau)] = \mathbf{P}_\beta \mathbf{R}_1(\tau, \xi_\beta) \quad (72a)$$

and

$$\mathbf{P}_\beta \sum_{\alpha=0}^N [\mathbf{B}_\alpha(\xi_\beta) \mathbf{d}_\alpha(\tau) - \mathbf{A}_\alpha(\xi_\beta) \mathbf{c}_\alpha(\tau)] = \mathbf{P}_\beta \mathbf{R}_2(\tau, \xi_\beta) \quad (72b)$$

for  $\beta = 1, 2, \dots, \aleph + M$  to find  $\{\mathbf{c}_\alpha(\tau)\}$  and  $\{\mathbf{d}_\alpha(\tau)\}$  for any  $\tau \in (0, \tau_0)$ . We note that only the right-hand sides of Eqs. (72) depend on  $\tau$ , and so only one matrix inversion (or LU factorization) is required for any number of values of  $\tau \in (0, \tau_0)$ .

Even though here we can have complex eigenvalues, we elect always to set up our linear systems, i.e., Eqs. (68) and (72), in real mode. Thus, in the event that there is a pair of complex eigenvalues, say  $v_\beta$  and  $v_\beta^*$ , we consider separately the real and imaginary parts of Eqs. (68) and (72) for  $\xi_\beta = v_\beta$  and neglect the equations resulting from considering  $\xi = v_\beta^*$ .

Although we now have available a complete solution, given by Eqs. (57), (58) and (59), for the required  $\mathbf{I}(0, -\mu)$ ,  $\mathbf{I}(\tau_0, \mu)$  and  $\mathbf{I}(\tau, \pm\mu)$ , for  $\mu \in [0, 1]$  and  $\tau \in (0, \tau_0)$ , we proceed to follow our

previous work<sup>12,13</sup> and use the idea of *postprocessing* to improve the accuracy of our  $F_N$  solution. After noting Eq. (49), we rewrite Eq. (62a) as

$$\mathbf{B}_\alpha(\mu) = 2P_\alpha(2\mu - 1)\mathbf{J} - \varpi\mathbf{C}_\alpha(\mu) - \mathbf{H}^m(\mu)P_\alpha(2\mu - 1) \quad (73)$$

for  $\mu \in [0, 1]$ . Here,

$$\mathbf{C}_\alpha(\mu) = \int_0^1 v[P_\alpha(2v - 1) - P_\alpha(2\mu - 1)]\mathbf{G}^T(\mu, v)\mathbf{E}^m(v)\mathbf{J} \frac{dv}{v - \mu} \quad (74)$$

and

$$\mathbf{H}^m(\mu) = \varpi\mathbf{E}^m(\mu)\mathbf{J} - \mathbf{A}_0(\mu)\mathbf{D}\mathbf{K}^m, \quad (75)$$

where

$$\mathbf{E}^0(\mu) = 2[\mathbf{G}_0^0(\mu)]^T\mathbf{B}_0^T \text{diag}\{1, 0, 0, 1\} + \frac{2}{5}[\mathbf{G}_2^0(\mu)]^T\mathbf{B}_2^T \text{diag}\{0, 1, 1, 0\}, \quad (76a)$$

$$\mathbf{E}^1(\mu) = \frac{2(2)^{1/2}}{3}[\mathbf{G}_1^1(\mu)]^T\mathbf{B}_1^T \text{diag}\{1, 0, 0, 1\} + \frac{2}{5}[\mathbf{G}_2^1(\mu)]^T\mathbf{B}_2^T \text{diag}\{0, 1, 1, 0\} \quad (76b)$$

and

$$\mathbf{E}^m(\mu) = \frac{2}{2m + 1}\Delta_m\mathbf{B}_m^T\Delta_m^{-1}, \quad m \geq 2. \quad (76c)$$

We can now use Eq. (73) in Eqs. (60), for  $\xi = \mu \in [0, 1]$ , to find

$$2\mathbf{J} \sum_{\alpha=0}^N \mathbf{a}_\alpha P_\alpha(2\mu - 1) = \mathbf{T}_1(0, \mu) + \mathbf{Q}_1(\mu) \quad (77a)$$

and

$$2\mathbf{J} \sum_{\alpha=0}^N \mathbf{b}_\alpha P_\alpha(2\mu - 1) = \mathbf{T}_2(\tau_0, \mu) + \mathbf{Q}_2(\mu), \quad (77b)$$

where

$$\mathbf{Q}_1(\mu) = \sum_{\alpha=0}^N \{[\varpi\mathbf{C}_\alpha(\mu) + \mathbf{H}^m(\mu)P_\alpha(2\mu - 1)]\mathbf{a}_\alpha - \exp\{-\tau_0/\mu\}[\mathbf{A}_\alpha(\mu) - \rho_\alpha\mathbf{B}_0(\mu)\mathbf{L}]\mathbf{b}_\alpha\} \quad (78a)$$

and

$$\mathbf{Q}_2(\mu) = \sum_{\alpha=0}^N \{[\varpi\mathbf{C}_\alpha(\mu) + \mathbf{H}^m(\mu)P_\alpha(2\mu - 1)]\mathbf{b}_\alpha - \exp\{-\tau_0/\mu\}\mathbf{A}_\alpha(\mu)\mathbf{a}_\alpha + \rho_\alpha\mathbf{A}_0(\mu)\mathbf{L}\mathbf{b}_\alpha\}. \quad (78b)$$

To find our final results for the boundary solutions, we use Eqs. (77) in Eqs. (58) and obtain

$$\mathbf{I}(0, -\mu) = \mathbf{I}_0(0, -\mu) + \frac{\varpi}{2}\mathbf{D}\mathbf{E}^m(\mu)\mathbf{J}[\mathbf{T}_1(0, \mu) + \mathbf{Q}_1(\mu)] \quad (79a)$$

and

$$\mathbf{I}(\tau_0, \mu) = \mathbf{I}_0(\tau_0, \mu) + \frac{\varpi}{2}\mathbf{E}^m(\mu)\mathbf{J}[\mathbf{T}_2(\tau_0, \mu) + \mathbf{Q}_2(\mu)] \quad (79b)$$

for  $\mu \in [0, 1]$ . In a similar manner, we can use Eq. (73) and Eqs. (70) in Eqs. (59) to find our final results for the interior quantities, viz.,

$$\mathbf{I}(\tau, -\mu) = \mathbf{I}_0(\tau, -\mu) + \frac{\varpi}{2}\mathbf{D}\mathbf{E}^m(\mu)\mathbf{J} \left[ \mathbf{R}_1(\tau, \mu) + \sum_{\alpha=0}^N \{[\varpi\mathbf{C}_\alpha(\mu) + \mathbf{H}^m(\mu)P_\alpha(2\mu - 1)]\mathbf{c}_\alpha(\tau) + \mathbf{A}_\alpha(\mu)\mathbf{d}_\alpha(\tau)\} \right] \quad (80a)$$

and

$$\mathbf{I}(\tau, \mu) = \mathbf{I}_0(\tau, \mu) + \frac{\varpi}{2} \boldsymbol{\Xi}^m(\mu) \mathbf{J} \left[ \mathbf{R}_2(\tau, \mu) + \sum_{\alpha=0}^N \{[\varpi \mathbf{C}_\alpha(\mu) + \mathbf{H}^m(\mu) P_\alpha(2\mu - 1)] \mathbf{d}_\alpha(\tau) + \mathbf{A}_\alpha(\mu) \mathbf{c}_\alpha(\tau)\} \right] \quad (80b)$$

for  $\mu \in [0, 1]$  and  $\tau \in (0, \tau_0)$ .

## 5. THE DISCRETE SPECTRUM

In order to implement our  $F_N$  solution to the considered polarization problem, we must first establish an algorithm for computing accurately the discrete spectrum (in the right half-plane) as defined by

$$A^m(\xi) \mathbf{M}(\xi) = \mathbf{0}, \quad \xi \in [0, 1]. \quad (81)$$

In Ref. 10, we proved that the required discrete spectrum could be approximated by those values of  $z$  that satisfy

$$\mathbf{G}_{M+1}^m(z) \mathbf{M}_{M+1}(z) = \mathbf{0}, \quad z \in [0, 1], \quad (82)$$

with improving accuracy as  $M$  tends to infinity.

In Ref. 1, we used a generalized spherical harmonics method to develop an approximate solution (of order  $N'$ , where  $N'$  is odd) of the polarization problem considered here. As part of our generalized spherical harmonics solution, we established an algorithm, based on computing the eigenvalues of a certain banded matrix, for finding all of the zeros of  $\det \mathbf{G}_{M+1}^m(z)$ , where  $M = m + N'$ . To define our initial approximation to the discrete spectrum we seek,  $\{v_\beta\}$ ,  $\beta = 1, 2, \dots, \aleph$ , we compute the zeros (in the right half-plane) of  $\det \mathbf{G}_{m+N'+1}^m(z)$  for increasing values of  $N'$  until a *stable* set of zeros  $\{z_\beta\}$ ,  $\beta = 1, 2, \dots, \aleph$ , is obtained.

Some comments in regard to our definition of a stable set  $\{z_\beta\}$  are in order. First of all we have found that our numerical methods taken from the EISPACK collection<sup>14,15</sup> yield some zeros of  $\det \mathbf{G}_{m+N'+1}^m(z)$  of the form  $z = x \pm ic$  with  $x \in (0, 1)$  and  $c$  small; however since, as shown in Ref. 10, the limiting values of the dispersion matrix  $[A^m(\tau)]^\pm$  cannot be singular for  $\tau \in (0, 1)$ , and since the zeros of  $\det \mathbf{G}_{m+N'+1}^m(z)$  of this form do not appear to converge as  $N'$  increases, these zeros are not included in the stable set  $\{z_\beta\}$ . Also, for the numerical data sets we have so far investigated, we have found that only zeros (in the right half-plane) of  $\det \mathbf{G}_{m+N'+1}^m(z)$  with real part  $> 1$  appear to converge as  $N'$  increases; however, we have no proof that such is the case. Finally we have observed that the zeros of  $\det \mathbf{G}_{m+N'+1}^m(z)$  with large magnitudes converge quickly as  $N'$  increases; on the other hand, the zeros with magnitudes just  $> 1$  converge so slowly with increasing  $N'$  that some refinement is required to get good accuracy before  $N'$  becomes unmanageably large.

Our procedure (to date) to compute the desired discrete spectrum can thus be stated as: (i) use the algorithm of Ref. 1 with increasing  $N'$  until the number  $\aleph$  of zeros with  $|z_\beta| \geq 1$  has converged. Of course a numerical implementation of the argument principle<sup>11</sup> can be used to confirm that the value of  $\aleph$  so obtained is correct. We also insist that  $N'$  be sufficiently large so that all zeros with  $|z_\beta|$  greater than, say,  $a = 2$  will have converged to a desired number (say 13) of significant figures. (ii) Use an iterative procedure, say Newton's method, to refine the zeros from the stable set  $\{z_\beta\}$  that are such that  $1 \leq |z_\beta| \leq a$ .

In order to develop our method for refining the accuracy of the eigenvalues  $v_\beta$ , where  $1 \leq |v_\beta| \leq a$ , we first write the dispersion matrix in the form

$$A^m(s) = \mathbf{I} - \varpi s \sum_{l=m}^L \mathbf{Q}_l^m(s) \mathbf{B}_l \mathbf{G}_l^m(s), \quad (83)$$

where

$$\mathbf{Q}_l^m(s) = \frac{1}{2} \int_{-1}^1 \boldsymbol{\Xi}^m(\mu) \boldsymbol{\Pi}_l^m(\mu) \frac{d\mu}{s - \mu}. \quad (84)$$

We note from Ref. 10 that the  $\mathbf{Q}$  matrices satisfy the recursion relation

$$\mathbf{Q}_l^m(\xi)[(2l+1)\xi\mathbf{I} + \mathbf{V}_l^m] = \mathbf{Q}_{l+1}^m(\xi)\mathbf{U}_{l+1}^m + \mathbf{Q}_{l-1}^m(\xi)\mathbf{U}_l^m \quad (85)$$

for  $l \geq n+1$ , where  $n = \max\{2, m\}$ . As we now intend to use backward recursion to compute the  $\mathbf{Q}$  matrices, we let

$$\mathbf{S}_l^m(\xi) = [\mathbf{Q}_l^m(\xi)]^{-1}\mathbf{Q}_{l+1}^m(\xi), \quad l \geq n, \quad (86)$$

and deduce from Eq. (85) that, for any  $M > n+1$ ,

$$\mathbf{S}_{l-1}^m(\xi) = \mathbf{U}_l^m[(2l+1)\xi\mathbf{I} + \mathbf{V}_l^m - \mathbf{S}_l^m(\xi)\mathbf{U}_{l+1}^m]^{-1}, \quad l = M, M-1, \dots, n+1. \quad (87)$$

Thus, for some sufficiently large  $M > L$  we take  $\mathbf{S}_M^m(\xi) = \mathbf{0}$  and use Eq. (87) to compute  $\mathbf{S}_l^m(\xi)$  for  $l = n, n+1, \dots, L-1$ . It follows that we can now write

$$\mathbf{Q}_{n+1}^m(\xi) = \mathbf{Q}_n^m(\xi)\mathbf{S}_n^m(\xi), \quad (88a)$$

$$\mathbf{Q}_{n+2}^m(\xi) = \mathbf{Q}_n^m(\xi)\mathbf{S}_n^m(\xi)\mathbf{S}_{n+1}^m(\xi) \quad (88b)$$

or, in general,

$$\mathbf{Q}_l^m(\xi) = \mathbf{Q}_n^m(\xi)\mathbf{X}_l^m(\xi), \quad l \geq n, \quad (88c)$$

where

$$\mathbf{X}_n^m(\xi) = \mathbf{I}, \quad (89a)$$

$$\mathbf{X}_{n+1}^m(\xi) = \mathbf{S}_n^m(\xi) \quad (89b)$$

and, in general,

$$\mathbf{X}_l^m(\xi) = \mathbf{S}_n^m(\xi)\mathbf{S}_{n+1}^m(\xi) \cdots \mathbf{S}_{l-1}^m(\xi), \quad l \geq n+1. \quad (89c)$$

We thus can write Eq. (83) as

$$\mathbf{A}^m(s) = \mathbf{\Gamma}^m(s) - \varpi s \mathbf{Q}_n^m(s) \sum_{l=n}^L \mathbf{X}_l^m(s) \mathbf{B}_l \mathbf{G}_l^m(s), \quad (90)$$

where we have introduced

$$\mathbf{\Gamma}^m(s) = \mathbf{I} - \varpi s (\delta_{0,m} + \delta_{1,m}) \sum_{l=m}^1 \mathbf{Q}_l^m(s) \mathbf{B}_l \mathbf{G}_l^m(s). \quad (91)$$

