J. Quant. Spectrosc. Radiat. Transfer Vol. 41, No. 6, pp. 467–481, 1989
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0022-4073/89 $3.00 + 0.00
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THE SEARCHLIGHT PROBLEM IN RADIATIVE TRANSFER

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(*Received* 20 *May* 1988; *received for publication* 27 *September* 1988)

Abstract—Integral transformation techniques and the F_N method are used to develop, for the case of a finite plane-parallel layer, general results for the mean intensity and the net flux relevant to the classic searchlight problem in radiative transfer. The special case of a normally incident beam is then considered, and the resulting expressions for the mean intensity and the net flux are reduced to one-dimensional inversion integrals that are evaluated numerically to yield accurate numerical results for several test cases.

1. INTRODUCTION

In two previous papers[1,2] concerning radiation transport through a finite plane-parallel layer with nonuniform surface illumination, we used two-dimensional Fourier transformation techniques and the F_N method[3] to establish, for the classic searchlight problem of Chandrasekhar,[4] some analytical and computational results that are valid on the two boundaries of the layer. Here we continue our work on the searchlight problem and develop tractable expressions that yield the mean intensity and the net flux at any point within the layer as well as on the two boundaries. As our previous analysis of the searchlight problem was reported in detail in Refs. 1 and 2, we assume those works to be available and thus give here only a sketch of the material introductory to our current development.

We consider the equation of transfer

$$\mu \frac{\partial}{\partial z} I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) + \boldsymbol{\omega} \cdot \frac{\partial}{\partial \boldsymbol{\rho}} I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) + I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) = \frac{\varpi}{4\pi} \iint I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}') \, d\boldsymbol{\Omega}' \tag{1}$$

and the boundary conditions

$$I[0, \boldsymbol{\rho}, \boldsymbol{\Omega}(\mu, \phi)] = \frac{1}{2\pi\rho} \delta(\rho) \delta(\mu - \mu_0) \delta(\phi - \phi_0) \tag{2a}$$

and

$$I[z_0, \boldsymbol{\rho}, \boldsymbol{\Omega}(-\mu, \phi)] = 0 \tag{2b}$$

for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$. We follow the notation of Rybicki[5] and note that $z \in [0, z_0]$ and $\boldsymbol{\rho}$, which lies in the x–y plane, locate in optical units the position in the homogeneous medium, and $\boldsymbol{\Omega} = \boldsymbol{\Omega}(\mu, \phi)$, with $\mu = \cos\theta$, is a unit vector that defines the direction of propagation (see Fig. 1). In addition, $\boldsymbol{\omega}$ is the projection of $\boldsymbol{\Omega}$ in the x–y plane, $\boldsymbol{\Omega}_0 = \boldsymbol{\Omega}(\mu_0, \phi_0)$ defines the direction of the incident beam and $\varpi < 1$ is the albedo for single scattering.

In this work we seek the mean intensity

$$J(z, \boldsymbol{\rho}) = \frac{1}{4\pi} \iint I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) \, d\boldsymbol{\Omega} \tag{3a}$$

and the net flux

$$F(z, \boldsymbol{\rho}) = \frac{1}{\pi} \iint I(z, \boldsymbol{\rho}, \boldsymbol{\Omega}) \mu \, d\boldsymbol{\Omega} \tag{3b}$$

for all $\boldsymbol{\rho}$ and for all $z \in [0, z_0]$.

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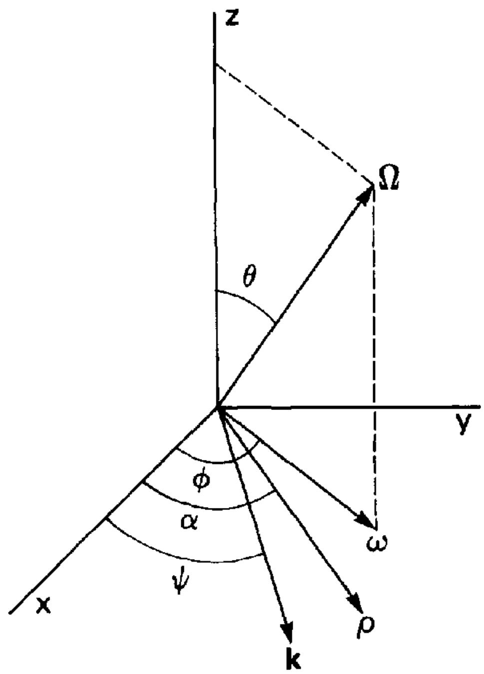

Fig. 1. The geometry for $\mathbf{\Omega}$, ω, ρ and $\mathbf{k}$.

As shown in Ref. 1, the Fourier transform

$$\Psi(z, \mu, \phi) = \iint I(z, \boldsymbol{\rho}, \mathbf{\Omega}) \exp\{i\mathbf{k}\cdot\boldsymbol{\rho}\}\, \mathrm{d}\boldsymbol{\rho} \tag{4}$$

satisfies

$$\mu \frac{\partial}{\partial z} \Psi(z, \mu, \phi) + u(\mu, \phi)\Psi(z, \mu, \phi) = \frac{\varpi}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \Psi(z, \mu', \phi')\, \mathrm{d}\phi'\, \mathrm{d}\mu' \tag{5}$$

and the boundary conditions

$$\Psi(0, \mu, \phi) = \delta(\mu - \mu_0)\delta(\phi - \phi_0) \tag{6a}$$

and

$$\Psi(z_0, -\mu, \phi) = 0 \tag{6b}$$

for $\mu \in [0,1]$ and $\phi \in [0, 2\pi]$. Here we use $k = |\mathbf{k}|$ and

$$u(\mu, \phi) = 1 - ik(1 - \mu^2)^{1/2} \cos(\phi - \psi). \tag{7}$$

We can also take Fourier transforms of Eqs. (3) and use Eq. (4) to obtain

$$\iint J(z, \boldsymbol{\rho}) \exp\{i\mathbf{k}\cdot\boldsymbol{\rho}\}\, \mathrm{d}\boldsymbol{\rho} = \frac{1}{4\pi} \Psi(z) \tag{8a}$$

and

$$\iint F(z, \boldsymbol{\rho}) \exp\{i\mathbf{k}\cdot\boldsymbol{\rho}\}\, \mathrm{d}\boldsymbol{\rho} = \frac{1}{\pi} \Upsilon(z) \tag{8b}$$

where

$$\Psi(z) = \int_{-1}^{1} \int_{0}^{2\pi} \Psi(z, \mu, \phi)\, \mathrm{d}\phi\, \mathrm{d}\mu \tag{9a}$$

and

$$\Upsilon(z) = \int_{-1}^{1} \int_{0}^{2\pi} \mu\Psi(z, \mu, \phi)\, \mathrm{d}\phi\, \mathrm{d}\mu. \tag{9b}$$

It follows that once $\Psi(z)$ and $\Upsilon(z)$ are available we can find the desired mean intensity and the net flux from the inversion integrals

$$J(z, \boldsymbol{\rho}) = \frac{1}{(2\pi)^2} \iint \frac{1}{4\pi} \Psi(z) \exp\{-i\mathbf{k}\cdot\boldsymbol{\rho}\}\, \mathrm{d}\mathbf{k} \tag{10a}$$

and

$$F(z, \boldsymbol{\rho}) = \frac{1}{(2\pi)^2} \iint \frac{1}{\pi} \Upsilon(z) \exp\{-i\mathbf{k}\cdot\boldsymbol{\rho}\}\, \mathrm{d}\mathbf{k} \tag{10b}$$

or

$$J(z, \rho, \alpha) = \frac{1}{16\pi^3} \int_0^\infty \int_0^{2\pi} k\Psi(z) \exp\{-ik\rho \cos(\alpha - \psi)\}\, \mathrm{d}\psi\, \mathrm{d}k \tag{11a}$$

and

$$F(z, \rho, \alpha) = \frac{1}{4\pi^3} \int_0^\infty \int_0^{2\pi} k\Upsilon(z) \exp\{-ik\rho \cos(\alpha - \psi)\}\, \mathrm{d}\psi\, \mathrm{d}k. \tag{11b}$$

We now proceed to obtain $\Psi(z)$ and $\Upsilon(z)$ so that we can use Eqs. (11) to find the mean intensity and the net flux.

