9. A Lattice Problem in One-Speed Neutron Transport Theory, G. R. Bond, C. E. Siewert (NC State U)

One of the more rigorous one-speed analyses for the disadvantage factors of isotropically scattering slab cells has been provided by Fergzger and Robinson. They employed Case's normal modes and solved numerically the ensuing Fredholm equations for the expansion coefficients. We have taken the next step toward generalization by treating the moderator as a linearly anisotropic scattering region.

If we denote the fuel and moderator regions by the subscripts 1 and 2, we can write the Boltzmann equation appropriate to the present case as

\[ \frac{\partial}{\partial x} \psi_i(x, \mu) + \psi_i(x, \mu) = \frac{c_i}{2} \int_{-1}^{1} \left( 1 + \omega \mu \mu' \sigma_{12} \right) \psi_i(x, \mu') \mathrm{d} \mu' + \frac{S}{\sigma_s} \delta_{12} \quad i = 1, 2, \]  

(1)

where \( \omega \) is the anisotropy coefficient, \( x \) is the optical variable measured from the center of the fuel region, and the other symbols have their usual meanings. We must solve Eq. (1) subject to the boundary conditions

\[ \psi_i(x, \mu) = \psi_i(0, \mu) \quad \mu \in (-1, 1) , \]  

(2a)

\[ \psi_i(x, \mu) = \psi_i(x, \mu) \quad \mu \in (-1, 1) , \]  

(2b)

and

\[ \psi_i(x, 0) = \psi_i(x, \mu) \quad \mu \in (-1, 1) , \]  

(2c)

where \( a \) and \( A \) are, respectively, the fuel and moderator half-thicknesses in optical units.

We write solutions to Eq. (1) in the forms

\[ \psi_i(x, \mu) = A_+ \phi_i(x) \exp(-x/\lambda_0) + A_- \phi_i(x) \exp(x/\lambda_0) \]  

(3)

\[ + \int_{-1}^{1} A(\nu) \phi_i(\nu) \exp(-x/\lambda_0) \mathrm{d} \nu \]

\[ \psi_i(x, \mu) = B_+ \chi_i(x) \exp(-x/\lambda_0) + B_- \chi_i(x) \exp(x/\lambda_0) \]  

(4)

\[ + \int_{-1}^{1} B(\eta) \chi_i(\eta) \exp(-x/\lambda_0) \mathrm{d} \eta + 1 . \]

Here \( \{ \phi_i(\mu) \} \) denotes the eigenfunctions developed by Case for isotropic scattering, and \( \{ \chi_i(\eta) \} \) represents the corresponding forms for linearly anisotropic scattering. Without loss of generality we have taken \( S = \sigma_s(1 - c_i) \).

Upon applying the boundary conditions, we arrive at the following equation from which the expansion coefficients are to be determined:

\[ A_+ \phi_i(\mu) \exp(-x/\lambda_0) + A_- \phi_i(\mu) \exp(x/\lambda_0) \]  

(5)

\[ + \int_{-1}^{1} A(\nu) \phi_i(\nu) \exp(-x/\lambda_0) \mathrm{d} \nu = 1 \]

\[ + B_+ \chi_i(x) \exp(-x/\lambda_0) + B_- \chi_i(x) \exp(x/\lambda_0) \]  

\[ + \int_{-1}^{1} B(\eta) \chi_i(\eta) \exp(-x/\lambda_0) \mathrm{d} \eta , \quad \mu \in (-1, 1) . \]

(6)

The usual technique for isolating the expansion coefficients, namely, employing orthogonality, yields two coupled singular integral equations for \( A(\nu) \) and \( B(\eta) \). Robinson was able to circumvent the singular nature of these equations by judicious algebraic manipulation. We take a slightly different tack, which yields similar results, by utilizing an integral operator \( H(\sigma, \beta) \); if \( G(\mu) \) is an arbitrary function defined for \( \mu(-1, 1) \), then

\[ H(\sigma, \beta) G(\mu) \]  

(7)

\[ + \Omega_2(\mu) \exp(-\beta/\alpha) \int_{0}^{1} \Omega_1(\mu) \mathrm{d} \mu , \]

where \( \Omega_2(\mu) \) represents either \( \{ \phi_i(\mu) \} \) or \( \{ \chi_i(\mu) \} \). We will always associate the parameter \( \nu \) with \( \phi_i(\mu) \) and \( \eta \) with \( \chi_i(\eta) \).