We also wish to avoid using forward recursion to compute the  $\mathbf{G}_l^m(s)$  for  $|s| > 1$ , and so we rewrite Eq. (81) as

$$\mathbf{A}^m(s)\mathbf{M}(s) = \mathbf{\Gamma}^m(s)\mathbf{M}(s) - \varpi s \mathbf{Q}_n^m(s) \sum_{l=n}^L \mathbf{X}_l^m(s) \mathbf{B}_l \mathbf{T}_l^m(s), \quad (92)$$

where

$$\mathbf{T}_l^m(s) = \mathbf{G}_l^m(s)\mathbf{M}(s). \quad (93)$$

As discussed in Ref. 1, we write

$$\mathbf{T}_{l+1}^m(\xi) = \mathbf{R}_l^m(\xi)\mathbf{T}_l^m(\xi), \quad (94)$$

define, for sufficiently large  $M > L$ ,  $\mathbf{R}_M^m(\xi) = \mathbf{0}$  and use

$$\mathbf{R}_{l-1}^m(\xi) = [\xi\mathbf{h}_l + \mathbf{V}_l^m - \mathbf{U}_{l+1}^m \mathbf{R}_l^m(\xi)]^{-1} \mathbf{U}_l^m, \quad l = M, M-1, \dots, n+1. \quad (95)$$

Having found the  $\mathbf{R}_l^m(\xi)$  for  $l = n, n+1, \dots, L-1$ , we write

$$\mathbf{G}_{n+1}^m(\xi) = \mathbf{R}_n^m(\xi)\mathbf{G}_n^m(\xi), \quad (96a)$$

$$\mathbf{G}_{n+2}^m(\xi) = \mathbf{R}_{n+1}^m(\xi)\mathbf{R}_n^m(\xi)\mathbf{G}_n^m(\xi) \quad (96b)$$

or, in general,

$$\mathbf{G}_l^m(\xi) = \mathbf{Y}_l^m(\xi)\mathbf{G}_n^m(\xi), \quad (96c)$$

where

$$\mathbf{Y}_n^m(\xi) = \mathbf{I}, \quad (97a)$$

$$\mathbf{Y}_{n+1}^m(\xi) = \mathbf{R}_n^m(\xi) \quad (97b)$$

and, in general,

$$\mathbf{Y}_l^m(\xi) = \mathbf{R}_{l-1}^m(\xi)\mathbf{R}_{l-2}^m(\xi) \cdots \mathbf{R}_n^m(\xi), \quad l \geq n+1. \quad (97c)$$

Using these definitions, we write Eq. (90) as

$$\mathbf{A}^m(s) = \mathbf{F}^m(s) - \varpi s \mathbf{Q}_n^m(s) \mathbf{W}(s) \mathbf{G}_n^m(s), \quad (98)$$

where

$$\mathbf{W}(s) = \sum_{l=n}^L \mathbf{X}_l^m(s) \mathbf{B}_l \mathbf{Y}_l^m(s). \quad (99)$$

As the starting values for the  $\mathbf{G}$  polynomials are available from Eqs. (29), we need now only the starting values for the  $\mathbf{Q}$  matrices to complete Eq. (98).

We consider first the case  $m = 0$ , and let

$$\mathcal{L}(\xi) = -\frac{1}{2} \log \left( \frac{\xi - 1}{\xi + 1} \right); \quad (100)$$

next, we write

$$\mathbf{Q}_0^0(\xi) = \mathcal{L}(\xi) \text{diag}\{1, 0, 0, 1\}, \quad (101a)$$

$$\mathbf{Q}_1^0(\xi) = [\xi \mathcal{L}(\xi) - 1] \text{diag}\{1, 0, 0, 1\} \quad (101b)$$

and

$$\mathbf{Q}_2^0(\xi) = \mathcal{L}(\xi) \mathbf{I}_2^0(\xi) + \mathbf{J}_2^0(\xi), \quad (101c)$$

where

$$\mathbf{I}_2^0(\xi) = \text{diag}\left\{\frac{1}{2}(3\xi^2 - 1), \frac{3}{8}(1 - \xi^2)^2, \frac{3}{8}(1 - \xi^2)^2, \frac{1}{2}(3\xi^2 - 1)\right\} \quad (102a)$$

and

$$\mathbf{J}_2^0(\xi) = \text{diag}\left\{-\frac{3}{2}\xi, \frac{3}{8}\xi(\xi^2 - \xi^2), \frac{3}{8}\xi(\xi^2 - \xi^2), -\frac{3}{2}\xi\right\}. \quad (102b)$$

For the case  $m = 1$ , we find

$$\mathbf{Q}_1^1(\xi) = 2^{-1/2}[(1 - \xi^2)\mathcal{L}(\xi) + \xi] \text{diag}\{1, 0, 0, 1\} \quad (103a)$$

and

$$\mathbf{Q}_2^1(\xi) = \mathbf{S}^{-1}[\mathcal{L}(\xi)\mathbf{I}_2^1(\xi) + \mathbf{J}_2^1(\xi)]\mathbf{S}, \quad (103b)$$

where

$$\mathbf{I}_2^1(\xi) = (1 - \xi^2) \text{diag}\left\{\frac{6^{1/2}}{2}\xi, \frac{1}{4}(\xi - 1)^2, \frac{1}{4}(\xi + 1)^2, \frac{6^{1/2}}{2}\xi\right\}, \quad (104a)$$

$$\mathbf{J}_2^1(\xi) = \text{diag}\left\{\frac{6^{1/2}}{2}(\xi^2 - \frac{2}{3}), \frac{1}{4}\xi^3 - \frac{1}{2}\xi^2 + \frac{1}{12}\xi + \frac{1}{3}, \frac{1}{4}\xi^3 + \frac{1}{2}\xi^2 + \frac{1}{12}\xi - \frac{1}{3}, \frac{6^{1/2}}{2}(\xi^2 - \frac{2}{3})\right\} \quad (104b)$$

and

$$\mathbf{S} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (105)$$

For the case  $m = 2$ , we find

$$\mathbf{Q}_2^z(\xi) = \mathbf{S}^{-1}[\mathcal{L}(\xi)\mathbf{I}_2^z(\xi) + \mathbf{J}_2^z(\xi)]\mathbf{S}, \quad (106)$$

where

$$\mathbf{I}_2^z(\xi) = \text{diag}\{(1 - \xi^2)^2, (1 - \xi)^4, (1 + \xi)^4, (1 - \xi^2)^2\}\Delta_2 \quad (107a)$$

and

$$\mathbf{J}_2^z(\xi) = \text{diag}\{\xi(\frac{5}{3} - \xi^2), \frac{16}{3} - \frac{19}{3}\xi + 4\xi^2 - \xi^3, -\frac{16}{3} - \frac{19}{3}\xi - 4\xi^2 - \xi^3, \xi(\frac{5}{3} - \xi^2)\}\Delta_2. \quad (107b)$$

Finally, for  $m > 2$ , we use

$$\mathbf{Q}_{\alpha+1}^z(\xi) = \Delta_\alpha^{-1}\Delta_{\alpha+1}\left[(1 - \xi^2)\mathbf{Q}_\alpha^z(\xi) + \frac{1}{2\alpha + 1}\Delta_\alpha^{-1}\left(\xi\mathbf{I} - \frac{1}{2\alpha + 1}\mathbf{V}_\alpha^z\right)\right], \quad (108)$$

for  $\alpha = 2, 3, \dots, m - 1$ , where  $\mathbf{V}_\alpha^z$  is defined by Eq. (28).

We note that, since we are concerned here with a refinement to our eigenvalue calculation when  $|v_\beta|$  is close to unity, we can experience a loss of accuracy when we must compute  $v_\beta - 1$  as required in Eq. (100). We find it convenient to follow Bond<sup>16</sup> and introduce the change of variables

$$s = (1 - \exp\{-u\})^{-1} \quad (109)$$

in Eq. (98). If we let

$$\Omega(u) = \det A^m[(1 - \exp\{-u\})^{-1}] \quad (110)$$

and consider that we have two initial approximations (from, say, using two values of  $N'$  in the method of Ref. 1), then we can use, for example,

$$u_{i+1} = u_i - \frac{u_i - u_{i-1}}{\Omega(u_i) - \Omega(u_{i-1})}\Omega(u_i) \quad (111)$$

until a result with the desired degree of accuracy is obtained.

Clearly once a discrete eigenvalue, say  $v_\beta$ , has been computed, we can use the established  $A(v_\beta)$  and, e.g., singular value decomposition<sup>17</sup> to compute the required null vector  $\mathbf{M}(v_\beta)$ .

We have found that the method discussed here works very well for the two test problems solved in Ref. 1. However, we note that in the event that one of the eigenvalues  $v_\beta$  is such that  $|v_\beta|$  is especially close to unity, say  $|v_\beta| - 1 < 10^{-6}$ , then the calculation can perhaps be made more efficient by using forward recursion to evaluate the  $\mathbf{G}$  and  $\mathbf{Q}$  matrices.

## 6. COMPUTATIONAL ASPECTS OF THE $F_N$ SOLUTION

Having developed our algorithm for computing the discrete spectrum, we now wish to discuss our methods for computing the matrix-valued functions  $\mathbf{A}_\alpha(\xi)$  and  $\mathbf{B}_\alpha(\xi)$  for  $\xi \in \mathcal{P} = \{v_\beta\} \cup [0, 1]$ .

First of all, we consider  $\mathbf{B}_\alpha(v)$ , as defined by Eq. (62a), for  $v \in [0, 1]$ . We can multiply Eq. (62a) by  $(2v - 1)$  and use the three-term recursion formula for the Legendre polynomials to find

$$\mathbf{B}_{\alpha+1}(v) = \frac{2\alpha + 1}{\alpha + 1}[(2v - 1)\mathbf{B}_\alpha(v) - 2\varpi\mathbf{Z}_\alpha(-v)] - \frac{\alpha}{\alpha + 1}\mathbf{B}_{\alpha-1}(v) \quad (112)$$

where, for  $\xi \in \mathcal{P}$ ,

$$\mathbf{Z}_\alpha(\xi) = \mathbf{E}(\xi)\mathbf{D}\mathbf{K}^m \int_0^1 \mu P_\alpha(2\mu - 1)\mathbf{G}^T(-\xi, \mu)\mathbf{J}\mathbf{E}^m(\mu) d\mu \quad (113a)$$

and

$$\mathbf{Z}_\alpha(-\xi) = \mathbf{E}(\xi) \int_0^1 \mu P_\alpha(2\mu - 1)\mathbf{G}^T(\xi, \mu)\mathbf{J}\mathbf{E}^m(\mu) d\mu \quad (113b)$$

with  $\mathbf{E}(\xi)$  defined by Eqs. (64). As we intend to use forward recursion to compute  $\mathbf{B}_\alpha(v)$ ,  $v \in [0, 1]$ , we now require the starting value  $\mathbf{B}_0(v)$ . After noting Eq. (73), we write

$$\mathbf{B}_0(v) = 2\mathbf{J} - \varpi\mathbf{E}^m(v)\mathbf{J} + \mathbf{A}_0(v)\mathbf{D}\mathbf{K}^m, \quad (114)$$

where the  $\mathbf{E}^m(v)$  are defined by Eqs. (76). We use Eqs. (62b) and (32) to write

$$\mathbf{A}_0(v) = \varpi \sum_{l=m}^L (-1)^{l-m} [\mathbf{G}_l^m(v)]^T \mathbf{DJB} \mathbf{H}_l^m(v) - \varpi v \log\left(1 + \frac{1}{v}\right) \mathbf{G}^T(v, v) \mathbf{E}^m(v) \mathbf{DK}^m \mathbf{J}, \quad (115)$$

where, in general,

$$\mathbf{H}_l^m(\xi) = \mathbf{\Pi}_l^m(-\xi) \mathbf{E}^m(-\xi) + \int_0^1 \mu [\mathbf{\Pi}_l^m(\mu) \mathbf{E}^m(\mu) - \mathbf{\Pi}_l^m(-\xi) \mathbf{E}^m(-\xi)] \frac{d\mu}{\mu + \xi}. \quad (116)$$

We note that we can, as an alternative to using Eq. (115), use backward recursion (to be discussed) to compute  $\mathbf{A}_0(v)$  for, say,  $v \in (0.001, 1]$ .

We can show that the  $\mathbf{H}_l^m(\xi)$  satisfy

$$[(2l+1)\xi \mathbf{I} - \mathbf{V}_l^m] \mathbf{H}_l^m(\xi) = (2l+1) \Delta_{0,l}^m - \mathbf{U}_{l+1}^m \mathbf{H}_{l+1}^m(\xi) - \mathbf{U}_l^m \mathbf{H}_{l-1}^m(\xi), \quad (117)$$

where, in general,

$$\Delta_{\alpha,l}^m = \int_0^1 \mu P_\alpha(2\mu-1) \mathbf{\Pi}_l^m(\mu) \mathbf{E}^m(\mu) d\mu. \quad (118)$$

We can write Eqs. (113) as

$$\mathbf{Z}_\alpha(v_\beta) = \sum_{l=m}^L (-1)^{l-m} [\mathbf{T}_l^m(v_\beta)]^T \mathbf{DJB}_l \Delta_{\alpha,l}^m \quad (119a)$$

and

$$\mathbf{Z}_\alpha(-v_\beta) = \sum_{l=m}^L [\mathbf{T}_l^m(v_\beta)]^T \mathbf{JB}_l \Delta_{\alpha,l}^m, \quad (119b)$$

where  $\mathbf{T}_l^m(v_\beta)$  is defined by Eq. (93), and we can write

$$\mathbf{Z}_\alpha(v) = \sum_{l=m}^L (-1)^{l-m} [\mathbf{G}_l^m(v)]^T \mathbf{DJB}_l \Delta_{\alpha,l}^m \quad (119c)$$

and

$$\mathbf{Z}_\alpha(-v) = \sum_{l=m}^L [\mathbf{G}_l^m(v)]^T \mathbf{JB}_l \Delta_{\alpha,l}^m \quad (119d)$$

for  $v \in [0, 1]$ . It is clear that in order to proceed with our  $F_N$  calculation, we must develop an efficient way to compute the matrices  $\Delta_{\alpha,l}^m$ .