2. THE PSEUDO PROBLEM

Rather than try to find $\Psi(z)$ and $\Upsilon(z)$ directly from the transport problem (in transform space) defined by Eqs. (5) and (6), we proceed, as we did in Refs. 1 and 2, to base our analysis on the pseudo problem introduced by Williams.[6] For the considered searchlight problem this pseudo problem is defined by

$$\mu(1 + k^2\mu^2)^{1/2} \frac{\partial}{\partial z} \Phi(z, \mu) + (1 + k^2\mu^2)\Phi(z, \mu) = \frac{\varpi}{2} \int_{-1}^{1} \Phi(z, \mu')\, \mathrm{d}\mu' + \tfrac{1}{2}F(z) \tag{12}$$

with

$$\Phi(0, \mu) = 0 \tag{13a}$$

and

$$\Phi(z_0, -\mu) = 0 \tag{13b}$$

for $\mu \in [0, 1]$. Here

$$F(z) = \exp\{-z/U_0\} \tag{14}$$

where $U_0 = \mu_0/u(\mu_0, \phi_0)$.

In order to express the mean intensity and the net flux in terms of the pseudo problem we first solve Eq. (5) to obtain, after we note Eqs. (6),

$$\Psi(z, \mu, \phi) = \delta(\mu - \mu_0)\delta(\phi - \phi_0) \exp\{-z/U_0\} + \frac{\varpi}{4\pi\mu} \int_0^z \Psi(z') \exp\{-(z - z')/U\}\, \mathrm{d}z' \tag{15a}$$

and

$$\Psi(z, -\mu, \phi) = \frac{\varpi}{4\pi\mu} \int_z^{z_0} \Psi(z') \exp\{-(z' - z)/U\}\, \mathrm{d}z', \tag{15b}$$

for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$, where $U = \mu/u(\mu, \phi)$. We can now integrate Eqs. (15) over μ from 0 to 1 and over ϕ from 0 to 2π and add the two resulting equations to obtain[5,6]

$$\Psi(z) = F(z) + \frac{\varpi}{2} \int_0^{z_0} \Psi(z')K(|z' - z|)\, \mathrm{d}z' \tag{16}$$

where

$$K(\xi)=\int_0^1 \exp\{-\xi/\mu\}J_0\left[\frac{k\xi}{\mu}(1-\mu^2)^{1/2}\right]\frac{d\mu}{\mu}. \tag{17}$$

Note that we use $J_0(x)$ to denote the zeroth-order Bessel function of the first kind.[7]

We now follow Rybicki[5] and integrate the identity (see Ref. 7, p. 1027, formula 29.3.92)

$$\int_1^\infty \exp\{-st\}J_0[\eta(t^2-1)^{1/2}]\,dt=\frac{\exp\{-(s^2+\eta^2)^{1/2}\}}{(s^2+\eta^2)^{1/2}} \tag{18}$$

over s from $s=\xi$ to $s=\infty$ to obtain (after some variable changes)

$$\int_0^1 \exp\{-\xi/\mu\}J_0\left[\frac{\eta}{\mu}(1-\mu^2)^{1/2}\right]\frac{d\mu}{\mu}=\int_0^1\frac{\exp\{-\xi(1+\eta^2\mu^2/\xi^2)^{1/2}/\mu\}}{(1+\eta^2\mu^2/\xi^2)^{1/2}}\frac{d\mu}{\mu}. \tag{19}$$

In a similar manner we can multiply Eq. (18) by s and integrate over s from $s=\xi$ to $s=\infty$ to obtain

$$\int_0^1\left(1+\frac{\xi}{\mu}\right)\exp\{-\xi/\mu\}J_0\left[\frac{\eta}{\mu}(1-\mu^2)^{1/2}\right]d\mu=\exp\{-(\xi^2+\eta^2)^{1/2}\}. \tag{20}$$

We therefore can use Eqs. (19) and (20) to find

$$\int_0^1 \exp\{-\xi/\mu\}J_0\left[\frac{\eta}{\mu}(1-\mu^2)^{1/2}\right]d\mu$$
$$=\exp\{-(\xi^2+\eta^2)^{1/2}\}-\xi\int_0^1\frac{\exp\{-\xi(1+\eta^2\mu^2/\xi^2)^{1/2}/\mu\}}{(1+\eta^2\mu^2/\xi^2)^{1/2}}\frac{d\mu}{\mu} \tag{21}$$

or, after an integration by parts,

$$\int_0^1 \exp\{-\xi/\mu\}J_0\left[\frac{\eta}{\mu}(1-\mu^2)^{1/2}\right]d\mu=\int_0^1\exp\{-\xi(1+\eta^2\mu^2/\xi^2)^{1/2}/\mu\}\,d\mu. \tag{22}$$

If we note Eq. (19) we can rewrite Eq. (17) as

$$K(\xi)=\int_0^1\frac{\exp\{-\xi(1+k^2\mu^2)^{1/2}/\mu\}}{(1+k^2\mu^2)^{1/2}}\frac{d\mu}{\mu}. \tag{23}$$

Now solving Eq. (12) and using Eqs. (13), we can write

$$\Phi(z,\mu)=\frac{1}{2\mu}(1+k^2\mu^2)^{-1/2}\int_0^z S(z')\exp\{-(z-z')(1+k^2\mu^2)^{1/2}/\mu\}\,dz' \tag{24a}$$

and

$$\Phi(z,-\mu)=\frac{1}{2\mu}(1+k^2\mu^2)^{-1/2}\int_z^{z_0} S(z')\exp\{-(z'-z)(1+k^2\mu^2)^{1/2}/\mu\}\,dz' \tag{24b}$$

for $\mu\in[0,1]$; here

$$S(z)=\varpi\Phi(z)+F(z) \tag{25}$$

with

$$\Phi(z)=\int_{-1}^1\Phi(z,\mu)\,d\mu. \tag{26}$$

Now since we can integrate Eqs. (24) over μ from 0 to 1 and add the resulting equations to obtain, after noting Eq. (23),

$$S(z)=F(z)+\frac{\varpi}{2}\int_0^{z_0}S(z')K(|z'-z|)\,dz', \tag{27}$$

we conclude, after viewing Eq. (16), that as noted by Williams[6]

$$\Psi(z)=F(z)+\varpi\Phi(z). \tag{28}$$

We can multiply Eqs. (15) by μ, integrate over μ from 0 to 1 and over ϕ from 0 to 2π and subtract the two resulting equations to obtain, after we note Eq. (9b),

$$\Upsilon(z) = \mu_0 F(z) + \frac{\varpi}{2}\int_0^{z_0} \Psi(z')\operatorname{sgn}(z - z')L(|z' - z|)\,\mathrm{d}z' \tag{29}$$

where

$$L(\xi) = \int_0^1 \exp\{-\xi/\mu\}J_0\left[\frac{k\xi}{\mu}(1-\mu^2)^{1/2}\right]\mathrm{d}\mu \tag{30}$$

or, in view of Eq. (22),

$$L(\xi) = \int_0^1 \exp\{-\xi(1 + k^2\mu^2)^{1/2}/\mu\}\,\mathrm{d}\mu. \tag{31}$$

