9. A Lattice Problem in One-Speed Neutron Transport Theory, *G. R. Bond, C. E. Siewert (NC State U)*

One of the more rigorous one-speed analyses for the disadvantage factors of isotropically scattering slab cells has been provided by Ferziger and Robinson.[1] They employed Case's normal modes and solved numerically the ensuing Fredholm equations for the expansion coefficients. We have taken the next step toward generalization by treating the moderator as a linearly anisotropic scattering region.

If we denote the fuel and moderator regions by the subscripts 1 and 2, we can write the Boltzmann equation appropriate to the present case as

$$\mu \frac{\partial}{\partial x} \Psi_i(x,\mu) + \Psi_i(x,\mu)$$
$$= \frac{c_i}{2} \int_{-1}^{1} (1 + \omega\mu\mu'\delta_{i2}) \Psi_i(x,\mu')\, d\mu' + \frac{S}{\sigma_2} \delta_{i2} \quad i = 1, 2, \tag{1}$$

where ω is the anisotropy coefficient, x is the optical variable measured from the center of the fuel region, and the other symbols have their usual meanings.[2] We must solve Eq. (1) subject to the boundary conditions

$$\Psi_1(a,\mu) = \Psi_2(a,\mu) \;, \quad \mu \in (-1,1) \;, \tag{2a}$$

$$\Psi_2(a + \Delta,\mu) = \Psi_2(a + \Delta,-\mu) \;, \quad \mu \in (-1,1) \;, \tag{2b}$$

and

$$\Psi_1(0,\mu) = \Psi_1(0,-\mu) \;, \quad \mu \in (-1,1) \;, \tag{2c}$$

where $\underline{a}$ and $\underline{\Delta}$ are, respectively, the fuel and moderator half-thicknesses in optical units.

We write solutions to Eq. (1) in the forms

$$\Psi_1(x,\mu) = A_+ \phi_+(\mu) \exp(-x/\nu_0) + A_- \phi_-(\mu) \exp(x/\nu_0)$$
$$+ \int_{-1}^{1} A(\nu)\phi_\nu(\mu) \exp(-x/\nu) d\nu \tag{3}$$

and

$$\Psi_2(x,\mu) = B_+ \chi_+(\mu) \exp[(a + \Delta - x)/\eta_0]$$
$$+ B_- \chi_-(\mu) \exp[-(a + \Delta - x)/\eta_0]$$
$$+ \int_{-1}^{1} B(\eta)\, \chi_\eta(\mu) \exp[(b - x)/\eta] d\eta + 1 \;. \tag{4}$$

Here $\{\phi_\nu(\mu)\}$ denotes the eigenfunctions developed by Case[2] for isotropic scattering, and $\{\chi_\eta(\mu)\}$ represents the corresponding forms for linearly anisotropic scattering.[3] Without loss of generality we have taken $S = \sigma_2(1 - c_2)$.

Upon applying the boundary conditions, we arrive at the following equation from which the expansion coefficients are to be determined:

$$A_+\{\phi_+(\mu) \exp(-a/\nu_0) + \phi_-(\mu) \exp(a/\nu_0)\}$$
$$+ \int_{-1}^{1} A(\nu) \exp(-a/\nu)\, \phi_\nu(\mu)\, d\nu = 1$$
$$+ B_+\{\chi_+(\mu) \exp(\Delta/\eta_0) + \chi_-(\mu) \exp(-\Delta/\eta_0)\}$$
$$+ \int_{-1}^{1} B(\eta) \exp(\Delta/\eta)\, \chi_\eta(\mu)\, d\eta \;, \quad \mu \in (-1,1) \;. \tag{5}$$

The usual technique for isolating the expansion coefficients, namely, employing orthogonality, yields two coupled singular integral equations for $A(\nu)$ and $B(\eta)$. Robinson[4] was able to circumvent the singular nature of these equations by judicious algebraic manipulation. We take a slightly different tack, which yields similar results, by utilizing an integral operator $H(\alpha,\beta)$; if $G(\mu)$ is an arbitrary function defined for $\mu(-1,1)$, then

$$H(\alpha,\beta)\, G(\mu) \triangleq \int_{-1}^{1} \mu\, [\Omega_\alpha(\mu) \exp(-\beta/\alpha)$$
$$+ \Omega_{-\alpha}(\mu) \exp(\beta/\alpha)]\, G(\mu)\, d\mu \;, \tag{6}$$

where $\Omega_\alpha(\mu)$ represents either $\{\phi_\nu(\mu)\}$ or $\{\chi_\eta(\mu)\}$. [We will always associate the parameter ν with $\phi_\nu(\mu)$ and η with $\chi_\eta(\mu)$.]