Proceeding now to report our way of computing the matrices  $\Delta_{\alpha,l}^m$ , we note first that the constants

$$X_{\alpha,l}^m = \left[ \frac{(l-m)!}{(l+m)!} \right]^{1/2} \int_0^1 \mu (1-\mu^2)^{m/2} P_\alpha(2\mu-1) P_l^m(\mu) d\mu \quad (120)$$

are basic to this development. We reported in Ref. 18 a scheme for computing some constants  $T_{\alpha,l}^m$  that correspond to the integral in Eq. (120). To be complete we summarize here our way of computing the  $X_{\alpha,l}^m$ . To start, for  $m=0$ , the first two rows of the  $\mathbf{X}$  matrix can be computed from

$$X_{0,l+1}^0 = \frac{2-l}{3+l} X_{0,l-1}^0, \quad (121a)$$

$$X_{1,l+1}^0 = \frac{4}{4+l} X_{0,l}^0 - X_{0,l+1}^0 \quad (121b)$$

and the initial values

$$X_{\alpha,0}^0 = \frac{1}{2}, \quad X_{0,1}^0 = \frac{1}{3} \quad \text{and} \quad X_{1,0}^0 = \frac{1}{6}. \quad (122a,b \text{ and } c)$$

The remaining non-zero  $X_{\alpha,l}^0$  (in general  $X_{\alpha,l}^m = 0$  for  $\alpha > l+m+1$ ) can be computed from<sup>13</sup>

$$X_{\alpha,l+1}^0 = \frac{1}{2} \left( \frac{2l+1}{l+1} \right) \left( \frac{\alpha}{2\alpha+1} X_{\alpha-1,l}^0 + X_{\alpha,l}^0 + \frac{\alpha+1}{2\alpha+1} X_{\alpha+1,l}^0 \right) - \frac{l}{l+1} X_{\alpha,l-1}^0. \quad (123)$$



For  $m > 0$ , we use<sup>18</sup>

$$(2l + 1)X_{\alpha,l}^m = [(l + m - 1)(l + m)]^{1/2} X_{\alpha,l-1}^m - [(l - m + 1)(l - m + 2)]^{1/2} X_{\alpha,l+1}^m. \quad (124)$$

Considering the  $X_{\alpha,l}^m$  to be now available, we find we can write

$$\Delta_{\alpha,l}^0 = \text{diag} \left\{ X_{\alpha,l}^0, \frac{6^{1/2}}{4} (1 - \delta_{0,l})(1 - \delta_{1,l}) X_{\alpha,l}^2, \frac{6^{1/2}}{4} (1 - \delta_{0,l})(1 - \delta_{1,l}) X_{\alpha,l}^2, X_{\alpha,l}^0 \right\}. \quad (125)$$

For  $m \geq 1$  we use a similarity transformation to reduce  $\Delta_{\alpha,l}^m$  to diagonal form. Thus, with  $S$  as given by Eq. (105) we find we can write

$$S \Delta_{\alpha,l}^m S^{-1} = \text{diag} \{ X_{\alpha,l}^m, Y_{\alpha,l}^m, Z_{\alpha,l}^m, X_{\alpha,l}^m \}, \quad (126)$$

where

$$Y_{\alpha,l}^m = \left[ \frac{(l - m)!}{(l + m)!} \right]^{1/2} \int_0^1 \mu (1 - \mu^2)^{m/2} P_{\alpha}(2\mu - 1) [R_l^m(\mu) - T_l^m(\mu)] \Theta_1^m(\mu) d\mu \quad (127a)$$

and

$$Z_{\alpha,l}^m = \left[ \frac{(l - m)!}{(l + m)!} \right]^{1/2} \int_0^1 \mu (1 - \mu^2)^{m/2} P_{\alpha}(2\mu - 1) [R_l^m(\mu) + T_l^m(\mu)] \Theta_2^m(\mu) d\mu \quad (127b)$$

with

$$\Theta_1^1(\mu) = \frac{1}{2}(1 - \mu) \quad \text{and} \quad \Theta_1^m(\mu) = \frac{1 - \mu}{1 + \mu}, \quad m \geq 2, \quad (128a \text{ and } b)$$

and

$$\Theta_2^1(\mu) = -\frac{1}{2}(1 + \mu) \quad \text{and} \quad \Theta_2^m(\mu) = \frac{1 + \mu}{1 - \mu}, \quad m \geq 2. \quad (129a \text{ and } b)$$

The constants  $Y_{\alpha,l}^m$  and  $Z_{\alpha,l}^m$  can be readily computed with recursion relations derived by expressing the functions  $R_l^m(\mu)$  and  $T_l^m(\mu)$  in terms of Jacobi polynomials and using some properties of these polynomials reported by Szegő.<sup>19</sup> In regard to the  $Y_{\alpha,l}^m$  constants, we find for  $m = 1$  that the first row of the  $Y$  matrix can be computed from

$$l(l + 5) \left[ \frac{(l - 2)(l - 1)l}{(l + 3)} \right]^{1/2} Y_{0,l+1}^1 = 4(2l + 1) \left( \frac{l - 2}{l + 2} \right)^{1/2} Y_{0,l}^1 - (l - 4)(l + 1)(l^2 - 1)^{1/2} Y_{0,l-1}^1, \quad (130)$$

with  $l = 4, 5, \dots$ , and the initial values

$$Y_{0,2}^1 = \frac{1}{60}, \quad Y_{0,3}^1 = \frac{29}{1680} \left( \frac{5}{2} \right)^{1/2} \quad \text{and} \quad Y_{0,4}^1 = \frac{2^{1/2}}{64}. \quad (131a, b \text{ and } c)$$

Noting that, for  $m = 1$ , the first column ( $l = 1$ ) of the  $Y$  matrix is zero, as are the elements  $Y_{\alpha,l}^1$  with  $\alpha > l + 3$ , we compute the remaining non-zero  $Y_{\alpha,l}^1$  from the general expression

$$a_{l+1}^m Y_{\alpha,l+1}^m = \frac{2l + 1}{2} \left\{ \frac{\alpha}{2\alpha + 1} Y_{\alpha-1,l}^m + \left[ 1 + \frac{4m}{l(l + 1)} \right] Y_{\alpha,l}^m + \frac{\alpha + 1}{2\alpha + 1} Y_{\alpha+1,l}^m \right\} - a_l^m Y_{\alpha,l-1}^m, \quad (132)$$

where

$$a_l^m = \frac{[(l^2 - m^2)(l^2 - 4)]^{1/2}}{l}. \quad (133)$$

In order to use Eq. (132) for  $m = 1$ , we require, in addition to the first row, the initial values

$$Y_{1,2}^1 = -\frac{1}{210}, \quad Y_{2,2}^1 = -\frac{1}{336}, \quad Y_{3,2}^1 = \frac{11}{5040}, \quad Y_{4,2}^1 = -\frac{1}{5040} \quad \text{and} \quad Y_{5,2}^1 = -\frac{1}{11088}. \quad (134a, b, c, d \text{ and } e)$$

For  $m = 2$ , we find that the first row of the  $Y$  matrix can be computed from

$$(l - 1)l(l + 5) Y_{0,l+1}^2 = 8(2l + 1) Y_{0,l}^2 - (l - 4)(l + 1)(l + 2) Y_{0,l-1}^2, \quad (135)$$

with  $l = 4, 5, \dots$ , and the initial values

$$Y_{0,2}^2 = \frac{1}{120}, \quad Y_{0,3}^2 = \frac{1}{42} \quad \text{and} \quad Y_{0,4}^2 = \frac{1}{32}. \quad (136a,b \text{ and } c)$$

In general, for  $m \geq 2$ ,  $Y_{\alpha,l}^m = 0$  for  $\alpha > l + m + 1$ , and so the remaining non-zero  $Y_{\alpha,l}^m$  can be computed by using Eq. (132) with  $m = 2$ , the first row, as given by Eqs. (135) and (136), and the initial values

$$Y_{1,2}^2 = -\frac{1}{280}, \quad Y_{2,2}^2 = -\frac{1}{1680}, \quad Y_{3,2}^2 = \frac{1}{720}, \quad Y_{4,2}^2 = -\frac{1}{1680} \quad \text{and} \quad Y_{5,2}^2 = \frac{1}{11088}. \quad (137a,b,c,d \text{ and } e)$$

Once the elements  $Y_{\alpha,l}^2$  are available, we use

$$(2l+1)Y_{\alpha,l}^{m+1} = \left(\frac{l+m+1}{l-m}\right)^{1/2} a_l^m Y_{\alpha,l-1}^m + \frac{2(2l+1)}{l(l+1)} [(l-m)(l+m+1)]^{1/2} Y_{\alpha,l}^m - \left(\frac{l-m}{l+m+1}\right)^{1/2} a_{l+1}^m Y_{\alpha,l+1}^m \quad (138)$$

to obtain all  $Y_{\alpha,l}^m$  for  $m > 2$ .

Turning our attention to the  $Z$  matrices, we find for  $m = 1$  that the first row can be computed from

$$l(l+5) \left[ \frac{(l-2)(l-1)l}{(l+3)} \right]^{1/2} Z_{0,l+1}^1 = -4(2l+1) \left(\frac{l-2}{l+2}\right)^{1/2} Z_{0,l}^1 - (l-4)(l+1)(l^2-1)^{1/2} Z_{0,l-1}^1, \quad (139)$$

with  $l = 4, 5, \dots$ , and the initial values

$$Z_{0,2}^1 = \frac{1}{20}, \quad Z_{0,3}^1 = \frac{33}{560} \left(\frac{5}{2}\right)^{1/2} \quad \text{and} \quad Z_{0,4}^1 = \frac{2^{1/2}}{64}. \quad (140a,b \text{ and } c)$$

Here, the first column ( $l = 1$ ) of the  $Z$  matrix for  $m = 1$  and all elements  $Z_{\alpha,l}^1$  with  $\alpha > l + 3$  are zero, and so the remaining non-zero  $Z_{\alpha,l}^1$  can be computed from the expression

$$a_{l+1}^m Z_{\alpha,l+1}^m = \frac{2l+1}{2} \left\{ \frac{\alpha}{2\alpha+1} Z_{\alpha-1,l}^m + \left[ 1 - \frac{4m}{l(l+1)} \right] Z_{\alpha,l}^m + \frac{\alpha+1}{2\alpha+1} Z_{\alpha+1,l}^m \right\} - a_l^m Z_{\alpha,l-1}^m, \quad (141)$$

with  $m = 1$ , the first row, as given by Eqs. (139) and (140), and the initial values

$$Z_{1,2}^1 = \frac{1}{35}, \quad Z_{2,2}^1 = -\frac{3}{112}, \quad Z_{3,2}^1 = -\frac{61}{5040}, \quad Z_{4,2}^1 = -\frac{1}{560} \quad \text{and} \quad Z_{5,2}^1 = -\frac{1}{11088}. \quad (142a,b,c,d \text{ and } e)$$

For  $m = 2$ , we find that the first row of the  $Z$  matrix can be computed from

$$(l-1)l(l+5)Z_{0,l+1}^2 = -8(2l+1)Z_{0,l}^2 - (l-4)(l+1)(l+2)Z_{0,l-1}^2, \quad (143)$$

with  $l = 4, 5, \dots$ , and the initial values

$$Z_{0,2}^2 = \frac{43}{40}, \quad Z_{0,3}^2 = \frac{5}{14} \quad \text{and} \quad Z_{0,4}^2 = \frac{1}{32}. \quad (144a,b \text{ and } c)$$

In general, for  $m \geq 2$ ,  $Z_{\alpha,l}^m = 0$  for  $\alpha > l + m + 1$ , and so the remaining non-zero  $Z_{\alpha,l}^m$  can be computed by using Eq. (141) with  $m = 2$ , the first row, as given by Eqs. (143) and (144), and the initial values

$$Z_{1,2}^2 = \frac{167}{280}, \quad Z_{2,2}^2 = \frac{101}{560}, \quad Z_{3,2}^2 = \frac{151}{5040}, \quad Z_{4,2}^2 = \frac{13}{5040} \quad \text{and} \quad Z_{5,2}^2 = \frac{1}{11088}. \quad (145a,b,c,d \text{ and } e)$$

Once the elements  $Z_{\alpha,l}^2$  are available, we use

$$(2l+1)Z_{\alpha,l}^{m+1} = \left(\frac{l+m+1}{l-m}\right)^{1/2} a_l^m Z_{\alpha,l-1}^m - \frac{2(2l+1)}{l(l+1)} [(l-m)(l+m+1)]^{1/2} Z_{\alpha,l}^m - \left(\frac{l-m}{l+m+1}\right)^{1/2} a_{l+1}^m Z_{\alpha,l+1}^m \quad (146)$$

to obtain all  $Z_{\alpha,l}^m$  for  $m > 2$ .

It is clear that numerical integration can be used as an alternative way of computing the constants  $\Delta_{x,l}^m$ , defined by Eq. (118), and so to be complete we report our scheme for computing the  $\Pi$  matrices. We write

$$\Pi_l^m(\mu) = \mathbf{S}^{-1} \text{diag}\{\mathcal{X}_l^m(\mu), \mathcal{Y}_l^m(\mu), \mathcal{Z}_l^m(\mu), \mathcal{X}_l^m(\mu)\}\mathbf{S}, \quad (147)$$

where  $\mathbf{S}$  is given by Eq. (105). In addition, we defined

$$\mathcal{X}_l^m(\mu) = \left[ \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(\mu), \quad l \geq m, \quad (148)$$

$$\mathcal{Y}_0^0(\mu) = 0, \quad \mathcal{Y}_1^0(\mu) = 0, \quad \mathcal{Y}_1^1(\mu) = 0, \quad (149a,b,c)$$

$$\mathcal{Y}_l^m(\mu) = \left[ \frac{(l-m)!}{(l+m)!} \right]^{1/2} [R_l^m(\mu) - T_l^m(\mu)], \quad l \geq \max\{2, m\}, \quad (149d)$$

$$\mathcal{Z}_0^0(\mu) = 0, \quad \mathcal{Z}_1^0(\mu) = 0, \quad \mathcal{Z}_1^1(\mu) = 0 \quad (150a,b,c)$$

and

$$\mathcal{Z}_l^m(\mu) = \left[ \frac{(l-m)!}{(l+m)!} \right]^{1/2} [R_l^m(\mu) + T_l^m(\mu)], \quad l \geq \max\{2, m\}. \quad (150d)$$

To compute the functions  $\mathcal{X}_l^m(\mu)$ ,  $\mathcal{Y}_l^m(\mu)$  and  $\mathcal{Z}_l^m(\mu)$  for  $\mu \in [0, 1]$ , we use the recursion formulae

$$[(l+1)^2 - m^2]^{1/2} \mathcal{X}_{l+1}^m(\mu) = (2l+1)\mu \mathcal{X}_l^m(\mu) - (l^2 - m^2)^{1/2} \mathcal{X}_{l-1}^m(\mu), \quad l \geq m, \quad (151)$$

$$a_{l+1}^m \mathcal{Y}_{l+1}^m(\mu) = (2l+1) \left[ \mu + \frac{2m}{l(l+1)} \right] \mathcal{Y}_l^m(\mu) - a_l^m \mathcal{Y}_{l-1}^m(\mu), \quad l \geq \max\{2, m\}, \quad (152)$$

and

$$a_{l+1}^m \mathcal{Z}_{l+1}^m(\mu) = (2l+1) \left[ \mu - \frac{2m}{l(l+1)} \right] \mathcal{Z}_l^m(\mu) - a_l^m \mathcal{Z}_{l-1}^m(\mu), \quad l \geq \max\{2, m\}, \quad (153)$$

along with the starting values

$$\mathcal{X}_m^m(\mu) = S_m(1 - \mu^2)^{m/2}, \quad (154)$$

$$\mathcal{Y}_2^0(\mu) = \frac{6^{1/2}}{4}(1 - \mu^2), \quad \mathcal{Y}_2^1(\mu) = \frac{1}{2}(1 - \mu)(1 - \mu^2)^{1/2} \quad \text{and} \quad \mathcal{Y}_m^m(\mu) = S_m R_m (1 - \mu)^2 (1 - \mu^2)^{(m/2)-1} \quad (155a,b \text{ and } c)$$

and

$$\mathcal{Z}_2^0(\mu) = \frac{6^{1/2}}{4}(1 - \mu^2), \quad \mathcal{Z}_2^1(\mu) = -\frac{1}{2}(1 + \mu)(1 - \mu^2)^{1/2} \quad \text{and} \quad \mathcal{Z}_m^m(\mu) = S_m R_m (1 + \mu)^2 (1 - \mu^2)^{(m/2)-1} \quad (156a,b \text{ and } c)$$

where  $S_m$  and  $R_m$  are given by Eqs. (30) and (31).