If we let

$$\Xi(z) = \int_1^1 \mu(1 + k^2\mu^2)^{1/2}\Phi(z,\mu)\,\mathrm{d}\mu \tag{32}$$

we can multiply Eqs. (24) by $\mu(1 + k^2\mu^2)^{1/2}$, integrate over μ from 0 to 1 and subtract the two resulting equations to find

$$\Xi(z) = \frac{1}{2}\int_0^{z_0} S(z')\operatorname{sgn}(z - z')L(|z' - z|)\,\mathrm{d}z'. \tag{33}$$

Upon comparing Eqs. (25), (28), (29) and (33), we conclude that

$$\Upsilon(z) = \mu_0 F(z) + \varpi\Xi(z). \tag{34}$$

As the desired results for the mean intensity and the net flux have now been expressed in terms of the pseudo problem, viz.,

$$J(z,\rho,\alpha) = \frac{1}{16\pi^3}\int_0^\infty\int_0^{2\pi} k\Psi(z)\exp\{-ik\rho\cos(\alpha - \psi)\}\,\mathrm{d}\psi\,\mathrm{d}k \tag{35a}$$

and

$$F(z,\rho,\alpha) = \frac{1}{4\pi^3}\int_0^\infty\int_0^{2\pi} k\Upsilon(z)\exp\{-ik\rho\cos(\alpha - \psi)\}\,\mathrm{d}\psi\,\mathrm{d}k \tag{35b}$$

where

$$\Psi(z) = \exp\{-z/U_0\} + \varpi\Phi(z) \tag{36a}$$

and

$$\Upsilon(z) = \mu_0\exp\{-z/U_0\} + \varpi\Xi(z), \tag{36b}$$

we proceed to develop our solution to the pseudo problem.

3. ANALYSIS OF THE PSEUDO PROBLEM

In this section we start with the pseudo problem defined by

$$\mu(1 + k^2\mu^2)^{1/2}\frac{\partial}{\partial z}\Phi(z,\mu) + (1 + k^2\mu^2)\Phi(z,\mu) = \frac{\varpi}{2}\int_{-1}^1 \Phi(z,\mu')\,\mathrm{d}\mu' + \tfrac{1}{2}F(z) \tag{37}$$

with

$$\Phi(0,\mu) = 0 \tag{38a}$$

and

$$\Phi(z_0, -\mu) = 0 \tag{38b}$$

for $\mu \in [0, 1]$. As we wish ultimately to use the F_N method[3] to solve this transport problem we now generalize the analysis reported in Refs. 1 and 8 in order to formulate here systems of singular integral equations and integral constraints that can be used to deduce $\Phi(0, -\mu)$ and $\Phi(z_0, \mu)$ for $\mu \in [0, 1]$ as well as $\Phi(z, \mu)$ for all $z \in (0, z_0)$ and all $\mu \in [-1, 1]$.

If we change μ to $-\mu$ in Eq. (37), multiply the resulting equation by $\exp\{-z/s\}$ and integrate over z from $z = a$ to $z = b$ we find, after an integration by parts,

$$W^{-1}(\mu, s) \int_a^b \exp\{-z/s\}\Phi(z, -\mu)\,\mathrm{d}z = \mu(1 + k^2\mu^2)^{1/2} \times [\Phi(b, -\mu)\exp\{-b/s\} - \Phi(a, -\mu)\exp\{-a/s\}] + \tfrac{1}{2}[\varpi\Phi^*(s) + F^*(s)] \quad (39)$$

where

$$W(\mu, s) = s[s(1 + k^2\mu^2) - \mu(1 + k^2\mu^2)^{1/2}]^{-1}, \quad (40)$$

$$\Phi^*(s) = \int_a^b \exp\{-z/s\}\Phi(z)\,\mathrm{d}z \quad (41)$$

and

$$F^*(s) = \int_a^b \exp\{-z/s\}F(z)\,\mathrm{d}z. \quad (42)$$

In order to keep our development general we do not, at this point, specify a and b more precisely than $a, b \in [0, z_0]$ and $a < b$. We now multiply Eq. (39) by $\varpi W(\mu, s)$ and integrate the resulting equation over μ from -1 to 1 to obtain

$$\Lambda(s)[\varpi\Phi^*(s) + F^*(s)] = F^*(s) + \varpi s \int_{-1}^{1} \frac{\mu[\Phi(a, -\mu)\exp\{-a/s\} - \Phi(b, -\mu)\exp\{-b/s\}]}{\mu - s(1 + k^2\mu^2)^{1/2}}\,\mathrm{d}\mu \quad (43)$$

where

$$\Lambda(s) = 1 + \frac{\varpi s}{2}\int_{-1}^{1} \frac{(1 + k^2\mu^2)^{-1/2}}{\mu - s(1 + k^2\mu^2)^{1/2}}\,\mathrm{d}\mu \quad (44)$$

or[1]

$$\Lambda(s) = 1 + \frac{\varpi s}{2}\int_{-\gamma}^{\gamma} \phi(\tau)\frac{\mathrm{d}\tau}{\tau - s} \quad (45)$$

where

$$\phi(\tau) = (1 - k^2\tau^2)^{-1/2} \quad (46)$$

and

$$\gamma = (1 + k^2)^{-1/2}. \quad (47)$$

Changing variables in the integral term in Eq. (43), we find the convenient form

$$\Lambda(s)[\varpi\Phi^*(s) + F^*(s)] = F^*(s) + \varpi s\Delta(s) \quad (48)$$

where

$$\Delta(s) = \int_0^{\gamma} \tau\phi^3(\tau)(\Phi[a, -p(\tau)]\exp\{-a/s\} - \Phi[b, -p(\tau)]\exp\{-b/s\})\frac{\mathrm{d}\tau}{\tau - s} + \int_0^{\gamma} \tau\phi^3(\tau)(\Phi[a, p(\tau)]\exp\{-a/s\} - \Phi[b, p(\tau)]\exp\{-b/s\})\frac{\mathrm{d}\tau}{\tau + s} \quad (49)$$

and

$$p(\tau) = \tau(1 - k^2\tau^2)^{-1/2}. \quad (50)$$

Rather than consider Eq. (48) for all s, we first multiply Eq. (48) by $\exp\{a/s\}$ and consider the resulting equation with $a = z$, $b = z_0$ and $\mathscr{R}s > 0$. Next we multiply Eq. (48), after changing s to $-s$, by $\exp\{-b/s\}$ and consider the resulting equation with $a = 0$, $b = z$ and $\mathscr{R}s > 0$. In this way we find, after we note Eqs. (38), for $\mathscr{R}s > 0$

$$\Lambda(s)I(s) = U(s) + \varpi s X(s) \tag{51a}$$

and

$$\Lambda(s)J(s) = V(s) + \varpi s Y(s) \tag{51b}$$

where

$$I(s) = \int_z^{z_0} [\varpi\Phi(\tau) + F(\tau)]\exp\{-(\tau - z)/s\}\, \mathrm{d}\tau, \tag{52a}$$

$$J(s) = \int_0^z [\varpi\Phi(\tau) + F(\tau)]\exp\{-(z - \tau)/s\}\, \mathrm{d}\tau, \tag{52b}$$

$$U(s) = \int_z^{z_0} F(\tau)\exp\{-(\tau - z)/s\}\, \mathrm{d}\tau, \tag{53a}$$

$$V(s) = \int_0^z F(\tau)\exp\{-(z - \tau)/s\}\, \mathrm{d}\tau, \tag{53b}$$

$$X(s) = \int_0^\gamma \tau\phi^3(\tau)\Phi[z, -p(\tau)]\frac{\mathrm{d}\tau}{\tau - s} + \int_0^\gamma \tau\phi^3(\tau)(\Phi[z, p(\tau)] - \Phi[z_0, p(\tau)]\exp\{-(z_0 - z)/s\})\frac{\mathrm{d}\tau}{\tau + s} \tag{54a}$$

and

$$Y(s) = \int_0^\gamma \tau\phi^3(\tau)\Phi[z, p(\tau)]\frac{\mathrm{d}\tau}{\tau - s} + \int_0^\gamma \tau\phi^3(\tau)(\Phi[z, -p(\tau)] - \Phi[0, -p(\tau)]\exp\{-z/s\})\frac{\mathrm{d}\tau}{\tau + s}. \tag{54b}$$