By operating on Eq. (5) with $H(\nu_0,\Delta)$, $H(\eta_0,-a)$, $H(\nu,\Delta)$, and $H(\eta,-a)$, and then rearranging the resulting equations, we obtain the following:

$$\tilde{A}_+ N_{+1} T(\nu_0) = Q(\nu_0,c_1,\Delta) + \tilde{B}_+ R(\eta_0,\nu_0,\Delta)$$
$$+ \int_0^1 \tilde{B}(\eta) R(\eta,\nu_0,\Delta)\, d\eta \;, \tag{7}$$

$$-\tilde{B}_+ N_{+2} T(\eta_0) = Q(\eta_0,c_2,a) + \tilde{A}_+ R(\nu_0,\eta_0,a)$$
$$+ \int_0^1 \tilde{A}(\nu) R(\nu,\eta_0,a)\, d\nu \;, \tag{8}$$

$$\tilde{A}(\nu) N_1(\nu) T(\nu) = Q(\nu,c_1,\Delta) + \tilde{B}_+ R(\eta_0,\nu,\Delta)$$
$$+ \int_0^1 \tilde{B}(\eta) R(\eta,\nu,\Delta)\, d\eta \;, \tag{9}$$

and

$$-\tilde{B}(\eta) N_2(\eta) T(\eta) = Q(\eta,c_2,a) + \tilde{A}_+(\nu) R(\nu_0,\eta,a)$$
$$+ \int_0^1 \tilde{A}(\nu) R(\nu,\eta,a)\, d\nu \;. \tag{10}$$

Here,

$$\tilde{A}_+ = A_+ \cosh a/\nu_0,\; \tilde{A}(\nu) = A(\nu) \cosh a/\nu \;, \tag{11a,b}$$

$$\tilde{B}_+ = B_+ \cosh \Delta/\eta_0,\; \tilde{B}(\eta) = B(\eta) \cosh \Delta/\eta \;, \tag{12a,b}$$

$$T(x) = \tanh a/x + \tanh \Delta/x \;, \tag{13}$$

$$Q(x,y,z) = x(1 - y) \tanh z/x \;, \tag{14}$$

and

$$R(x,y,z) = \frac{2xy}{x^2 - y^2} \left[x\, F(y^2) \tanh \frac{z}{y} - y\, F(x^2) \tanh \frac{z}{x} \right] \;. \tag{15}$$

We note that the function $R(x,y,z)$ is nonsingular, and thus standard quadrature formulae may be used to evaluate the integrals appearing in Eqs. (7) through (10). We have utilized an 81-point improved Gaussian quadrature formula[5] and numerically solved these equations for the expansion coefficients. The disadvantage factor is then readily obtained:

$$\zeta = \frac{a}{\Delta} \left\{ \frac{\eta_0 \tilde{B}_+ \tanh \Delta/\eta_0 + \int_0^1 \eta\, \tilde{B}(\eta) \tanh \Delta/\eta\, d\eta + \Delta}{\nu_0 \tilde{A}_+ \tanh a/\nu_0 + \int_0^1 \nu\, A(\nu) \tanh a/\nu\, dv} \right\} . \tag{16}$$

The results of our computations and P_1 theory estimates are shown in Table I. The basic cells are those of Theys.[6] As an indication of the accuracy of our calculations, we have numerically computed the first ten moments of the interface condition Eq. (5). We found that

$$\left| \frac{\int_{-1}^{1} \Psi_1(a,\mu)\, \mu^K\, d\mu}{\int_{-1}^{1} \Psi_2(a,\mu)\, \mu^K\, d\mu} - 1 \right| < 2 \times 10^{-6}$$

in all cases.

TABLE I

The Disadvantage Factor of Slab Cells with Linearly Anisotropic Scattering

Calculational Model	Anisotropy Coefficient ω	ζ, the disadvantage factor			
		Cell 1[a] a = 0.10 cm b = 0.45 cm	Cell 2[a] a = 0.20 cm b = 0.90 cm	Cell 3[a] a = 0.30 cm b = 1.35 cm	Cell 4[a] a = 0.40 cm b = 1.80 cm
P_1 Theory	0.0	1.028	1.113	1.253	1.447
Converged Solution	0.0	1.0978	1.2317	1.4077	1.6284
P_1 Theory	0.1	1.027	1.110	1.245	1.433
Converged Solution	0.1	1.0970	1.2283	1.4001	1.6151
P_1 Theory	0.3	1.026	1.103	1.230	1.407
Converged Solution	0.3	1.0953	1.2215	1.3849	1.5885
P_1 Theory	0.6	1.023	1.093	1.207	1.366
Converged Solution	0.6	1.0927	1.2113	1.3621	1.5485
P_1 Theory	0.9	1.021	1.082	1.184	1.326
Converged Solution	0.9	1.0901	1.2010	1.3392	1.5083

[a] $\Sigma_a^{Fuel} = 0.32\ cm^{-1}$ $\quad \Sigma_T^{Fuel} = 0.717\ cm^{-1}$

$\Sigma_a^{Mod} = 0.0195\ cm^{-1}$ $\quad \Sigma_T^{Mod} = 2.33\ cm^{-1}$

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