Returning now to our computation of the functions  $\mathbf{B}_x^m(v)$ , for  $v \in [0, 1]$ , we note that we next require the functions  $\mathbf{H}_l^m(v)$  as defined by Eq. (116). We write

$$\mathbf{H}_l^m(\xi) = \mathbf{S}^{-1} \text{diag}\{\mathcal{A}_l^m(\xi), \mathcal{B}_l^m(\xi), \mathcal{C}_l^m(\xi), \mathcal{A}_l^m(\xi)\}\mathbf{S}, \quad (157)$$

where the polynomials  $\mathcal{A}_l^m(\xi)$ ,  $\mathcal{B}_l^m(\xi)$  and  $\mathcal{C}_l^m(\xi)$  can be computed from recursion formulae. In order to evaluate the polynomials  $\mathcal{A}_l^m(\xi)$  we use, first of all,

$$\mathcal{A}_0^0(\xi) = 1 \quad (158)$$

and

$$(l+1)\mathcal{A}_{l+1}^0(\xi) = (2l+1)[X_{0,l}^0 - \xi \mathcal{A}_l^0(\xi)] - l \mathcal{A}_{l-1}^0(\xi) \quad (159)$$

for  $l = 0, 1, \dots$ . Subsequently, we use

$$(2l + 1)\mathcal{A}_l^m(\xi) = [(l + m - 1)(l + m)]^{1/2} \mathcal{A}_{l-1}^{m-1}(\xi) - [(l - m + 1)(l - m + 2)]^{1/2} \mathcal{A}_{l+1}^{m-1}(\xi) \quad (160)$$

for  $m \geq 1$  to find all other required values of the  $\mathcal{A}$  polynomials.

Considering now the  $\mathcal{B}$  and  $\mathcal{C}$  polynomials, we note that  $\mathcal{B}_0^0(\xi) = 0$ ,  $\mathcal{B}_1^0(\xi) = 0$  and

$$\mathcal{B}_l^0(\xi) = \frac{6^{1/2}}{4} \mathcal{A}_l^2(\xi) \quad (161)$$

for  $l \geq 2$ . Thus, since  $\mathcal{C}_l^0(\xi) = \mathcal{B}_l^0(\xi)$  for all  $l \geq 0$ , we now consider the cases of  $m \geq 1$ . For  $m = 1$  and  $m = 2$  we use the starting values  $\mathcal{B}_1^1(\xi) = 0$ ,  $\mathcal{C}_1^1(\xi) = 0$ ,

$$\mathcal{B}_2^1(\xi) = \frac{1}{4} \left( \frac{3}{10} + \frac{19}{12}\xi + \frac{2}{3}\xi^2 - \frac{3}{2}\xi^3 - \xi^4 \right), \quad (162a)$$

$$\mathcal{C}_2^1(\xi) = \frac{1}{4} \left( \frac{13}{10} - \frac{13}{12}\xi - \frac{4}{3}\xi^2 + \frac{5}{2}\xi^3 - \xi^4 \right), \quad (162b)$$

$$\mathcal{B}_2^2(\xi) = \frac{1}{4} \left( \frac{1}{3} + \frac{25}{12}\xi + \frac{13}{3}\xi^2 + \frac{7}{3}\xi^3 + \xi^4 \right) \quad (163a)$$

and

$$\mathcal{C}_2^2(\xi) = \frac{1}{4} \left( \frac{31}{5} - \frac{103}{12}\xi + \frac{25}{3}\xi^2 - \frac{9}{2}\xi^3 + \xi^4 \right) \quad (163b)$$

along with

$$a_{l+1}^m \mathcal{B}_{l+1}^m(\xi) = (2l + 1)Y_{0,l}^m - (2l + 1) \left[ \xi - \frac{2m}{l(l+1)} \right] \mathcal{B}_l^m(\xi) - a_l^m \mathcal{B}_{l-1}^m(\xi), \quad l \geq \max\{2, m\}, \quad (164a)$$

and

$$a_{l+1}^m \mathcal{C}_{l+1}^m(\xi) = (2l + 1)Z_{0,l}^m - (2l + 1) \left[ \xi + \frac{2m}{l(l+1)} \right] \mathcal{C}_l^m(\xi) - a_l^m \mathcal{C}_{l-1}^m(\xi), \quad l \geq \max\{2, m\}, \quad (164b)$$

to evaluate the required polynomials  $\mathcal{B}$  and  $\mathcal{C}$ . Finally, once the polynomials  $\mathcal{B}_l^2(\xi)$  and  $\mathcal{C}_l^2(\xi)$  are available, we use

$$(2l + 1)\mathcal{B}_l^{m+1}(\xi) = \left( \frac{l + m + 1}{l - m} \right)^{1/2} a_l^m \mathcal{B}_{l-1}^m(\xi) + \frac{2(2l + 1)}{l(l + 1)} \times [(l - m)(l + m + 1)]^{1/2} \mathcal{B}_l^m(\xi) - \left( \frac{l - m}{l + m + 1} \right)^{1/2} a_{l+1}^m \mathcal{B}_{l+1}^m(\xi) \quad (165a)$$

and

$$(2l + 1)\mathcal{C}_l^{m+1}(\xi) = \left( \frac{l + m + 1}{l - m} \right)^{1/2} a_l^m \mathcal{C}_{l-1}^m(\xi) - \frac{2(2l + 1)}{l(l + 1)} \times [(l - m)(l + m + 1)]^{1/2} \mathcal{C}_l^m(\xi) - \left( \frac{l - m}{l + m + 1} \right)^{1/2} a_{l+1}^m \mathcal{C}_{l+1}^m(\xi) \quad (165b)$$

to obtain all additional  $\mathcal{B}$  and  $\mathcal{C}$  polynomials that are required. We have found, for  $\xi = v \in [0, 1]$  or  $\xi = v_\beta$  with  $|v_\beta| \in [1, 1.001]$ , that the foregoing scheme yields the desired  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  polynomials without significant loss of accuracy.

Summarizing our procedure for evaluating the functions  $\mathbf{B}_\alpha(v)$ , for  $v \in [0, 1]$ , we note that the functions  $\mathbf{G}_l^m(v)$ , for  $v \in [0, 1]$ , can be accurately evaluated by forward recursion. We then use Eqs. (114) and (115) to establish the starting value  $\mathbf{B}_0(v)$ . We subsequently use Eq. (112) in the forward direction to compute  $\mathbf{B}_\alpha(v)$  for  $\alpha = 1, 2, \dots, N$ .

If we let

$$\mathbf{A}_\alpha(-v_\beta) = -\varpi \mathbf{M}^T(v_\beta) \int_0^1 \mu P_\alpha(2\mu - 1) \mathbf{G}^T(v_\beta, \mu) \mathbf{J} \Xi^m(\mu) \frac{d\mu}{v_\beta - \mu}, \quad (166)$$

then clearly

$$\mathbf{B}_x(v_\beta) = -\mathbf{A}_x(-v_\beta) \quad (167)$$

for  $\beta = 1, 2, \dots, N$ . We thus summarize our way of computing  $\mathbf{A}_x(\xi)$  for  $\xi \in \{\pm v_\beta\} \cup [0, 1]$ . Our general procedure for computing the  $\mathbf{A}_x(\xi)$  is to use backward recursion; however, for  $\xi$  near to 0 or  $-1$ , say for example  $\xi = v \in [0, 0.001]$  or  $\xi = -v_\beta$  with  $|v_\beta| \in [1, 1.001]$ , we use forward recursion. Thus, for  $|v_\beta| \in [1, 1.001]$ , we use

$$\mathbf{B}_0(v_\beta) = \mathbf{M}^T(v_\beta)[2\mathbf{J} - \varpi \mathbf{E}^m(v_\beta)\mathbf{J}] + \mathbf{A}_0(v_\beta)\mathbf{DK}^m \quad (168)$$

with

$$\begin{aligned} \mathbf{A}_0(v_\beta) = \varpi \sum_{l=m}^L (-1)^{l-m} [\mathbf{T}_l^m(v_\beta)]^T \mathbf{DJB} \mathbf{H}_l^m(v_\beta) \\ - \varpi v_\beta \log\left(1 + \frac{1}{v_\beta}\right) \mathbf{M}^T(v_\beta) \mathbf{G}^T(v_\beta, v_\beta) \mathbf{E}^m(v_\beta) \mathbf{DK}^m \mathbf{J}, \end{aligned} \quad (169)$$

where  $\mathbf{T}_l^m(v_\beta)$  is defined by Eq. (93), and

$$\mathbf{B}_{\alpha+1}(v_\beta) = \frac{2\alpha+1}{\alpha+1} [(2v_\beta-1)\mathbf{B}_\alpha(v_\beta) - 2\varpi \mathbf{Z}_\alpha(-v_\beta)] - \frac{\alpha}{\alpha+1} \mathbf{B}_{\alpha-1}(v_\beta) \quad (170)$$

for  $\alpha = 0, 1, \dots, N-1$ . We note that we can also use backward recursion, rather than Eq. (169), to deduce  $\mathbf{A}_0(v_\beta)$ .

For  $v \in [0, 0.001]$  and  $\alpha = 0, 1, \dots, N-1$ , we use

$$\mathbf{A}_{\alpha+1}(v) = \frac{2\alpha+1}{\alpha+1} [2\varpi \mathbf{Z}_\alpha(v) - (2v+1)\mathbf{A}_\alpha(v)] - \frac{\alpha}{\alpha+1} \mathbf{A}_{\alpha-1}(v), \quad (171)$$

where the starting value is provided by Eq. (115).

We now wish to describe the method of backward recursion we use to compute the functions  $\mathbf{A}_x(\xi)$  for  $\xi \neq v \in [0, 0.001]$  or  $\xi \neq -v_\beta$  with  $|v_\beta| \in [1, 1.001]$ , and for  $\alpha = 0, 1, \dots, N$ . As in Ref. 18, we use the starting value

$$U_0(\xi) = \frac{1}{2\xi+1} \quad (172a)$$

and forward recursion, i.e.,

$$U_\alpha(\xi) = \left[ \frac{2\alpha+1}{\alpha+1} (2\xi+1) - \frac{\alpha}{\alpha+1} U_{\alpha-1}(\xi) \right]^{-1} \quad (172b)$$

for  $\alpha = 1, 2, \dots$  to compute the ratios

$$U_\alpha(\xi) = P_\alpha(2\xi+1) [P_{\alpha+1}(2\xi+1)]^{-1}. \quad (173)$$

Then we use forward recursion, i.e.,

$$\mathbf{V}_0(\xi) = U_0(\xi) \mathbf{Z}_0(\xi) \quad (174a)$$

and

$$\mathbf{V}_x(\xi) = -U_x(\xi) [\mathbf{V}_{x-1}(\xi) - (2\alpha+1) \mathbf{Z}_x(\xi)] \quad (174b)$$

for  $\alpha = 1, 2, \dots$  to compute the functions  $\mathbf{V}_x(\xi)$  which define the inhomogeneous term in the two-term recursion formula

$$\mathbf{A}_x(\xi) = -U_x(\xi) \mathbf{A}_{x+1}(\xi) + \frac{2\varpi}{\alpha+1} \mathbf{V}_x(\xi). \quad (175)$$

It is clear that we now require a starting value  $\mathbf{A}_{M+1}(\xi)$  with  $M+1 \geq N$  in order to compute the required  $\mathbf{A}_x(\xi)$  from Eq. (175) by backward recursion. For  $M+1 > L + \max\{2, m\}$ , we can deduce from Eqs. (61b), (62b) and (166) that

$$\mathbf{A}_{M+1}(\xi) = 2\mathbf{F}_{M+1}(\xi) \mathbf{E}(\xi) [\mathbf{\Psi}^m(\xi)]^T \mathbf{K}^m \mathbf{DJ} \quad (176a)$$

for  $\xi \in \mathcal{P}$  and

$$\mathbf{A}_{M+1}(-v_\beta) = 2F_{M+1}(-v_\beta)\mathbf{M}^T(v_\beta)[\Psi^m(v_\beta)]^T\mathbf{J}. \quad (176b)$$

Here, the characteristic matrix is expressed as

$$\Psi^m(\xi) = \frac{\varpi}{2} \Xi^m(\xi)\mathbf{G}(\xi, \xi), \quad (177)$$

and

$$F_{M+1}(\xi) = \int_0^1 \mu P_{M+1}(2\mu - 1) \frac{d\mu}{\mu + \xi}. \quad (178)$$

Our general procedure for computing the functions  $\mathbf{A}_x(\xi)$  from Eq. (175) can be stated as follows. We compute  $F_{M+1}(\xi)$  by backward recursion as in Ref. 18 and use the result in Eqs. (176) to establish the starting value  $\mathbf{A}_{M+1}(\xi)$ , for  $M+1 > \max\{N, L+n\}$ , where  $n = \max\{2, m\}$ . We then use Eq. (175) to compute the required  $\mathbf{A}_x(\xi)$ .

Now in regard to the  $\mathbf{T}_l^m(\xi)$  required for  $\xi \in \{v_\beta\}$ , we first note that the initial values for the  $\mathbf{T}$  vectors can be found by multiplying the initial values of the  $\mathbf{G}$  matrices given by Eqs. (29) by the null vector  $\mathbf{M}(\xi)$  found as discussed in Sec. 5. For  $\xi = v_\beta$ ,  $|v_\beta| \in [1, 1.001]$ , we use the computed initial values for  $\mathbf{T}_l^m(\xi)$  along with the recursion relation

$$\mathbf{T}_{l+1}^m(\xi) = (\mathbf{U}_{l+1}^m)^{-1} [(\xi \mathbf{h}_l + \mathbf{V}_l^m)\mathbf{T}_l^m(\xi) - \mathbf{U}_l^m \mathbf{T}_{l-1}^m(\xi)] \quad (179)$$

in the forward direction. For  $\xi = v_\beta$ ,  $|v_\beta| > 1.001$ , we use backward recursion for computing the ratios  $\mathbf{R}_l^m(\xi)$  as discussed in Sec. 5 and then take

$$\mathbf{T}_{l+1}^m(\xi) = \mathbf{R}_l^m(\xi)\mathbf{T}_l^m(\xi) \quad (180)$$

for  $l = n, n+1, \dots, N-1$ .