As noted in Ref. 1, the dispersion function $\Lambda(s)$ has two zeros $\pm s_0$ in the complex plane cut from $-\gamma$ to γ along the real axis, and so we can first evaluate Eqs. (51) at $s = s_0$ to obtain the two integral constraints

$$U(s_0) + \varpi s_0 X(s_0) = 0 \tag{55a}$$

and

$$V(s_0) + \varpi s_0 Y(s_0) = 0. \tag{55b}$$

We can also let $s \to v \in [0, \gamma)$ from above (+) and below (−) the branch cut of $\Lambda(s)$ and use the Plemelj formulae[9] to deduce from Eqs. (51) that

$$\left[\lambda(v) \pm \frac{\varpi}{2}\pi i v\phi(v)\right]I(v) = U(v) + \varpi v X^{\pm}(v) \tag{56a}$$

and

$$\left[\lambda(v) \pm \frac{\varpi}{2}\pi i v\phi(v)\right]J(v) = V(v) + \varpi v Y^{\pm}(v). \tag{56b}$$

Here

$$\lambda(v) = 1 + \frac{\varpi v}{2}⨍_{-\gamma}^{\gamma} \phi(\tau)\frac{\mathrm{d}\tau}{\tau - v} \tag{57}$$

where the symbol ⨍ implies that the integral is to be evaluated in the Cauchy principal-value sense,[9]

$$X^{\pm}(v) = x(v) \pm \pi i v\phi^3(v)\Phi[z, -p(v)] \tag{58a}$$

and

$$Y^{\pm}(v) = y(v) \pm \pi i v\phi^3(v)\Phi[z, p(v)] \tag{58b}$$

where

$$x(\nu) = \int_0^{\gamma} \tau\phi^3(\tau)\Phi[z, -p(\tau)]\frac{\mathrm{d}\tau}{\tau - \nu} + \int_0^{\gamma} \tau\phi^3(\tau)(\Phi[z, p(\tau)] - \Phi[z_0, p(\tau)]\exp\{-(z_0 - z)/\nu\})\frac{\mathrm{d}\tau}{\tau + \nu} \tag{59a}$$

and

$$y(\nu) = \int_0^{\gamma} \tau\phi^3(\tau)\Phi[z, p(\tau)]\frac{\mathrm{d}\tau}{\tau - \nu} + \int_0^{\gamma} \tau\phi^3(\tau)(\Phi[z, -p(\tau)] - \Phi[0, -p(\tau)]\exp\{-z/\nu\})\frac{\mathrm{d}\tau}{\tau + \nu}. \tag{59b}$$

We can now eliminate $I(\nu)$ between the two versions of Eq. (56a) and $J(\nu)$ between the two versions of Eq. (56b) to find

$$2\nu\lambda(\nu)\phi^2(\nu)\Phi[z, -p(\nu)] = U(\nu) + \varpi\nu x(\nu) \tag{60a}$$

and

$$2\nu\lambda(\nu)\phi^2(\nu)\Phi[z, p(\nu)] = V(\nu) + \varpi\nu y(\nu) \tag{60b}$$

for all $\nu \in [0, \gamma]$. Equations (55) and (60) define a system of integral constraints and singular integral equations we can solve to establish $\Phi(0, -\mu)$ and $\Phi(z_0, \mu)$ for $\mu \in [0, 1]$ and $\Phi(z, \pm\mu)$ for $z \in (0, z_0)$ and $\mu \in [0, 1]$.

Intending first to establish $\Phi(0, -\mu)$ and $\Phi(z_0, \mu)$ for $\mu \in [0, 1]$, we deduce from Eqs. (55a) and (60a), with $z = 0$, and Eqs. (55b) and (60b), with $z = z_0$, the system of equations

$$\varpi s_0 \int_0^{\gamma} \tau\phi^3(\tau)\Phi[0, -p(\tau)]\frac{\mathrm{d}\tau}{s_0 - \tau} + \varpi s_0 \exp\{-z_0/s_0\} \times \int_0^{\gamma} \tau\phi^3(\tau)\Phi[z_0, p(\tau)]\frac{\mathrm{d}\tau}{s_0 + \tau} = s_0 U_0 S(z_0; s_0, U_0) \tag{61a}$$

and

$$\varpi s_0 \int_0^{\gamma} \tau\phi^3(\tau)\Phi[z_0, p(\tau)]\frac{\mathrm{d}\tau}{s_0 - \tau} + \varpi s_0 \exp\{-z_0/s_0\} \times \int_0^{\gamma} \tau\phi^3(\tau)\Phi[0, -p(\tau)]\frac{\mathrm{d}\tau}{s_0 + \tau} = s_0 U_0 C(z_0; s_0, U_0) \tag{61b}$$

and, for $\nu \in [0, \gamma]$,

$$2\nu\lambda(\nu)\phi^2(\nu)\Phi[0, -p(\nu)] + \varpi\nu \int_0^{\gamma} \tau\phi^3(\tau)\Phi[0, -p(\tau)]\frac{\mathrm{d}\tau}{\nu - \tau} + \varpi\nu \exp\{-z_0/\nu\} \int_0^{\gamma} \tau\phi^3(\tau)\Phi[z_0, p(\tau)]\frac{\mathrm{d}\tau}{\nu + \tau} = \nu U_0 S(z_0; \nu, U_0) \tag{62a}$$

and

$$2\nu\lambda(\nu)\phi^2(\nu)\Phi[z_0, p(\nu)] + \varpi\nu \int_0^{\gamma} \tau\phi^3(\tau)\Phi[z_0, p(\tau)]\frac{\mathrm{d}\tau}{\nu - \tau} + \varpi\nu \exp\{-z_0/\nu\} \int_0^{\gamma} \tau\phi^3(\tau)\Phi[0, -p(\tau)]\frac{\mathrm{d}\tau}{\nu + \tau} = \nu U_0 C(z_0; \nu, U_0) \tag{62b}$$

where, in general,

$$S(a; x, y) = \frac{1 - \exp\{-a/x\}\exp\{-a/y\}}{x + y} \tag{63a}$$

and

$$C(a; x, y) = \frac{\exp\{-a/x\} - \exp\{-a/y\}}{x - y}. \tag{63b}$$

The integral constraints and the singular integral equations given respectively by Eqs. (61) and (62) are to be solved to yield the boundary values $\Phi(0, -\mu)$ and $\Phi(z_0, \mu)$ for $\mu \in [0, 1]$.