By operating on Eq. (5) with \( H(\lambda_0, \Delta) \), \( H(\lambda_0, -\lambda) \), \( H(\lambda, \Delta) \), and \( H(\eta, -\lambda) \), and then rearranging the resulting equations, we obtain the following:

\[ \tilde{A} + \bar{N}_1 T(\lambda_0) = Q(\lambda_0, c_1, \Delta) + \tilde{B} + R(\lambda_0, \lambda_0, \Delta) \]  

(8)

\[ + \int_{0}^{1} \tilde{A}(\nu) R(\nu, \lambda_0) \mathrm{d} \nu , \]

\[ \tilde{B} + N_1 T(\lambda) = Q(\lambda, c_2, \Delta) + \tilde{A} \]  

(9)

\[ + \int_{0}^{1} \tilde{B}(\eta) R(\eta, \lambda_0) \mathrm{d} \eta , \]

and

\[ -\tilde{B} + N_2 T(\eta) = Q(\eta, c_2, \Delta) + \tilde{A}(\nu) R(\nu, \eta) \]  

(10)

\[ + \int_{0}^{1} \tilde{B}(\eta) R(\eta, \nu) \mathrm{d} \eta \]

Here,

\[ \tilde{A} = A_+ \cosh a/\lambda_0, \quad \tilde{B}(\nu) = A(\nu) \cosh a/\nu \]  

(11a, b)

\[ \tilde{B}_+ = B_+ \cosh \Delta/\eta_0, \quad \tilde{B}(\eta) = B(\eta) \cosh \Delta/\eta \]  

(12a, b)

\[ T(x) = \tanh a/x + \tanh \Delta/x \]  

(13)

\[ Q(x,y,z) = x(1 - y) \tanh z/\xi \]  

(14)

and

\[ R(x,y,z) = \frac{2xy}{x^2 - y^2} \left[ x F(y^2) \tanh \frac{z}{y} - y F(x^2) \tanh \frac{z}{x} \right] \]  

(15)

We note that the function \( R(x,y,z) \) is nonsingular, and thus standard quadrature formulae may be used to evaluate the integrals appearing in Eqs. (7) through (10). We have utilized an 81-point improved Gaussian quadrature formula and numerically solved these equations for the expansion coefficients. The disadvantage factor is then readily obtained:

\[ \xi = \frac{a}{\Delta} \left\{ \frac{\sigma_0 B_+ \tanh \Delta/\eta_0 + \int_{0}^{1} \tilde{B}(\eta) \tanh \Delta/\eta \mathrm{d} \eta + \Delta}{\nu_0 A_+ \tanh a/\nu_0 + \int_{0}^{1} \nu A(\nu) \tanh a/\nu \mathrm{d} \nu} \right\} \]  

(16)

The results of our computations and \( P_1 \) theory estimates are shown in Table I. The basic cells are those of Thiesa. As an indication of the accuracy of our calculations, we have numerically computed the first ten moments of the interface condition Eq. (5). We found that

\[ \left| \int_{-1}^{1} \psi_i(\mu) \mu K \mathrm{d} \mu - 1 \right| < 2 \times 10^{-8} \]

in all cases.
### TABLE I
The Disadvantage Factor of Slab Cells with Linearly Anisotropic Scattering

<table>
<thead>
<tr>
<th>Calculational Model</th>
<th>Anisotropy Coefficient $\omega$</th>
<th>$\zeta$, the disadvantage factor</th>
<th>Cell 1&lt;sup&gt;a&lt;/sup&gt;</th>
<th>Cell 2&lt;sup&gt;a&lt;/sup&gt;</th>
<th>Cell 3&lt;sup&gt;a&lt;/sup&gt;</th>
<th>Cell 4&lt;sup&gt;a&lt;/sup&gt;</th>
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<td>1.113</td>
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</table>

$a\Sigma_a^{\text{Fuel}} = 0.32 \text{ cm}^{-1} \quad \Sigma_T^{\text{Fuel}} = 0.717 \text{ cm}^{-1}$

$\Sigma_a^{\text{Mod}} = 0.0195 \text{ cm}^{-1} \quad \Sigma_T^{\text{Mod}} = 2.33 \text{ cm}^{-1}$