To complete this section, we note here that the polynomial matrices  $\mathbf{C}_x(\xi)$  defined by Eq. (74) for  $\xi \in [0, 1]$  can be generated in the forward direction from

$$\mathbf{C}_{x+1}(\xi) = \frac{2x+1}{x+1} [(2\xi-1)\mathbf{C}_x(\xi) + 2\mathbf{Z}_x(-\xi)] - \frac{x}{x+1} \mathbf{C}_{x-1}(\xi) \quad (181)$$

with  $\mathbf{C}_0(\xi) = \mathbf{0}$ .

## 7. NUMERICAL RESULTS

In order to demonstrate that our  $F_N$  solution and the numerical techniques developed here do, in fact, define an especially accurate computational method, we now wish to report some numerical results for two test problems that have been defined in the literature.<sup>1</sup> As mentioned in Ref. 1, the *Greek constants* for these two model problems were computed by de Rooij and van der Stap,<sup>20</sup> were used in Refs. 21, 22 and 23 and are tabulated in Ref. 21; we refer to these two test problems as the  $L = 13$  problem and the  $L = 60$  problem. We should note that these *Greek constants* were calculated for the case  $\varpi = 1$ ; however to allow some absorption in the models we follow Refs. 1, 22 and 23 and consider these test problems with  $\varpi = 0.99$ .

As the collection of eigenvalues plays a crucial role in our  $F_N$  solution, we list, first of all, in Tables 1 and 2 the results of our use of the methods described in Sec. 5 to compute the discrete spectrum for each of the two test problems.

To report some specific results we follow Ref. 1 and consider a layer with optical thickness  $\tau_0 = 1$  and take the Lambertian ground to be defined by  $\lambda_0 = 0.1$ . We consider the case of a beam of unpolarized light, i.e.,  $\mathbf{F} = |1\ 0\ 0\ 0|^T$ , impinging on the layer at an angle defined by  $\mu_0 = 0.2$ . We report in Tables 3–10, for the  $L = 13$  problem, and in Tables 11–18, for the  $L = 60$  problem, our results for the diffuse field

$$\mathbf{I}_*(\tau, \mu, \phi) = \mathbf{I}(\tau, \mu, \phi) - \pi\delta(\mu - \mu_0)\delta(\phi - \phi_0)\mathbf{F} \exp\{-\tau/\mu\}. \quad (182)$$

All of our reported results were based on a collocation scheme defined by the discrete spectrum and the zeros of the Chebyshev polynomials of the first kind mapped onto the interval  $[0, 1]$ .

Table 1. The discrete spectrum for the  $L = 13$  problem with  $\omega = 0.99$ .

$m = 0$	$m = 1$	$m = 2$	$m = 3-13$
1.019535723105	1.041343962201	1.871962367868	no eigenvalues
1.896261279525	1.332260070864		
8.052579861387			

Table 2. The discrete spectrum for the  $L = 60$  problem with  $\omega = 0.99$ .

$m = 0$	$m = 1$
1.030780435341	1.004925017990
1.038869411971	1.043469589064
1.139072646734	1.077150068731
1.153451577057	1.145908071442
1.392584925216	1.230376882947
1.454513843050	2.013271330065
3.285868706929	1.002301759862 ± i 3.403411378895(-4)
1.015382838541(1)	1.064999343900 ± i 1.103104743978(-3)
1.028957515744 ± i 4.810303487412(-3)	1.199828052746 ± i 3.733810152033(-4)
1.133801054671 ± i 2.359049300604(-3)	1.620846488454 ± i 4.724528394054(-3)
1.964325970300 ± i 1.068168843828(-2)	
$m = 2$	$m = 3$
1.022703844098	1.003853554612
1.058875984710	1.043368723889
1.091809309717	1.113136643723
1.092247059833	1.118590280523
1.121273563429	1.158491006003
1.235686328631	1.409665375044
1.240548506477	1.022088568749 ± i 2.858576481365(-4)
1.322837891914	
2.907999608542	
1.011015913492 ± i 3.157260456160(-4)	
$m = 4$	$m = 5$
1.001950207882	1.011427141836
1.042223383173	1.094765195251
1.044042778925	
1.067285231607	
1.192485908200	
$m = 6$	$m = 7-60$
1.030378652765	no eigenvalues

We note that the results listed in Tables 3–18 were obtained with  $N \leq 400$  and are thought to be correct within  $\pm 1$  in the last digits given. To establish confidence in the accuracy of our results, we have observed what appears to be convergence in the numerical solutions as  $N$  increases. Of course, our results here show, in general, agreement to four or five significant figures with those

Table 3. The diffuse intensity  $I_*(\tau, \mu, \phi)$  for the  $L = 13$  problem with  $\phi - \phi_0 = 0$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	5.49566(-2)	4.87338(-2)	4.93458(-2)	3.47494(-2)	1.95908(-2)	1.28200(-2)	8.74694(-3)
-0.9	9.04913(-2)	7.81831(-2)	6.77241(-2)	5.15295(-2)	2.52424(-2)	1.47808(-2)	8.74694(-3)
-0.8	1.26601(-1)	1.07512(-1)	9.22144(-2)	6.87261(-2)	3.15057(-2)	1.71323(-2)	8.74694(-3)
-0.7	1.87810(-1)	1.42808(-1)	1.21728(-1)	8.95325(-2)	3.92821(-2)	2.01700(-2)	8.74694(-3)
-0.6	2.19343(-1)	1.85923(-1)	1.57808(-1)	1.15037(-1)	4.90477(-2)	2.41486(-2)	8.74694(-3)
-0.5	2.82944(-1)	2.39151(-1)	2.02365(-1)	1.46608(-1)	6.14581(-2)	2.94669(-2)	8.74694(-3)
-0.4	3.82682(-1)	3.05880(-1)	2.58221(-1)	1.88227(-1)	7.74894(-2)	3.68024(-2)	8.74694(-3)
-0.3	4.65232(-1)	3.91625(-1)	3.29894(-1)	2.36974(-1)	9.86196(-2)	4.73928(-2)	8.74694(-3)
-0.2	6.02877(-1)	5.06471(-1)	4.25517(-1)	3.04077(-1)	1.26933(-1)	6.36368(-2)	8.74694(-3)
-0.1	8.02239(-1)	6.72666(-1)	5.63009(-1)	3.98605(-1)	1.64452(-1)	8.94606(-2)	8.74694(-3)
-0.0	1.11633	9.49048(-1)	7.92855(-1)	5.54308(-1)	2.16685(-1)	1.19876(-1)	8.74694(-3)
0.0		9.40048(-1)	7.92855(-1)	5.54308(-1)	2.16685(-1)	1.19876(-1)	6.94687(-2)
0.1		4.12616(-1)	5.98687(-1)	6.37328(-1)	3.09925(-1)	1.63366(-1)	9.57901(-2)
0.2		2.32932(-1)	3.77201(-1)	4.93650(-1)	3.64184(-1)	2.20560(-1)	1.33509(-1)
0.3		1.58655(-1)	2.67847(-1)	3.80677(-1)	3.53420(-1)	2.48804(-1)	1.66798(-1)
0.4		1.16441(-1)	2.00962(-1)	2.98793(-1)	3.18326(-1)	2.49367(-1)	1.83225(-1)
0.5		8.80974(-2)	1.54206(-1)	2.36105(-1)	2.76095(-1)	2.33632(-1)	1.84462(-1)
0.6		6.70328(-2)	1.18554(-1)	1.85498(-1)	2.32659(-1)	2.08978(-1)	1.74765(-1)
0.7		5.02977(-2)	8.97207(-2)	1.42929(-1)	1.90031(-1)	1.79442(-1)	1.57561(-1)
0.8		3.63518(-2)	6.53713(-2)	1.05918(-1)	1.48706(-1)	1.47070(-1)	1.35063(-1)
0.9		2.41589(-2)	4.38523(-2)	7.24332(-2)	1.08011(-1)	1.12263(-1)	1.08064(-1)
1.0		9.89221(-3)	1.83913(-2)	3.18369(-2)	5.42520(-2)	6.22536(-2)	6.53180(-2)

Table 4. The Stokes parameter  $Q(\tau, \mu, \phi)$  for the  $L = 13$  problem with  $\phi - \phi_0 = 0$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	-2.16088(-2)	-1.76341(-2)	-1.44003(-2)	-9.66520(-3)	-3.04103(-3)	-1.01473(-3)	
-0.9	-3.25810(-2)	-2.64804(-2)	-2.15679(-2)	-1.44270(-2)	-4.59292(-3)	-1.60062(-3)	
-0.8	-3.50476(-2)	-2.82563(-2)	-2.28372(-2)	-1.50735(-2)	-4.71192(-3)	-1.68885(-3)	
-0.7	-3.49497(-2)	-2.78598(-2)	-2.22808(-2)	-1.44203(-2)	-4.35775(-3)	-1.61066(-3)	
-0.6	-3.27675(-2)	-2.56795(-2)	-2.02040(-2)	-1.26615(-2)	-3.57482(-3)	-1.37404(-3)	
-0.5	-2.86635(-2)	-2.18244(-2)	-1.86774(-2)	-9.81783(-3)	-2.33324(-3)	-9.55030(-4)	
-0.4	-2.27539(-2)	-1.63484(-2)	-1.17142(-2)	-5.85389(-3)	-5.58327(-4)	-3.00637(-4)	
-0.3	-1.52402(-2)	-9.34559(-3)	-5.34144(-3)	-7.18670(-4)	1.87565(-3)	6.98247(-4)	
-0.2	-0.64174(-3)	-1.10608(-3)	2.28653(-3)	5.58742(-3)	5.15012(-3)	2.24308(-3)	
-0.1	1.43775(-3)	7.42188(-3)	1.05703(-2)	1.28186(-2)	9.38543(-3)	4.79920(-3)	
-0.0	-8.02959(-4)	1.18520(-2)	1.69093(-2)	1.99136(-2)	1.42875(-2)	8.33828(-3)	
0.0		1.18520(-2)	1.69093(-2)	1.99136(-2)	1.42875(-2)	8.33828(-3)	2.50377(-3)
0.1		6.12928(-3)	1.23090(-2)	1.99811(-2)	1.89218(-2)	1.21064(-2)	6.01031(-3)
0.2		3.77753(-3)	8.06480(-3)	1.49476(-2)	1.93038(-2)	1.45870(-2)	8.82235(-3)
0.3		1.79323(-3)	4.37000(-3)	9.40770(-3)	1.57884(-2)	1.39516(-2)	9.85097(-3)
0.4		-8.70005(-5)	8.64268(-4)	3.78743(-3)	1.01387(-2)	1.06075(-2)	8.54172(-3)
0.5		-1.87938(-3)	-2.47178(-3)	-1.73375(-3)	3.37923(-3)	5.39011(-3)	5.18067(-3)
0.6		-3.51990(-3)	-5.56080(-3)	-6.96537(-3)	-3.83064(-3)	-9.79612(-4)	2.78207(-4)
0.7		-4.95456(-3)	-8.27784(-3)	-1.16812(-2)	-1.09689(-2)	-7.89307(-3)	-5.63231(-3)
0.8		-6.07845(-3)	-1.04497(-2)	-1.55948(-2)	-1.75331(-2)	-1.47367(-2)	-1.20240(-2)
0.9		-6.72849(-3)	-1.17916(-2)	-1.82594(-2)	-2.28983(-2)	-2.10359(-2)	-1.83355(-2)
1.0		-5.91556(-3)	-1.06101(-2)	-1.71825(-2)	-2.45082(-2)	-2.48013(-2)	-2.34397(-2)

Table 5. The intensity  $I(\tau, \mu, \phi)$  for the  $L = 13$  problem with  $\phi - \phi_0 = \pi/2$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	5.49566(-2)	4.87338(-2)	4.35458(-2)	3.47494(-2)	1.95908(-2)	1.28200(-2)	8.74694(-3)
-0.9	6.22106(-2)	5.53025(-2)	4.92512(-2)	3.94892(-2)	2.19069(-2)	1.37972(-2)	8.74694(-3)
-0.8	7.05637(-2)	6.29038(-2)	5.61233(-2)	4.50657(-2)	2.47364(-2)	1.50411(-2)	8.74694(-3)
-0.7	8.02020(-2)	7.17474(-2)	6.41625(-2)	5.16620(-2)	2.82193(-2)	1.66414(-2)	8.74694(-3)
-0.6	9.14350(-2)	8.21057(-2)	7.36303(-2)	5.95182(-2)	3.25520(-2)	1.87343(-2)	8.74694(-3)
-0.5	1.04612(-1)	9.43314(-2)	8.48659(-2)	6.89494(-2)	3.80166(-2)	2.15380(-2)	8.74694(-3)
-0.4	1.20182(-1)	1.08871(-1)	9.82988(-2)	8.03555(-2)	4.50202(-2)	2.54236(-2)	8.74694(-3)
-0.3	1.38680(-1)	1.26264(-1)	1.14439(-1)	9.41968(-2)	5.41233(-2)	3.10651(-2)	8.74694(-3)
-0.2	1.60702(-1)	1.47140(-1)	1.39847(-1)	1.10879(-1)	6.59017(-2)	3.97436(-2)	8.74694(-3)
-0.1	1.87015(-1)	1.72614(-1)	1.57516(-1)	1.30858(-1)	7.99602(-2)	5.33616(-2)	8.74694(-3)
-0.0	2.09394(-1)	2.03761(-1)	1.86141(-1)	1.56874(-1)	9.51064(-2)	6.73319(-2)	8.74694(-3)
0.0		2.03761(-1)	1.86141(-1)	1.56874(-1)	9.51064(-2)	6.73319(-2)	4.43026(-2)
0.1		8.19211(-2)	1.26719(-1)	1.54567(-1)	1.13939(-1)	8.09648(-2)	5.73829(-2)
0.2		4.59131(-2)	7.90658(-2)	1.15815(-1)	1.18517(-1)	9.31576(-2)	6.96471(-2)
0.3		3.17120(-2)	5.67928(-2)	8.96445(-2)	1.10572(-1)	9.67375(-2)	7.80744(-2)
0.4		2.41370(-2)	4.40866(-2)	7.24049(-2)	9.99036(-2)	9.45579(-2)	8.16010(-2)
0.5		1.94371(-2)	3.58981(-2)	6.03744(-2)	8.96369(-2)	8.98139(-2)	8.17113(-2)
0.6		1.62477(-2)	3.01964(-2)	5.15439(-2)	8.04918(-2)	8.41481(-2)	7.98013(-2)
0.7		1.39543(-2)	2.60145(-2)	4.48079(-2)	7.25131(-2)	7.83120(-2)	7.67724(-2)
0.8		1.22395(-2)	2.28352(-2)	3.95208(-2)	6.55766(-2)	7.26385(-2)	7.31607(-2)
0.9		1.09223(-2)	2.03568(-2)	3.52835(-2)	5.95338(-2)	6.72682(-2)	6.92805(-2)
1.0		9.89221(-3)	1.83913(-2)	3.18369(-2)	5.42520(-2)	6.22536(-2)	6.53180(-2)