If we consider now that $\Phi(0, -\mu)$ and $\Phi(z_0, \mu)$, for $\mu \in [0, 1]$, are known we can, for $z \in (0, z_0)$, rewrite Eqs. (55) and (60) as

$$\varpi s_0 \int_0^\gamma \tau\phi^3(\tau)\Phi[z, -p(\tau)]\frac{d\tau}{s_0 - \tau} - \varpi s_0 \int_0^\gamma \tau\phi^3(\tau)\Phi[z, p(\tau)]\frac{d\tau}{s_0 + \tau} = W_1(z, s_0) \tag{64a}$$

and

$$\varpi s_0 \int_0^\gamma \tau\phi^3(\tau)\Phi[z, p(\tau)]\frac{d\tau}{s_0 - \tau} - \varpi s_0 \int_0^\gamma \tau\phi^3(\tau)\Phi[z, -p(\tau)]\frac{d\tau}{s_0 + \tau} = W_2(z, s_0) \tag{64b}$$

and, for $\nu \in [0, \gamma]$,

$$2\nu\lambda(\nu)\phi^2(\nu)\Phi[z, -p(\nu)] + \varpi\nu \fint_0^\gamma \tau\phi^3(\tau)\Phi[z, -p(\tau)]\frac{d\tau}{\nu - \tau} - \varpi\nu \int_0^\gamma \tau\phi^3(\tau)\Phi[z, p(\tau)]\frac{d\tau}{\nu + \tau} = W_1(z, \nu) \tag{65a}$$

and

$$2\nu\lambda(\nu)\phi^2(\nu)\Phi[z, p(\nu)] + \varpi\nu \fint_0^\gamma \tau\phi^3(\tau)\Phi[z, p(\tau)]\frac{d\tau}{\nu - \tau} - \varpi\nu \int_0^\gamma \tau\phi^3(\tau)\Phi[z, -p(\tau)]\frac{d\tau}{\nu + \tau} = W_2(z, \nu) \tag{65b}$$

where

$$W_1(z, s) = sU_0\exp\{-z/U_0\}S(z_0 - z; s, U_0) - \varpi s \exp\{-(z_0 - z)/s\} \int_0^\gamma \tau\phi^3(\tau)\Phi[z_0, p(\tau)]\frac{d\tau}{s + \tau} \tag{66a}$$

and

$$W_2(z, s) = sU_0C(z; s, U_0) - \varpi s \exp\{-z/s\} \int_0^\gamma \tau\phi^3(\tau)\Phi[0, -p(\tau)]\frac{d\tau}{s + \tau}. \tag{66b}$$

Of course once $\Phi(0, -\mu)$ and $\Phi(z_0, \mu)$, for $\mu \in [0, 1]$, are known, the right-hand sides of Eqs. (64) and (65) become known. It follows that the integral constraints and the singular integral equations given respectively by Eqs. (64) and (65) are to be solved to yield $\Phi(z, \pm\mu)$ for all $z \in (0, z_0)$ and $\mu \in [0, 1]$.

4. AN F_N SOLUTION TO THE PSEUDO PROBLEM

In Ref. 2 an F_N solution for the boundary values $\Phi(0, -\mu)$ and $\Phi(z_0, \mu)$, $\mu \in [0, 1]$, was reported, and so here we generalize that work in order to provide a tractable solution for $\Phi(z, \pm\mu)$, for $z \in (0, z_0)$ and $\mu \in [0, 1]$, as well as the boundary values. First of all we let $\Phi_0[z, -p(\nu)]$ and $\Phi_0[z, p(\nu)]$, for $z \in [0, z_0]$ and $\nu \in [0, \gamma]$, denote the desired solutions for the case $\varpi = 0$. We find from Eqs. (62) and (65) that

$$\Phi_0[z, -p(\nu)] = \tfrac{1}{2}U_0\phi^{-2}(\nu)\exp\{-z/U_0\}S(z_0 - z; \nu, U_0) \tag{67a}$$

and

$$\Phi_0[z, p(\nu)] = \tfrac{1}{2}U_0\phi^{-2}(\nu)C(z; \nu, U_0) \tag{67b}$$

for $z \in [0, z_0]$ and $\nu \in [0, \gamma]$.

Now for the general case ($\varpi \neq 0$) we use the approximations

$$\Phi[0, -p(\nu)] = \Phi_0[0, -p(\nu)] + \frac{\varpi}{2}\gamma\phi^{-2}(\nu)\sum_{\alpha=0}^{N} a_\alpha H_\alpha(\nu/\gamma) \tag{68a}$$

and

$$\Phi[z_0, p(\nu)] = \Phi_0[z_0, p(\nu)] + \frac{\varpi}{2}\gamma\phi^{-2}(\nu)\sum_{\alpha=0}^{N} b_\alpha H_\alpha(\nu/\gamma), \tag{68b}$$

for $\nu \in [0, \gamma]$, and

$$\Phi[z, -p(\nu)] = \Phi_0[z, -p(\nu)] + \frac{\varpi}{2}\gamma\phi^{-2}(\nu)\sum_{\alpha=0}^{N} c_\alpha(z) H_\alpha(\nu/\gamma) \tag{69a}$$

and

$$\Phi[z, p(\nu)] = \Phi_0[z, p(\nu)] + \frac{\varpi}{2}\gamma\phi^{-2}(\nu)\sum_{\alpha=0}^{N} d_\alpha(z) H_\alpha(\nu/\gamma), \tag{69b}$$

for $\nu \in [0, \gamma]$ and $z \in (0, z_0)$. Here the basis functions $H_\alpha(\mu)$ are to be chosen, and the constants a_α, b_α, $c_\alpha(z)$ and $d_\alpha(z)$ are to be found so that the approximations given by Eqs. (68) and (69) will satisfy Eqs. (61) and (62) and Eqs. (64) and (65) at $N + 1$ values of $\zeta \in [0, \gamma] \cup s_0$.

Substituting Eqs. (68) into Eqs. (61) and (62), we find,[2] after letting $\eta = \nu/\gamma$ and $\eta_0 = s_0/\gamma$,

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi) + b_\alpha A_\alpha(\xi)\exp\{-z_0/(\gamma\xi)\}] = U_0 R_1(0, \xi) \tag{70a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi) + a_\alpha A_\alpha(\xi)\exp\{-z_0/(\gamma\xi)\}] = U_0 R_2(z_0, \xi) \tag{70b}$$

for $\xi = \eta \in [0, 1]$ or $\xi = \eta_0$, which we hereafter abbreviate as $\xi \in \mathscr{P}$. We note that, in general,

$$R_1(z, \xi) = \exp\{-z/U_0\}\left\{\int_0^1 x\phi(\gamma x)[S(z_0 - z; \gamma x, U_0) - S(z_0 - z; \gamma\xi, U_0)]\frac{dx}{x - \xi} + W(\xi)\right.$$
$$\left.\times S(z_0 - z; \gamma\xi, U_0)\right\} + \int_0^1 x\phi(\gamma x)[C(z; \gamma x, U_0) - \exp\{-(z_0 - z)/(\gamma\xi)\}C(z_0; \gamma x, U_0)]\frac{dx}{x + \xi} \tag{71a}$$

and

$$R_2(z, \xi) = W(\xi)C(z; \gamma\xi, U_0) + \int_0^1 x\phi(\gamma x)[C(z; \gamma x, U_0) - C(z; \gamma\xi, U_0)]\frac{dx}{x - \xi}$$
$$+ \int_0^1 x\phi(\gamma x)[\exp\{-z/U_0\}S(z_0 - z; \gamma x, U_0) - \exp\{-z/(\gamma\xi)\}S(z_0; \gamma x, U_0)]\frac{dx}{x + \xi} \tag{71b}$$

where

$$W(\xi) = 2\int_0^1 \phi(\gamma x)\,dx - \int_0^1 x\phi(\gamma x)\frac{dx}{x + \xi}. \tag{72}$$

In addition,[2]

$$A_\alpha(\xi) = \varpi\gamma\int_0^1 x\phi(\gamma x)H_\alpha(x)\frac{dx}{x + \xi}, \quad \xi \in \mathscr{P}, \tag{73}$$

$$B_\alpha(\eta) = 2\lambda(\gamma\eta)H_\alpha(\eta) + \varpi\gamma\,⨍_0^1 x\phi(\gamma x)H_\alpha(x)\frac{dx}{\eta - x}, \tag{74a}$$

for $\eta \in [0, 1]$, and

$$B_\alpha(\eta_0) = \varpi\gamma\int_0^1 x\phi(\gamma x)H_\alpha(x)\frac{dx}{\eta_0 - x}. \tag{74b}$$

To complete our boundary solution, we consider Eqs. (70) at $N+1$ selected values of $\xi \in \mathscr{P}$, say ξ_β, $\beta = 0, 1, \ldots, N$, and solve the linear algebraic equations

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\xi_\beta) + b_\alpha A_\alpha(\xi_\beta)\exp\{-z_0/(\gamma\xi_\beta)\}] = U_0 R_1(0, \xi_\beta) \tag{75a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\xi_\beta) + a_\alpha A_\alpha(\xi_\beta)\exp\{-z_0/(\gamma\xi_\beta)\}] = U_0 R_2(z_0, \xi_\beta) \tag{75b}$$

to find the constants a_α and b_α, $\alpha = 0, 1, \ldots, N$.

Considering now that we have solved Eqs. (75) to find the constants a_α and b_α, we substitute Eqs. (69) into Eqs. (64) and (65) to obtain

$$\sum_{\alpha=0}^{N} [c_\alpha(z)B_\alpha(\xi) - d_\alpha(z)A_\alpha(\xi)] = U_0 R_1(z, \xi) - \exp\{-(z_0 - z)/(\gamma\xi)\} \sum_{\alpha=0}^{N} b_\alpha A_\alpha(\xi) \tag{76a}$$

and

$$\sum_{\alpha=0}^{N} [d_\alpha(z)B_\alpha(\xi) - c_\alpha(z)A_\alpha(\xi)] = U_0 R_2(z, \xi) - \exp\{-z/(\gamma\xi)\} \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi) \tag{76b}$$

for $\xi \in \mathscr{P}$. To complete the desired solution for any $z \in (0, z_0)$ we consider Eqs. (76) at $N+1$ selected values of $\xi \in \mathscr{P}$, say ξ_β, $\beta = 0, 1, \ldots, N$, and solve the linear algebraic equations

$$\sum_{\alpha=0}^{N} [c_\alpha(z)B_\alpha(\xi_\beta) - d_\alpha(z)A_\alpha(\xi_\beta)] = U_0 R_1(z, \xi_\beta) - \exp\{-(z_0 - z)/(\gamma\xi_\beta)\} \sum_{\alpha=0}^{N} b_\alpha A_\alpha(\xi_\beta) \tag{77a}$$

and

$$\sum_{\alpha=0}^{N} [d_\alpha(z)B_\alpha(\xi_\beta) - c_\alpha(z)A_\alpha(\xi_\beta)] = U_0 R_2(z, \xi_\beta) - \exp\{-z/(\gamma\xi_\beta)\} \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi_\beta) \tag{77b}$$

to find the constants $c_\alpha(z)$ and $d_\alpha(z)$, $\alpha = 0, 1, \ldots, N$. It is important to note that only the right-hand side of the linear system given by Eqs. (77) depends on z, and so one matrix inversion (or LU factorization) is sufficient for any number of values of z.

Now that all of the required constants are available we can use Eqs. (38) and the approximate solutions given by Eqs. (68) and (69) to obtain from Eqs. (26) and (32)

$$\Phi(0) = \frac{\gamma}{2} U_0 \int_0^1 \phi(\gamma x) S(z_0; \gamma x, U_0)\,dx + \frac{\varpi}{2}\gamma^2 \sum_{\alpha=0}^{N} a_\alpha I_\alpha \tag{78a}$$

and

$$\Xi(0) = -\frac{\gamma}{2} U_0 \int_0^1 \gamma x \phi^3(\gamma x) S(z_0; \gamma x, U_0)\,dx - \frac{\varpi}{2}\gamma^2 \sum_{\alpha=0}^{N} a_\alpha J_\alpha, \tag{78b}$$

$$\Phi(z) = \frac{\gamma}{2} U_0 \int_0^1 \phi(\gamma x)[C(z; \gamma x, U_0) + \exp\{-z/U_0\} \times S(z_0 - z; \gamma x, U_0)]\,dx + \frac{\varpi}{2}\gamma^2 \sum_{\alpha=0}^{N} [d_\alpha(z) + c_\alpha(z)] I_\alpha \tag{79a}$$

and

$$\Xi(z) = \frac{\gamma}{2} U_0 \int_0^1 \gamma x \phi^3(\gamma x)[C(z; \gamma x, U_0) - \exp\{-z/U_0\} \times S(z_0 - z; \gamma x, U_0)]\,dx + \frac{\varpi}{2}\gamma^2 \sum_{\alpha=0}^{N} [d_\alpha(z) - c_\alpha(z)] J_\alpha, \tag{79b}$$

for $z \in (0, z_0)$, and

$$\Phi(z_0) = \frac{\gamma}{2} U_0 \int_0^1 \phi(\gamma x) C(z_0; \gamma x, U_0)\,dx + \frac{\varpi}{2}\gamma^2 \sum_{\alpha=0}^{N} b_\alpha I_\alpha \tag{80a}$$

and

$$\Xi(z_0) = \frac{\gamma}{2} U_0 \int_0^1 \gamma x \phi^3(\gamma x) C(z_0; \gamma x, U_0)\, dx + \frac{\varpi}{2}\gamma^2 \sum_{\alpha=0}^{N} b_\alpha J_\alpha \tag{80b}$$

where we have defined

$$I_\alpha = \int_0^1 \phi(\gamma x) H_\alpha(x)\, dx \tag{81a}$$

and

$$J_\alpha = \int_0^1 \gamma x \phi^3(\gamma x) H_\alpha(x)\, dx. \tag{81b}$$

It is clear that we can now use Eqs. (78)–(80) in Eqs. (36) in order to establish, by way of Eqs. (35), the desired solutions.

5. THE FOURIER INVERSION

As a first demonstration that our developed solution to the searchlight problem can be implemented to yield what we believe to be accurate numerical results, we focus our attention now on the special case of a normally incident beam. It follows, since $\mu_0 = 1$ for this special case, that $U_0 = 1$ and subsequently that $\Phi(z)$ and $\Xi(z)$ are real valued and independent of the angle ψ. Since $\Phi(z)$ and $\Xi(z)$ are, for the case $\mu_0 = 1$, independent of ψ we use Eqs. (36) and carry out the integration over ψ in Eqs. (11) to find

$$4\pi J(z, \rho) = \frac{1}{2\pi} \int_0^\infty k[\exp\{-z\} + \varpi\Phi(z; k)] J_0(k\rho)\, dk \tag{82a}$$

and

$$\pi F(z, \rho) = \frac{1}{2\pi} \int_0^\infty k[\exp\{-z\} + \varpi\Xi(z; k)] J_0(k\rho)\, dk \tag{82b}$$

where we have written $\Phi(z;k)$ and $\Xi(z;k)$, rather than simply $\Phi(z)$ and $\Xi(z)$, in order to note explicitly the dependence on the transform variable k.