Table 6. The Stokes parameter  $Q(\tau, \mu, \phi)$  for the  $L = 13$  problem with  $\phi - \phi_0 = \pi/2$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	2.16088(-2)	1.76341(-2)	1.44003(-2)	9.66520(-3)	3.04103(-3)	1.01473(-3)	
-0.9	2.57044(-2)	2.11945(-2)	1.74722(-2)	1.19308(-2)	3.87944(-3)	1.28060(-3)	
-0.8	3.04695(-2)	2.53678(-2)	2.10986(-2)	1.46450(-2)	4.94340(-3)	1.64642(-3)	
-0.7	3.60466(-2)	3.02851(-2)	2.53986(-2)	1.79064(-2)	6.29144(-3)	2.14400(-3)	
-0.6	4.26327(-2)	3.61265(-2)	3.05357(-2)	2.18501(-2)	8.00638(-3)	2.82121(-3)	
-0.5	5.05054(-2)	4.31465(-2)	3.87399(-2)	2.66653(-2)	1.02094(-2)	3.75441(-3)	
-0.4	6.00866(-2)	5.17127(-2)	4.45419(-2)	3.26216(-2)	1.30824(-2)	5.07374(-3)	
-0.3	7.19145(-2)	6.23730(-2)	5.38290(-2)	4.01056(-2)	1.68992(-2)	7.01847(-3)	
-0.2	8.69871(-2)	7.59944(-2)	6.59568(-2)	4.96748(-2)	2.20321(-2)	1.00586(-2)	
-0.1	1.06899(-1)	9.42038(-2)	8.21481(-2)	6.22840(-2)	2.87427(-2)	1.50077(-2)	
-0.0	1.29957(-1)	1.19868(-1)	1.05706(-1)	8.06362(-2)	3.73032(-2)	2.10559(-2)	
0.0		1.19868(-1)	1.05706(-1)	8.06362(-2)	3.73032(-2)	2.10559(-2)	1.02824(-2)
0.1		4.93085(-2)	7.40852(-2)	8.48915(-2)	4.97448(-2)	2.85037(-2)	1.60315(-2)
0.2		2.78350(-2)	4.63966(-2)	6.43737(-2)	5.51476(-2)	3.64863(-2)	2.25248(-2)
0.3		1.90643(-2)	3.32533(-2)	4.99088(-2)	5.26704(-2)	3.99634(-2)	2.76720(-2)
0.4		1.44862(-2)	2.57594(-2)	4.02377(-2)	4.79392(-2)	4.00015(-2)	3.03236(-2)
0.5		1.16445(-2)	2.09190(-2)	3.34310(-2)	4.29812(-2)	3.82956(-2)	3.10371(-2)
0.6		9.71773(-3)	1.75459(-2)	2.84076(-2)	3.83730(-2)	3.58145(-2)	3.05070(-2)
0.7		8.33569(-3)	1.50746(-2)	2.45644(-2)	3.42426(-2)	3.30402(-2)	2.92244(-2)
0.8		7.30699(-3)	1.32018(-2)	2.15465(-2)	3.05838(-2)	3.02106(-2)	2.75009(-2)
0.9		6.52258(-3)	1.17504(-2)	1.91337(-2)	2.73545(-2)	2.74442(-2)	2.55315(-2)
1.0		5.91556(-3)	1.06101(-2)	1.71825(-2)	2.45082(-2)	2.48013(-2)	2.34397(-2)



Table 7. The Stokes parameter  $U(\tau, \mu, \phi)$  for the  $L = 13$  problem with  $\phi - \phi_0 = \pi/2$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	-5.98947(-3)	-5.28505(-3)	-4.65053(-3)	-3.59821(-3)	-1.60408(-3)	-6.42915(-4)	
-0.8	-9.18687(-3)	-8.08939(-3)	-7.14214(-3)	-5.56189(-3)	-2.54182(-3)	-1.04486(-3)	
-0.7	-1.21087(-2)	-1.07565(-2)	-9.52904(-3)	-7.47350(-3)	-3.50471(-3)	-1.48113(-3)	
-0.6	-1.51870(-2)	-1.35363(-2)	-1.20324(-2)	-9.50624(-3)	-4.58405(-3)	-1.99908(-3)	
-0.5	-1.85266(-2)	-1.68681(-2)	-1.47777(-2)	-1.17637(-2)	-5.85100(-3)	-2.64843(-3)	
-0.4	-2.22612(-2)	-1.99732(-2)	-1.78745(-2)	-1.43390(-2)	-7.38794(-3)	-3.50430(-3)	
-0.3	-2.65347(-2)	-2.38797(-2)	-2.14355(-2)	-1.73235(-2)	-9.29697(-3)	-4.69756(-3)	
-0.2	-3.15343(-2)	-2.84474(-2)	-2.55874(-2)	-2.07928(-2)	-1.16682(-2)	-6.46988(-3)	
-0.1	-3.76315(-2)	-3.40142(-2)	-3.05931(-2)	-2.48538(-2)	-1.43822(-2)	-9.14872(-3)	
-0.0	-4.47971(-2)	-4.14321(-2)	-3.73441(-2)	-3.01838(-2)	-1.72176(-2)	-1.16918(-2)	
0.0		-4.14321(-2)	-3.73441(-2)	-3.01838(-2)	-1.72176(-2)	-1.16918(-2)	-7.55875(-3)
0.1		-1.69321(-2)	-2.58330(-2)	-3.02653(-2)	-2.08041(-2)	-1.41059(-2)	-9.66419(-3)
0.2		-9.40192(-3)	-1.58483(-2)	-2.25128(-2)	-2.16191(-2)	-1.62462(-2)	-1.16874(-2)
0.3		-6.36561(-3)	-1.11530(-2)	-1.70704(-2)	-1.98156(-2)	-1.66331(-2)	-1.29335(-2)
0.4		-4.69014(-3)	-8.37767(-3)	-1.33380(-2)	-1.73404(-2)	-1.57800(-2)	-1.31399(-2)
0.5		-3.59899(-3)	-6.49672(-3)	-1.05929(-2)	-1.48317(-2)	-1.43054(-2)	-1.25689(-2)
0.6		-2.79909(-3)	-5.08982(-3)	-8.42859(-3)	-1.24294(-2)	-1.25185(-2)	-1.14690(-2)
0.7		-2.16126(-3)	-3.94620(-3)	-6.60317(-3)	-1.01136(-2)	-1.05313(-2)	-9.97453(-3)
0.8		-1.60215(-3)	-2.93221(-3)	-4.94095(-3)	-7.78634(-3)	-8.32487(-3)	-8.10055(-3)
0.9		-1.04314(-3)	-1.91119(-3)	-3.23517(-3)	-5.21028(-3)	-5.69113(-3)	-5.66308(-3)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 8. The Stokes parameter  $V(\tau, \mu, \phi)$  for the  $L = 13$  problem with  $\phi - \phi_0 = \pi/2$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	-5.68780(-5)	-5.54639(-5)	-5.24407(-5)	-4.50153(-5)	-2.39976(-5)	-1.02045(-5)	
-0.8	-6.80642(-5)	-6.81943(-5)	-6.58296(-5)	-5.83259(-5)	-3.32067(-5)	-1.49583(-5)	
-0.7	-6.74931(-5)	-7.02403(-5)	-6.97035(-5)	-6.42630(-5)	-3.94180(-5)	-1.88294(-5)	
-0.6	-5.86564(-5)	-6.48030(-5)	-6.60417(-5)	-6.51007(-5)	-4.36257(-5)	-2.21777(-5)	
-0.5	-4.27016(-5)	-5.29051(-5)	-5.84437(-5)	-6.15184(-5)	-4.60868(-5)	-2.51182(-5)	
-0.4	-1.97812(-5)	-3.46525(-5)	-4.42852(-5)	-5.35396(-5)	-4.67513(-5)	-2.76919(-5)	
-0.3	1.07632(-5)	-9.39495(-6)	-2.38660(-5)	-4.06483(-5)	-4.52572(-5)	-2.98803(-5)	
-0.2	5.05939(-5)	2.44972(-5)	4.27960(-5)	-2.16526(-5)	-4.06704(-5)	-3.14603(-5)	
-0.1	1.02780(-4)	7.03444(-5)	4.30616(-5)	5.51945(-6)	-3.10435(-5)	-3.09344(-5)	
-0.0	1.57669(-4)	1.34397(-4)	9.97707(-5)	4.59654(-5)	-1.55081(-5)	-2.20465(-5)	
0.0		1.34397(-4)	9.97707(-5)	4.59654(-5)	-1.55081(-5)	-2.20465(-5)	-2.85101(-5)
0.1		5.39811(-5)	7.66513(-5)	7.23288(-5)	8.03903(-6)	-8.57659(-6)	-1.26108(-5)
0.2		2.55971(-5)	4.17134(-5)	5.15114(-5)	2.33141(-5)	6.64074(-6)	1.08760(-6)
0.3		1.30888(-5)	2.27313(-5)	3.16549(-5)	2.31074(-5)	1.43924(-5)	1.17515(-5)
0.4		5.64837(-6)	1.03649(-5)	1.57240(-5)	1.56430(-5)	1.44405(-5)	1.67103(-5)
0.5		6.50936(-7)	1.57871(-6)	3.01926(-6)	5.52131(-6)	9.60200(-6)	1.65153(-5)
0.6		-2.79490(-6)	-4.74235(-6)	-6.86807(-6)	-4.85968(-6)	2.30996(-6)	1.26700(-5)
0.7		-5.00757(-6)	-8.97061(-6)	-1.39360(-5)	-1.38831(-5)	-5.48639(-6)	6.76720(-6)
0.8		-6.01735(-6)	-1.10430(-5)	-1.77447(-5)	-1.99688(-5)	-1.19288(-5)	4.30942(-7)
0.9		-5.52307(-6)	-1.03074(-5)	-1.69706(-5)	-2.06371(-5)	-1.44803(-5)	-4.31012(-6)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 9. The intensity  $I(\tau, \mu, \phi)$  for the  $L = 13$  problem with  $\phi - \phi_0 = \pi$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	5.49566(-2)	4.87338(-2)	4.33458(-2)	3.47494(-2)	1.95908(-2)	1.28200(-2)	8.74694(-3)
-0.9	5.50852(-2)	4.79827(-2)	4.34281(-2)	3.58681(-2)	2.11988(-2)	1.37157(-2)	8.74694(-3)
-0.8	5.86882(-2)	5.39270(-2)	4.84701(-2)	4.02796(-2)	2.38216(-2)	1.49660(-2)	8.74694(-3)
-0.7	6.56541(-2)	5.98875(-2)	5.45982(-2)	4.55733(-2)	2.69741(-2)	1.65134(-2)	8.74694(-3)
-0.6	7.36789(-2)	6.74528(-2)	6.16745(-2)	5.17181(-2)	3.07652(-2)	1.84642(-2)	8.74694(-3)
-0.5	8.27548(-2)	7.60558(-2)	6.97620(-2)	5.88228(-2)	3.53875(-2)	2.09963(-2)	8.74694(-3)
-0.4	9.29345(-2)	8.57835(-2)	7.89701(-2)	6.70365(-2)	4.11195(-2)	2.44135(-2)	8.74694(-3)
-0.3	1.04215(-1)	9.66756(-2)	8.93590(-2)	7.64598(-2)	4.83197(-2)	2.92647(-2)	8.74694(-3)
-0.2	1.16408(-1)	1.08606(-1)	1.00801(-1)	8.69498(-2)	5.72424(-2)	3.65762(-2)	8.74694(-3)
-0.1	1.29132(-1)	1.21441(-1)	1.13121(-1)	9.80516(-2)	6.70544(-2)	4.77366(-2)	8.74694(-3)
-0.0	1.36294(-1)	1.34765(-1)	1.26966(-1)	1.10589(-1)	7.59353(-2)	5.79795(-2)	8.74694(-3)
0.0		1.34765(-1)	1.26966(-1)	1.10589(-1)	7.59353(-2)	5.79795(-2)	4.02506(-2)
0.1		5.20197(-2)	3.15088(-2)	1.02605(-1)	8.50700(-2)	6.60242(-2)	4.98242(-2)
0.2		2.80778(-2)	4.90075(-2)	7.39267(-2)	8.36593(-2)	7.14511(-2)	5.71725(-2)
0.3		1.86034(-2)	3.38324(-2)	5.50909(-2)	7.48381(-2)	7.07012(-2)	6.09061(-2)
0.4		1.35418(-2)	2.51815(-2)	4.28060(-2)	6.51989(-2)	6.64881(-2)	6.11026(-2)
0.5		1.04213(-2)	1.96531(-2)	3.43509(-2)	5.66281(-2)	6.11613(-2)	5.92140(-2)
0.6		8.35819(-3)	1.59094(-2)	2.83560(-2)	4.95110(-2)	5.58939(-2)	5.64243(-2)
0.7		6.98236(-3)	1.33638(-2)	2.41261(-2)	4.39279(-2)	5.13121(-2)	5.35577(-2)
0.8		6.16506(-3)	1.18154(-2)	2.14546(-2)	4.00878(-2)	4.76698(-2)	5.13467(-2)
0.9		6.03307(-3)	1.15068(-2)	2.07943(-2)	3.89221(-2)	4.70233(-2)	5.10398(-2)
1.0		9.89221(-3)	1.83913(-2)	3.18369(-2)	5.42520(-2)	6.22536(-2)	6.53180(-2)

Table 10. The Stokes parameter  $Q(\tau, \mu, \phi)$  for the  $L = 13$  problem with  $\phi - \phi_0 = \pi$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	-2.16088(-2)	-1.76341(-2)	-1.44003(-2)	-9.66520(-3)	-8.04103(-3)	-1.01473(-3)	
-0.9	-8.90179(-3)	-6.88027(-3)	-5.30594(-3)	-3.14151(-3)	-6.60049(-4)	-2.03154(-4)	
-0.8	-3.30780(-3)	-2.00374(-3)	-1.07397(-3)	2.99033(-5)	5.87863(-4)	2.18491(-4)	
-0.7	1.21562(-3)	2.00978(-3)	2.46094(-3)	2.74466(-3)	1.70874(-3)	6.03016(-4)	
-0.6	5.21702(-3)	5.62166(-3)	5.68635(-3)	5.27786(-3)	2.80468(-3)	9.87693(-4)	
-0.5	8.87560(-3)	8.98983(-3)	8.73978(-3)	7.73348(-3)	3.92266(-3)	1.39212(-3)	
-0.4	1.22307(-2)	1.21619(-2)	1.16706(-2)	1.01586(-2)	5.09728(-3)	1.83498(-3)	
-0.3	1.52030(-2)	1.50988(-2)	1.44625(-2)	1.25621(-2)	6.36365(-3)	2.34410(-3)	
-0.2	1.75007(-2)	1.76154(-2)	1.69920(-2)	1.48899(-2)	7.75839(-3)	2.97509(-3)	
-0.1	1.82261(-2)	1.91420(-2)	1.88619(-2)	1.89233(-2)	9.26804(-3)	3.85730(-3)	
-0.0	1.21265(-2)	1.72782(-2)	1.85778(-2)	1.79542(-2)	1.06288(-2)	5.01473(-3)	
0.0		1.72782(-2)	1.85778(-2)	1.79542(-2)	1.06288(-2)	5.01473(-3)	-3.35051(-4)
0.1		4.97438(-3)	9.04365(-3)	1.33750(-2)	1.10924(-2)	5.83430(-3)	1.01219(-3)
0.2		1.83216(-3)	3.91078(-3)	7.20131(-3)	8.63581(-3)	5.33383(-3)	1.35034(-3)
0.3		4.10570(-4)	1.25609(-3)	3.03572(-3)	4.83152(-3)	3.14958(-3)	4.12909(-4)
0.4		-5.03882(-4)	-5.23358(-4)	-3.84805(-5)	9.20208(-4)	8.55875(-5)	-1.60117(-3)
0.5		-1.21381(-3)	-1.92045(-3)	-2.54837(-3)	-2.83411(-3)	-3.34722(-3)	-4.29760(-3)
0.6		-1.83933(-3)	-3.14724(-3)	-4.77595(-3)	-6.44225(-3)	-6.94192(-3)	-7.41247(-3)
0.7		-2.45123(-3)	-4.33221(-3)	-6.91090(-3)	-9.99773(-3)	-1.06489(-2)	-1.08121(-2)
0.8		-3.11770(-3)	-5.59778(-3)	-9.14010(-3)	-1.36572(-2)	-1.45199(-2)	-1.44615(-2)
0.9		-3.95864(-3)	-7.15385(-3)	-1.17804(-2)	-1.77371(-2)	-1.87503(-2)	-1.84379(-2)
1.0		-5.91556(-3)	-1.06101(-2)	-1.71825(-2)	-2.45082(-2)	-2.48013(-2)	-2.34397(-2)