We note now that the integral terms in Eqs. (78), (79) and (80) correspond to the once-collided components (see Refs. 1, 2 and 10) of the required solutions. We find subsequently that these components can be inverted analytically so that we can simplify Eqs. (82) and express our results as

$$4\pi J(z, \rho) = \frac{\delta(\rho)}{2\pi\rho} \exp\{-z\} + \frac{\varpi}{4\pi\rho} J_1(z, \rho) \exp\{-z\} + \frac{\varpi^2}{4\pi} J_2(z, \rho) \tag{83a}$$

and

$$\pi F(z, \rho) = \frac{\delta(\rho)}{2\pi\rho} \exp\{-z\} + \frac{\varpi}{4\pi\rho} F_1(z, \rho) \exp\{-z\} + \frac{\varpi^2}{4\pi} F_2(z, \rho). \tag{83b}$$

Here

$$J_1(z, \rho) = \int_0^{\alpha(z,\rho)} \exp\left\{-\rho\left(\frac{1-\mu}{1+\mu}\right)^{1/2}\right\} \frac{d\mu}{(1-\mu^2)^{1/2}} + \int_0^{\beta(z,\rho)} \exp\left\{-\rho\left(\frac{1+\mu}{1-\mu}\right)^{1/2}\right\} \frac{d\mu}{(1-\mu^2)^{1/2}} \tag{84a}$$

and

$$F_1(z, \rho) = \int_0^{\alpha(z,\rho)} \mu \exp\left\{-\rho\left(\frac{1-\mu}{1+\mu}\right)^{1/2}\right\} \frac{d\mu}{(1-\mu^2)^{1/2}} - \int_0^{\beta(z,\rho)} \mu \exp\left\{-\rho\left(\frac{1+\mu}{1-\mu}\right)^{1/2}\right\} \frac{d\mu}{(1-\mu^2)^{1/2}} \tag{84b}$$

where

$$\alpha(z, \rho) = z(\rho^2 + z^2)^{-1/2} \tag{85a}$$

and

$$\beta(z, \rho) = (z_0 - z)[\rho^2 + (z_0 - z)^2]^{-1/2}. \tag{85b}$$

To complete the solutions given by Eqs. (83) we require the F_N components which we elect to write as

$$J_2(z, \rho) = \int_0^\infty \frac{x}{x^2 + \rho^2}[\mathscr{L}(z; x/\rho) - \Theta(\rho - \rho_*)\mathscr{L}(z; \infty)]J_0(x)\,\mathrm{d}x + \Theta(\rho - \rho_*)\mathscr{L}(z; \infty)K_0(\rho) \tag{86a}$$

and

$$F_2(z, \rho) = \int_0^\infty \frac{x}{x^2 + \rho^2}[\mathscr{H}(z; x/\rho) - \Theta(\rho - \rho_*)\mathscr{H}(z; \infty)]J_0(x)\,\mathrm{d}x + \Theta(\rho - \rho_*)\mathscr{H}(z; \infty)K_0(\rho) \tag{86b}$$

where $K_0(x)$ denotes the modified Bessel function[7] and

$$\mathscr{L}(0; k) = \sum_{\alpha=0}^{N} a_\alpha I_\alpha \tag{87a}$$

and

$$\mathscr{H}(0; k) = -\sum_{\alpha=0}^{N} a_\alpha J_\alpha, \tag{87b}$$

$$\mathscr{L}(z; k) = \sum_{\alpha=0}^{N} [d_\alpha(z) + c_\alpha(z)]I_\alpha \tag{88a}$$

and

$$\mathscr{H}(z; k) = \sum_{\alpha=0}^{N} [d_\alpha(z) - c_\alpha(z)]J_\alpha, \tag{88b}$$

for $z \in (0, z_0)$, and

$$\mathscr{L}(z_0; k) = \sum_{\alpha=0}^{N} b_\alpha I_\alpha \tag{89a}$$

and

$$\mathscr{H}(z_0; k) = \sum_{\alpha=0}^{N} b_\alpha J_\alpha. \tag{89b}$$

As we intend to make use of some asymptotic analysis for small ρ, we have introduced

$$\Theta(\rho - \rho_*) = \begin{cases} 1, & \text{for} \quad 0 < \rho \leqslant \rho_*, \\ 0, & \text{for} \quad \rho > \rho_*, \end{cases} \tag{90}$$

where ρ_* is to be selected after some numerical experimentation. For basis functions $H_\alpha(x)$ that have a nonzero limit as $k \to \infty$ we can investigate Eqs. (70) and (76) in the limit $k \to \infty$ to find

$$\mathscr{L}(0; \infty) = \frac{7\pi^2}{48} + \frac{1}{2}\int_0^1 \ln(1 + x)\frac{\mathrm{d}x}{x} \tag{91a}$$

and

$$\mathscr{H}(0; \infty) = -\left(\frac{\pi}{4} + \frac{1}{2}\right), \tag{91b}$$

$$\mathscr{L}(z; \infty) = \frac{\pi^2}{2}\exp\{-z\} \tag{92a}$$

and

$$\mathscr{H}(z;\infty)=0, \tag{92b}$$

for $z\in(0,z_0)$, and

$$\mathscr{L}(z_0;\infty)=\mathscr{L}(0;\infty)\exp\{-z_0\} \tag{93a}$$

and

$$\mathscr{H}(z_0;\infty)=-\mathscr{H}(0;\infty)\exp\{-z_0\}. \tag{93b}$$

To evaluate the integrals in Eqs. (86) we follow Longman[11] and use the zeros of $J_0(x)$ as break points to subdivide the integration interval $[0,\infty)$; subsequently an Euler transformation[12] is used to sum the resulting slowly converging series in a more rapidly convergent manner.

6. COMPUTATIONAL ASPECTS OF THE F_N SOLUTION: NUMERICAL RESULTS

After having tried various different sets of basic functions, and considering the ease with which the F_N solution could be evaluated for these basis functions and the accuracy obtained with the different basis functions, we elect here to use two different schemes.

First of all we take $\rho_*=0.1$, and for all $\rho\leqslant\rho_*$ we follow Ref. 2 and choose to use the basis functions $H_\alpha(x)=x^\alpha$. Having made this choice of basis functions, we refer to Ref. 2 for a complete description of our way of evaluating the functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$ for $\xi\in\mathscr{P}$. We also reported in Ref. 2 convenient ways to compute

$$M_\alpha=\int_0^1 x^{\alpha+1}\phi(\gamma x)\,\mathrm{d}x \tag{94}$$

for $\alpha=0,1,2,\ldots$. Thus since, for $H_\alpha(x)=x^\alpha$,

$$I_\alpha=M_{\alpha-1} \tag{95}$$

we require only

$$I_0=\frac{\sin^{-1}r}{r}, \tag{96}$$

where $r=\gamma k$, to have available the I_α, for $\alpha=0,1,2,\ldots$, required in Eqs. (87a), (88a) and (89a). Also, we can, for the case $H_\alpha(x)=x^\alpha$, evaluate the integral in Eq. (81b) to find

$$J_\alpha=1-(\alpha+1)(1-r^2)^{1/2}M_\alpha \tag{97}$$

for $\alpha=0,1,2,\ldots$.

Table 1. The computed value of $4\pi J(z,\rho)$ for $\mu_0=1$, $\varpi=0.8$ and $z_0=1$.

ρ	$z/z_0=0$	$z/z_0=0.05$	$z/z_0=0.1$	$z/z_0=0.2$	$z/z_0=0.5$	$z/z_0=0.75$	$z/z_0=1$
0.001	9.9687(1)	1.8940(2)	1.8083(2)	1.6396(2)	1.2164(2)	9.4723(1)	3.7008(1)
0.01	9.7637	1.7924(1)	1.7703(1)	1.6356(1)	1.2291(1)	9.5620	3.8185
0.1	8.4077(−1)	1.1361	1.3086	1.4089	1.1991	9.2630(−1)	4.2588(−1)
0.2	3.6477(−1)	4.4621(−1)	5.0743(−1)	5.7604(−1)	5.4481(−1)	4.2286(−1)	2.1656(−1)
0.4	1.3991(−1)	1.6123(−1)	1.7785(−1)	2.0195(−1)	2.1090(−1)	1.7022(−1)	1.0015(−1)
0.6	7.2196(−2)	8.1428(−2)	8.8471(−2)	9.9111(−2)	1.0661(−1)	8.9826(−2)	5.7469(−2)
0.8	4.2094(−2)	4.6948(−2)	5.0567(−2)	5.6058(−2)	6.0809(−2)	5.2919(−2)	3.5779(−2)
1.0	2.6269(−2)	2.9097(−2)	3.1167(−2)	3.4295(−2)	3.7280(−2)	3.3174(−2)	2.3298(−2)
1.2	1.7130(−2)	1.8887(−2)	2.0154(−2)	2.2058(−2)	2.3981(−2)	2.1669(−2)	1.5634(−2)
1.4	1.1523(−2)	1.2663(−2)	1.3476(−2)	1.4692(−2)	1.5964(−2)	1.4581(−2)	1.0729(−2)
1.6	7.9341(−3)	8.6977(−3)	9.2373(−3)	1.0041(−2)	1.0902(−2)	1.0036(−2)	7.4936(−3)
1.8	5.5627(−3)	6.0867(−3)	6.4543(−3)	7.0001(−3)	7.5946(−3)	7.0314(−3)	5.3094(−3)
2.0	3.9573(−3)	4.3237(−3)	4.5792(−3)	4.9575(−3)	5.3745(−3)	4.9978(−3)	3.8072(−3)
2.2	2.8490(−3)	3.1092(−3)	3.2896(−3)	3.5562(−3)	3.8527(−3)	3.5949(−3)	2.7577(−3)
2.4	2.0718(−3)	2.2587(−3)	2.3879(−3)	2.5782(−3)	2.7915(−3)	2.6117(−3)	2.0149(−3)
2.6	1.5194(−3)	1.6552(−3)	1.7486(−3)	1.8861(−3)	2.0409(−3)	1.9137(−3)	1.4833(−3)
2.8	1.1225(−3)	1.2219(−3)	1.2902(−3)	1.3904(−3)	1.5037(−3)	1.4125(−3)	1.0991(−3)
3.0	8.3450(−4)	9.0791(−4)	9.5815(−4)	1.0318(−3)	1.1154(−3)	1.0493(−3)	8.1920(−4)
4.0	2.0375(−4)	2.2122(−4)	2.3307(−4)	2.5035(−4)	2.7012(−4)	2.5532(−4)	2.0153(−4)
5.0	5.4060(−5)	5.8629(−5)	6.1708(−5)	6.6182(−5)	7.1317(−5)	6.7579(−5)	5.3665(−5)

Table 2. The computed value of $\pi F(z, \rho)$ for $\mu_0 = 1$, $\varpi = 0.8$ and $z_0 = 1$.

ρ	$z/z_0 = 0$	$z/z_0 = 0.05$	$z/z_0 = 0.1$	$z/z_0 = 0.2$	$z/z_0 = 0.5$	$z/z_0 = 0.75$	$z/z_0 = 1$
0.001	−6.3217(1)	−5.5478(−1)	8.8956(−2)	3.7952(−1)	4.4737(−1)	4.2432(−1)	2.3563(1)
0.01	−6.0631	−8.0834(−1)	−1.7275(−1)	1.4020(−1)	2.6978(−1)	2.8594(−1)	2.4292
0.1	−4.7074(−1)	−3.7797(−1)	−2.4177(−1)	−6.1321(−2)	9.5914(−2)	1.4392(−1)	2.6473(−1)
0.2	−1.9033(−1)	−1.6963(−1)	−1.3636(−1)	−6.4882(−2)	4.9976(−2)	9.4158(−2)	1.3034(−1)
0.4	−6.5932(−2)	−6.0425(−2)	−5.3057(−2)	−3.4794(−2)	1.6980(−2)	4.4123(−2)	5.6066(−2)
0.6	−3.1437(−2)	−2.8851(−2)	−2.5732(−2)	−1.8235(−2)	6.6958(−3)	2.2312(−2)	2.9825(−2)
0.8	−1.7167(−2)	−1.5717(−2)	−1.4061(−2)	−1.0224(−2)	2.9517(−3)	1.2089(−2)	1.7259(−2)
1.0	−1.0138(−2)	−9.2563(−3)	−8.2814(−3)	−6.0868(−3)	1.4202(−3)	6.9498(−3)	1.0510(−2)
1.2	−6.3099(−3)	−5.7477(−3)	−5.1392(−3)	−3.7982(−3)	7.3166(−4)	4.1954(−3)	6.6435(−3)
1.4	−4.0799(−3)	−3.7097(−3)	−3.3148(−3)	−2.4583(−3)	3.9791(−4)	2.6353(−3)	4.3262(−3)
1.6	−2.7157(−3)	−2.4659(−3)	−2.2023(−3)	−1.6371(−3)	2.2604(−4)	1.7094(−3)	2.8865(−3)
1.8	−1.8494(−3)	−1.6775(−3)	−1.4976(−3)	−1.1153(−3)	1.3307(−4)	1.1382(−3)	1.9652(−3)
2.0	−1.2829(−3)	−1.1627(−3)	−1.0376(−3)	−7.7387(−4)	8.0708(−5)	7.7433(−4)	1.3609(−3)
2.2	−9.0339(−4)	−8.1822(−4)	−7.3004(−4)	−5.4510(−4)	5.0192(−5)	5.3620(−4)	9.5601(−4)
2.4	−6.4424(−4)	−5.8319(−4)	−5.2024(−4)	−3.8882(−4)	3.1891(−5)	3.7686(−4)	6.7990(−4)
2.6	−4.6435(−4)	−4.2017(−4)	−3.7475(−4)	−2.8030(−4)	2.0641(−5)	2.6822(−4)	4.8867(−4)
2.8	−3.3776(−4)	−3.0551(−4)	−2.7245(−4)	−2.0392(−4)	1.3576(−5)	1.9296(−4)	3.5446(−4)
3.0	−2.4762(−4)	−2.2391(−4)	−1.9966(−4)	−1.4952(−4)	9.0570(−6)	1.4010(−4)	2.5917(−4)
4.0	−5.7322(−5)	−5.1778(−5)	−4.6150(−5)	−3.4621(−5)	1.4082(−6)	3.1342(−5)	5.9338(−5)
5.0	−1.4673(−5)	−1.3246(−5)	−1.1803(−5)	−8.8619(−6)	2.6435(−7)	7.8622(−6)	1.5076(−5)

For all $\rho > \rho_*$ we use the basis functions $H_\alpha(x) = \phi^{-1}(\gamma x)P_\alpha(2x-1)$. Here $P_\alpha(x)$ is used to denote the Legendre polynomials,[7] and so with this choice of basis functions we can express the required $A_\alpha(\xi)$ and $B_\alpha(\xi)$ for $\xi \in \mathscr{P}$ as well as the integrals I_α and J_α in terms of Legendre functions of the second kind[7] which can be evaluated without significant loss of accuracy in a recursive manner.[13]

To complete our F_N solution of the pseudo problem we must simply specify a collocation strategy. We again follow Ref. 2 and use $\xi_0 = \eta_0$ and the zeros of the Chebyshev polynomial of the first kind $T_N(2x-1)$, i.e.,

$$\xi_\beta = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{(2\beta-1)\pi}{2N}\right], \quad \beta = 1, 2, \ldots, N. \tag{98}$$

We list in Tables 1 and 2 our numerical results deduced from the formalism herein discussed. To establish our belief that the reported results are correct to within ± 1 in the last digits given, we have used several orders of the F_N approximation and several variants (defined by the specific zero of $J_0(x)$ where we first employ the method and the total number of terms used) of Longman's method to evaluate the integrals in Eqs. (86). Finally we have gained additional confidence by finding agreement (to, say, two significant figures) with two independent Monte Carlo calculations.[14,15]

Acknowledgements—The authors would like to express their appreciation to C. R. Drumm, J. H. Renken, and A. M. Yacout for several helpful discussions concerning this work and for communicating Monte Carlo results that helped the authors establish their confidence in the numerical results in Tables 1 and 2. This work was supported by the National Science Foundation and the Air Force Office of Scientific Research.

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