Table 11. The diffuse intensity  $I_*(\tau, \mu, \phi)$  for the  $L = 60$  problem with  $\phi - \phi_0 = 0$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	3.87835(-2)	3.56999(-2)	3.28307(-2)	2.78696(-2)	1.78899(-2)	1.29857(-2)	9.87579(-3)
-0.9	6.38819(-2)	5.74757(-2)	5.15888(-2)	4.16484(-2)	2.29735(-2)	1.46962(-2)	9.87579(-3)
-0.8	9.35680(-2)	8.33824(-2)	7.40383(-2)	5.83588(-2)	2.94898(-2)	1.70420(-2)	9.87579(-3)
-0.7	1.35703(-1)	1.20162(-1)	1.05910(-1)	8.20902(-2)	3.88354(-2)	2.04843(-2)	9.87579(-3)
-0.6	1.96525(-1)	1.73245(-1)	1.51892(-1)	1.16318(-1)	5.24325(-2)	2.56125(-2)	9.87579(-3)
-0.5	2.84900(-1)	2.50051(-1)	2.18669(-1)	1.66032(-1)	7.24332(-2)	3.33822(-2)	9.87579(-3)
-0.4	4.14018(-1)	3.63003(-1)	3.16223(-1)	2.38750(-1)	1.02328(-1)	4.56096(-2)	9.87579(-3)
-0.3	6.06206(-1)	5.31486(-1)	4.62421(-1)	3.48040(-1)	1.48676(-1)	6.57007(-2)	9.87579(-3)
-0.2	9.30289(-1)	8.21274(-1)	7.15306(-1)	5.36726(-1)	2.28727(-1)	1.04206(-1)	9.87579(-3)
-0.1	1.74996	1.56325	1.35999	1.00481	4.04835(-1)	1.93985(-1)	9.87579(-3)
-0.0	4.40394	4.06555	3.52371	2.53541	9.00996(-1)	4.08404(-1)	9.87579(-3)
0.0		4.06555	3.52371	2.53541	9.00996(-1)	4.08404(-1)	1.81246(-1)
0.1		3.31433	4.82115	5.15523	2.26859	9.64121(-1)	4.30266(-1)
0.2		2.41107	3.88359	5.01983	3.41129	1.79299	8.87863(-1)
0.3		1.23246	2.07136	2.92364	2.61006	1.71425	1.03987
0.4		4.04965(-1)	7.06500(-1)	1.07516	1.21185	9.69717(-1)	7.12871(-1)
0.5		1.21465(-1)	2.21386(-1)	3.65241(-1)	5.13117(-1)	4.86244(-1)	4.18122(-1)
0.6		6.28108(-2)	1.16248(-1)	1.97346(-1)	3.01975(-1)	3.08349(-1)	2.86126(-1)
0.7		4.06633(-2)	7.55262(-2)	1.29378(-1)	2.05486(-1)	2.18095(-1)	2.11198(-1)
0.8		2.43812(-2)	4.56794(-2)	7.96344(-2)	1.33470(-1)	1.48149(-1)	1.49888(-1)
0.9		1.31646(-2)	2.49432(-2)	4.44409(-2)	7.93301(-2)	9.25615(-2)	9.81535(-2)
1.0		4.42618(-3)	8.57341(-3)	1.59032(-2)	3.16195(-2)	3.99547(-2)	4.55071(-2)

Table 12. The Stokes parameter  $Q(\tau, \mu, \phi)$  for the  $L = 60$  problem with  $\phi - \phi_0 = 0$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	3.20870(-3)	2.77800(-3)	2.39491(-3)	1.77047(-3)	6.81639(-4)	2.45234(-4)	
-0.9	5.64374(-3)	4.84758(-3)	4.14106(-3)	3.00253(-3)	1.09652(-3)	3.82882(-4)	
-0.8	7.89013(-3)	6.73064(-3)	5.72611(-3)	4.10900(-3)	1.48897(-3)	5.14544(-4)	
-0.7	9.99484(-3)	8.49239(-3)	7.18139(-3)	5.11100(-3)	1.80444(-3)	6.40306(-4)	
-0.6	1.16127(-2)	9.80470(-3)	8.24686(-3)	5.81814(-3)	2.03327(-3)	7.35736(-4)	
-0.5	1.21988(-2)	1.02052(-2)	8.51978(-3)	5.93948(-3)	2.04885(-3)	7.61942(-4)	
-0.4	1.10913(-2)	9.13593(-3)	7.52545(-3)	5.12671(-3)	1.70915(-3)	6.62892(-4)	
-0.3	8.32710(-3)	6.64667(-3)	5.29261(-3)	3.35818(-3)	9.31096(-4)	3.84707(-4)	
-0.2	7.35083(-3)	5.58008(-3)	4.12971(-3)	2.10659(-3)	-5.72342(-6)	-6.74884(-5)	
-0.1	1.91833(-2)	1.46008(-2)	1.08795(-2)	5.66365(-3)	-1.10472(-4)	-5.81224(-4)	
-0.0	5.11370(-2)	3.87330(-2)	2.89938(-2)	1.57032(-2)	9.25964(-4)	-1.14525(-3)	
0.0		3.87330(-2)	2.89938(-2)	1.57032(-2)	9.25964(-4)	-1.14525(-3)	-5.50332(-4)
0.1		1.02703(-2)	1.33068(-2)	1.06373(-2)	-8.53532(-4)	-2.76550(-3)	-2.16073(-3)
0.2		-1.08488(-4)	-6.36059(-4)	-2.25633(-3)	-5.38404(-3)	-5.10232(-3)	-3.79784(-3)
0.3		4.23777(-3)	6.77101(-3)	8.44883(-3)	3.96587(-3)	2.61057(-4)	-1.35778(-3)
0.4		5.23725(-3)	8.74033(-3)	1.21409(-2)	9.91429(-3)	5.52859(-3)	2.44648(-3)
0.5		2.11257(-3)	3.63484(-3)	5.36201(-3)	5.33434(-3)	3.65588(-3)	2.16911(-3)
0.6		1.12187(-3)	2.01318(-3)	3.18849(-3)	3.86885(-3)	3.21727(-3)	2.43612(-3)
0.7		1.56028(-3)	2.82082(-3)	4.56638(-3)	6.00302(-3)	5.46757(-3)	4.59909(-3)
0.8		1.74770(-3)	3.18327(-3)	5.24013(-3)	7.40473(-3)	7.19419(-3)	6.45756(-3)
0.9		1.44550(-3)	2.66107(-3)	4.48223(-3)	6.83103(-3)	7.08096(-3)	6.76792(-3)
1.0		6.15029(-4)	1.16210(-3)	2.06227(-3)	3.66526(-3)	4.28541(-3)	4.57234(-3)

Table 13. The intensity  $I(\tau, \mu, \phi)$  for the  $L = 60$  problem with  $\phi - \phi_0 = \pi/2$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	3.87835(-2)	3.56990(-2)	3.28307(-2)	2.78696(-2)	1.78899(-2)	1.29857(-2)	9.87579(-3)
-0.9	4.37020(-2)	4.02365(-2)	3.69821(-2)	3.12978(-2)	1.96139(-2)	1.37000(-2)	9.87579(-3)
-0.8	4.97010(-2)	4.58062(-2)	4.21083(-2)	3.55757(-2)	2.18288(-2)	1.46402(-2)	9.87579(-3)
-0.7	5.70375(-2)	5.26685(-2)	4.84650(-2)	4.09431(-2)	2.47026(-2)	1.58974(-2)	9.87579(-3)
-0.6	6.60347(-2)	6.11865(-2)	5.63857(-2)	4.77210(-2)	2.84777(-2)	1.76111(-2)	9.87579(-3)
-0.5	7.70990(-2)	7.17014(-2)	6.63111(-2)	5.63494(-2)	3.35224(-2)	2.00108(-2)	9.87579(-3)
-0.4	9.06982(-2)	8.48276(-2)	7.87979(-2)	6.74219(-2)	4.04383(-2)	2.35232(-2)	9.87579(-3)
-0.3	1.07091(-1)	1.00955(-1)	9.43698(-2)	8.16136(-2)	5.02168(-2)	2.90526(-2)	9.87579(-3)
-0.2	1.25177(-1)	1.19540(-1)	1.12803(-1)	9.91086(-2)	6.41015(-2)	3.85660(-2)	9.87579(-3)
-0.1	1.39358(-1)	1.36872(-1)	1.31339(-1)	1.18065(-1)	8.17278(-2)	5.53340(-2)	9.87579(-3)
-0.0	1.22471(-1)	1.39367(-1)	1.40873(-1)	1.33368(-1)	9.87228(-2)	7.43740(-2)	9.87579(-3)
0.0		1.39367(-1)	1.40873(-1)	1.33368(-1)	9.87228(-2)	7.43740(-2)	4.65115(-2)
0.1		4.75141(-2)	8.00363(-2)	1.12403(-1)	1.10523(-1)	8.89512(-2)	6.69373(-2)
0.2		2.41209(-2)	4.50045(-2)	7.52803(-2)	1.02761(-1)	9.45471(-2)	7.87061(-2)
0.3		1.57140(-2)	3.01708(-2)	5.36575(-2)	8.68136(-2)	8.92729(-2)	8.15145(-2)
0.4		1.15717(-2)	2.24467(-2)	4.09544(-2)	7.23791(-2)	8.00010(-2)	7.82271(-2)
0.5		9.09989(-3)	1.77384(-2)	3.27942(-2)	6.09820(-2)	7.06247(-2)	7.24731(-2)
0.6		7.46571(-3)	1.45807(-2)	2.71464(-2)	5.21129(-2)	6.23034(-2)	6.61903(-2)
0.7		6.32544(-3)	1.23494(-2)	2.30548(-2)	4.51409(-2)	5.51923(-2)	6.01609(-2)
0.8		5.50095(-3)	1.07197(-2)	2.00060(-2)	3.96051(-2)	4.91924(-2)	5.46633(-2)
0.9		4.88913(-3)	9.50108(-3)	1.76893(-2)	3.51783(-2)	4.41603(-2)	4.97789(-2)
1.0		4.42618(-3)	8.57341(-3)	1.59032(-2)	3.16195(-2)	3.99547(-2)	4.55071(-2)

Table 14. The Stokes parameter  $Q(\tau, \mu, \phi)$  for the  $L = 60$  problem with  $\phi - \phi_0 = \pi/2$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	-3.20870(-3)	-2.77800(-3)	-2.39491(-3)	-1.77047(-3)	-6.81639(-4)	-2.45234(-4)	
-0.9	-3.77758(-3)	-3.28853(-3)	-2.84919(-3)	-2.12471(-3)	-8.30775(-4)	-2.98217(-4)	
-0.8	-4.51730(-3)	-3.95908(-3)	-3.45121(-3)	-2.60191(-3)	-1.04084(-3)	-3.75266(-4)	
-0.7	-5.46982(-3)	-4.83166(-3)	-4.24203(-3)	-3.23951(-3)	-1.33506(-3)	-4.87640(-4)	
-0.6	-6.68414(-3)	-5.95746(-3)	-5.27214(-3)	-4.08422(-3)	-1.74435(-3)	-6.51607(-4)	
-0.5	-8.22113(-3)	-7.39906(-3)	-6.60423(-3)	-5.19608(-3)	-2.31347(-3)	-8.92932(-4)	
-0.4	-1.01530(-2)	-9.23518(-3)	-8.32018(-3)	-6.65902(-3)	-3.11598(-3)	-1.25801(-3)	
-0.3	-1.25509(-2)	-1.15587(-2)	-1.05256(-2)	-8.59223(-3)	-4.27825(-3)	-1.84333(-3)	
-0.2	-1.53625(-2)	-1.43981(-2)	-1.32896(-2)	-1.11039(-2)	-5.97986(-3)	-2.86515(-3)	
-0.1	-1.79192(-2)	-1.73738(-2)	-1.63732(-2)	-1.40856(-2)	-8.28388(-3)	-4.72407(-3)	
-0.0	-1.60856(-2)	-1.84457(-2)	-1.84471(-2)	-1.68681(-2)	-1.08257(-2)	-7.09728(-3)	
0.0		-1.84457(-2)	-1.84471(-2)	-1.68681(-2)	-1.08257(-2)	-7.09728(-3)	-3.54850(-3)
0.1		-6.29397(-3)	-1.05645(-2)	-1.45718(-2)	-1.29303(-2)	-9.28025(-3)	-6.08767(-3)
0.2		-3.17352(-3)	-5.89729(-3)	-9.72077(-3)	-1.23160(-2)	-1.04082(-2)	-7.83275(-3)
0.3		-2.05311(-3)	-3.92272(-3)	-6.87437(-3)	-1.04262(-2)	-1.00157(-2)	-8.46729(-3)
0.4		-1.50434(-3)	-2.89975(-3)	-5.20506(-3)	-8.64205(-3)	-8.99561(-3)	-8.25282(-3)
0.5		-1.17915(-3)	-2.27856(-3)	-4.13284(-3)	-7.20988(-3)	-7.89207(-3)	-7.65330(-3)
0.6		-9.67953(-4)	-1.86782(-3)	-3.39753(-3)	-6.09056(-3)	-6.88708(-3)	-6.93963(-3)
0.7		-8.28882(-4)	-1.58713(-3)	-2.87944(-3)	-5.22188(-3)	-6.02634(-3)	-6.23516(-3)
0.8		-7.28716(-4)	-1.39265(-3)	-2.51113(-3)	-4.55386(-3)	-5.31367(-3)	-5.59357(-3)
0.9		-6.61605(-4)	-1.25696(-3)	-2.24872(-3)	-4.04599(-3)	-4.73856(-3)	-5.03708(-3)
1.0		-6.15029(-4)	-1.16210(-3)	-2.08227(-3)	-3.66526(-3)	-4.28541(-3)	-4.57234(-3)

Table 15. The Stokes parameter  $U(\tau, \mu, \phi)$  for the  $L = 60$  problem with  $\phi - \phi_0 = \pi/2$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	8.29638(-4)	7.48355(-4)	6.72506(-4)	5.39631(-4)	2.57586(-4)	1.04932(-4)	
-0.8	1.30678(-3)	1.18820(-3)	1.07571(-3)	8.75007(-4)	4.32423(-4)	1.80711(-4)	
-0.7	1.78782(-3)	1.63895(-3)	1.49509(-3)	1.23320(-3)	6.31693(-4)	2.71298(-4)	
-0.6	2.31267(-3)	2.13822(-3)	1.96589(-3)	1.64505(-3)	8.75127(-4)	3.87370(-4)	
-0.5	2.90416(-3)	2.70932(-3)	2.51161(-3)	2.13393(-3)	1.18336(-3)	5.42528(-4)	
-0.4	3.57762(-3)	3.37037(-3)	3.15280(-3)	2.72409(-3)	1.58697(-3)	7.60792(-4)	
-0.3	4.32598(-3)	4.12240(-3)	3.89650(-3)	3.43471(-3)	2.13316(-3)	1.09202(-3)	
-0.2	5.06242(-3)	4.90243(-3)	4.69628(-3)	4.24269(-3)	2.86960(-3)	1.64032(-3)	
-0.1	5.50318(-3)	5.49481(-3)	5.37217(-3)	5.00440(-3)	3.72736(-3)	2.55853(-3)	
-0.0	4.66795(-3)	5.35153(-3)	5.52061(-3)	5.44846(-3)	4.41800(-3)	3.47199(-3)	
0.0		5.35153(-3)	5.52061(-3)	5.44846(-3)	4.41800(-3)	3.47199(-3)	2.21245(-3)
0.1		1.80180(-3)	3.06428(-3)	4.42284(-3)	4.74188(-3)	4.01686(-3)	3.13690(-3)
0.2		9.11890(-4)	1.70854(-3)	2.91142(-3)	4.24578(-3)	4.08982(-3)	3.53390(-3)
0.3		5.88508(-4)	1.13045(-3)	2.03500(-3)	3.46871(-3)	3.70574(-3)	3.49360(-3)
0.4		4.26111(-4)	8.23339(-4)	1.51089(-3)	2.78010(-3)	3.17213(-3)	3.18647(-3)
0.5		3.26151(-4)	6.29926(-4)	1.16243(-3)	2.22271(-3)	2.64250(-3)	2.77407(-3)
0.6		2.55979(-4)	4.92393(-4)	9.07099(-4)	1.76598(-3)	2.15545(-3)	2.33398(-3)
0.7		2.01147(-4)	3.84557(-4)	7.04018(-4)	1.37701(-3)	1.70877(-3)	1.89169(-3)
0.8		1.52958(-4)	2.90415(-4)	5.26968(-4)	1.02575(-3)	1.28478(-3)	1.44462(-3)
0.9		1.02858(-4)	1.93962(-4)	3.48433(-4)	6.70493(-4)	8.42818(-4)	9.57462(-4)
1.0		0.0	0.0	0.0	0.0	0.0	0.0

Table 16. The Stokes parameter  $V(\tau, \mu, \phi)$  for the  $L = 60$  problem with  $\phi - \phi_0 = \pi/2$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	0.0	0.0	0.0	0.0	0.0	0.0	
-0.9	-2.32984(-5)	-2.42828(-5)	-2.44542(-5)	-2.33025(-5)	-1.51629(-5)	-7.37680(-6)	
-0.8	-3.31595(-5)	-3.49066(-5)	-3.53836(-5)	-3.40042(-5)	-2.24874(-5)	-1.11095(-5)	
-0.7	-4.06056(-5)	-4.32698(-5)	-4.41949(-5)	-4.28703(-5)	-2.88592(-5)	-1.45241(-5)	
-0.6	-4.83302(-5)	-5.01262(-5)	-5.16468(-5)	-5.05993(-5)	-3.47077(-5)	-1.78692(-5)	
-0.5	-5.01681(-5)	-5.53272(-5)	-5.75725(-5)	-5.69592(-5)	-3.97091(-5)	-2.09874(-5)	
-0.4	-5.20304(-5)	-5.86689(-5)	-6.16289(-5)	-6.13179(-5)	-4.27559(-5)	-2.30599(-5)	
-0.3	-5.38703(-5)	-6.12927(-5)	-6.42718(-5)	-6.31349(-5)	-4.20980(-5)	-2.23295(-5)	
-0.2	-6.16942(-5)	-6.70538(-5)	-6.77784(-5)	-6.28124(-5)	-3.65581(-5)	-1.67751(-5)	
-0.1	-8.07117(-5)	-8.12180(-5)	-7.66798(-5)	-6.33756(-5)	-2.69250(-5)	-5.55081(-6)	
-0.0	-8.00327(-5)	-9.71645(-5)	-9.12124(-5)	-6.93495(-5)	-1.61811(-5)	7.98160(-6)	
0.0		-9.71645(-5)	-9.12124(-5)	-6.93495(-5)	-1.61811(-5)	7.98160(-6)	3.69463(-5)
0.1		-3.11162(-5)	-4.80208(-5)	-5.28586(-5)	-7.42165(-6)	2.14860(-5)	4.27527(-5)
0.2		-1.30178(-5)	-1.93521(-5)	-2.00899(-5)	9.67449(-6)	3.45182(-5)	5.45054(-5)
0.3		-6.46369(-6)	-7.85741(-6)	-3.27456(-6)	2.53938(-5)	4.65670(-5)	6.34534(-5)
0.4		-3.24284(-6)	-2.80436(-6)	3.45680(-6)	3.19415(-5)	5.17765(-5)	6.63396(-5)
0.5		-1.54826(-6)	-4.57717(-7)	5.72865(-6)	3.18042(-5)	5.01280(-5)	6.32349(-5)
0.6		-7.09485(-7)	5.00626(-7)	5.94728(-6)	2.83284(-5)	4.44417(-5)	5.61957(-5)
0.7		-3.64617(-7)	6.79123(-7)	5.07957(-6)	2.82876(-5)	3.67822(-5)	4.69668(-5)
0.8		-2.62978(-7)	4.80107(-7)	3.69926(-6)	1.74880(-5)	2.80674(-5)	3.63666(-5)
0.9		-2.82014(-7)	1.82378(-7)	2.15763(-6)	1.11165(-5)	1.82702(-5)	2.41140(-5)
1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 17. The intensity  $I(\tau, \mu, \phi)$  for the  $L = 60$  problem with  $\phi - \phi_0 = \pi$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	3.87835(-2)	3.56999(-2)	3.28307(-2)	2.78696(-2)	1.78899(-2)	1.29857(-2)	9.87579(-3)
-0.9	4.14090(-2)	3.84698(-2)	3.56850(-2)	3.07293(-2)	1.99618(-2)	1.40332(-2)	9.87579(-3)
-0.8	5.19431(-2)	4.80474(-2)	4.43601(-2)	3.78020(-2)	2.35197(-2)	1.55656(-2)	9.87579(-3)
-0.7	6.81338(-2)	6.25938(-2)	5.73776(-2)	4.81726(-2)	2.84658(-2)	1.76451(-2)	9.87579(-3)
-0.6	9.59381(-2)	8.68551(-2)	7.85074(-2)	6.42007(-2)	3.53570(-2)	2.04292(-2)	9.87579(-3)
-0.5	1.32649(-1)	1.18498(-1)	1.05700(-1)	8.42897(-2)	4.35651(-2)	2.37473(-2)	9.87579(-3)
-0.4	1.37799(-1)	1.25652(-1)	1.13865(-1)	9.29855(-2)	4.99203(-2)	2.70733(-2)	9.87579(-3)
-0.3	1.43386(-1)	1.33494(-1)	1.22970(-1)	1.03137(-1)	5.85573(-2)	3.21972(-2)	9.87579(-3)
-0.2	1.91758(-1)	1.76331(-1)	1.60731(-1)	1.32814(-1)	7.51588(-2)	4.22407(-2)	9.87579(-3)
-0.1	1.93222(-1)	1.86898(-1)	1.75965(-1)	1.51814(-1)	9.35798(-2)	5.89230(-2)	9.87579(-3)
-0.0	2.09480(-1)	2.28426(-1)	2.22113(-1)	1.95484(-1)	1.19942(-1)	8.04142(-2)	9.87579(-3)
0.0		2.28426(-1)	2.22113(-1)	1.95484(-1)	1.19942(-1)	8.04142(-2)	4.61692(-2)
0.1		1.02867(-1)	1.63341(-1)	2.08037(-1)	1.52486(-1)	1.02300(-1)	6.77372(-2)
0.2		4.65090(-2)	8.30090(-2)	1.27890(-1)	1.39683(-1)	1.09771(-1)	8.06147(-2)
0.3		2.50473(-2)	4.32443(-2)	7.32295(-2)	1.02614(-1)	9.50162(-2)	7.94389(-2)
0.4		1.35445(-2)	2.58922(-2)	4.57408(-2)	7.35893(-2)	7.60521(-2)	7.02288(-2)
0.5		8.75601(-3)	1.69347(-2)	3.07141(-2)	5.39401(-2)	6.00223(-2)	5.95405(-2)
0.6		5.95515(-3)	1.16405(-2)	2.15834(-2)	4.05682(-2)	4.77275(-2)	4.99994(-2)
0.7		4.29903(-3)	8.46290(-3)	1.59461(-2)	3.15941(-2)	3.88241(-2)	4.24544(-2)
0.8		3.36660(-3)	6.64383(-3)	1.26294(-2)	2.59202(-2)	3.28890(-2)	3.71594(-2)
0.9		2.98650(-3)	5.87818(-3)	1.11704(-2)	2.32477(-2)	3.00379(-2)	3.46583(-2)
1.0		4.42618(-3)	8.57341(-3)	1.59032(-2)	3.16195(-2)	3.99547(-2)	4.55071(-2)

Table 18. The Stokes parameter  $Q(\tau, \mu, \phi)$  for the  $L = 60$  problem with  $\phi - \phi_0 = \pi$ .

$\mu$	$\tau/\tau_0 = 0$	$\tau/\tau_0 = 0.05$	$\tau/\tau_0 = 0.1$	$\tau/\tau_0 = 0.2$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1$
-1.0	3.20870(-3)	2.77800(-3)	2.39491(-3)	1.77047(-3)	6.81639(-4)	2.45234(-4)	
-0.9	3.46968(-3)	2.95677(-3)	2.50939(-3)	1.79792(-3)	6.35777(-4)	2.20227(-4)	
-0.8	3.99931(-3)	3.33701(-3)	2.76883(-3)	1.88921(-3)	5.68476(-4)	1.76027(-4)	
-0.7	3.26610(-3)	2.66280(-3)	2.14700(-3)	1.36440(-3)	2.97546(-4)	6.65204(-5)	
-0.6	1.00310(-3)	7.68545(-4)	5.59430(-4)	2.45337(-4)	-9.12834(-5)	-6.31666(-5)	
-0.5	3.46629(-3)	2.60519(-3)	1.92309(-3)	9.74068(-4)	-4.60994(-5)	-8.50925(-5)	
-0.4	1.16486(-2)	8.87269(-3)	6.69423(-3)	3.67097(-3)	2.85390(-4)	-8.24999(-5)	
-0.3	7.36016(-3)	5.22778(-3)	3.56049(-3)	1.32583(-3)	-6.61654(-4)	-4.22384(-4)	
-0.2	-3.57940(-3)	-3.72232(-3)	-3.74909(-3)	-3.50991(-3)	-1.98085(-3)	-7.84136(-4)	
-0.1	1.20763(-2)	8.35959(-3)	5.66201(-3)	2.29798(-3)	-5.45270(-4)	-3.29379(-4)	
-0.0	3.57307(-2)	2.75101(-2)	2.11071(-2)	1.22667(-2)	1.96369(-3)	3.54266(-4)	
0.0		2.75101(-2)	2.11071(-2)	1.22667(-2)	1.96369(-3)	3.54266(-4)	6.75218(-4)
0.1		5.41761(-3)	7.69642(-3)	7.64814(-3)	2.10940(-3)	4.61906(-4)	4.95429(-4)
0.2		5.38656(-4)	6.28211(-4)	8.87345(-4)	5.28772(-4)	3.52170(-4)	5.83175(-4)
0.3		7.42369(-4)	1.27087(-3)	1.85467(-3)	1.89771(-3)	1.56239(-3)	1.43200(-3)
0.4		1.19475(-3)	2.06518(-3)	3.15055(-3)	3.66879(-3)	3.15446(-3)	2.67546(-3)
0.5		1.08945(-3)	1.95384(-3)	3.12732(-3)	4.11152(-3)	3.83633(-3)	3.41332(-3)
0.6		8.52033(-4)	1.56294(-3)	2.60024(-3)	3.78830(-3)	3.81730(-3)	3.81063(-3)
0.7		6.48590(-4)	1.21031(-3)	2.07843(-3)	3.30904(-3)	3.56542(-3)	3.56443(-3)
0.8		5.04398(-4)	9.54912(-4)	1.66656(-3)	2.90763(-3)	3.32015(-3)	3.48093(-3)
0.9		4.22254(-4)	8.08091(-4)	1.46002(-3)	2.68726(-3)	3.21300(-3)	3.49547(-3)
1.0		6.15029(-4)	1.16210(-3)	2.06227(-3)	3.66526(-3)	4.28541(-3)	4.57234(-3)

reported in Ref. 1. We have also found excellent agreement with reference results made available by de Haan<sup>24</sup> and discussed in Ref. 1.

## 8. CONCLUDING REMARKS

In this work, the  $F_N$  method has been used to solve the complete and general polarization problem for a plane-parallel layer. In addition to allowing for a scattering matrix more general than that required by the Mie scattering theory,<sup>25</sup> we have allowed the incident beam to be arbitrarily polarized.

Our program runs in core on a small IBM mainframe computer (4361), and in low order, say  $N \leq 19$ , and modest values of  $L$ , say  $L = 13$ , will produce results for all  $m \leq L$  accurate to three or four figures in something like 15 min. By way of comparing the generalized spherical harmonics (GSH) solution reported in Ref. 1 with our  $F_N$  solution, we can make several general comments. The  $F_N$  method is conceptually more complicated and somewhat more difficult to implement numerically than is the GSH method. On the other hand, we have found that the  $F_N$  method, in general, yields more accurate results and requires less computer time than does the GSH method. Also as pointed out in Ref. 1, there is in the GSH solution a complication concerning nearly repeated eigenvalues in the interval  $[0, 1]$  in either of the limits  $\omega \rightarrow 0$  or  $m \rightarrow L$  that must be resolved in a computational manner. It is noteworthy that this type of complication is not present in our  $F_N$  solution.

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