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A PARTICULAR SOLUTION FOR THE P_N METHOD IN RADIATIVE TRANSFER

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Abstract—A particular solution basic to a P_N solution of the radiative transfer equation is reported.

INTRODUCTION

In a paper published in 1983, Benassi et al[1] used the P_N method and an exact particular solution (for a simple inhomogeneous source term) to compute the partial heat fluxes for a class of radiative transfer problems. Although the solution formulated in Ref. 1 refers to a very general case, that work[1] still requires a particular solution of the equation of transfer for the case of a general inhomogeneous source term. In this brief communication, the particular solution that is required in a P_N solution of the radiative transfer problem (for the case of a general inhomogeneous term) is given.

We consider the equation of transfer[1,2]

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau, \mu')\,d\mu' + (1 - \varpi) \frac{\sigma}{\pi} T^4(\tau), \tag{1}$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$, and the boundary conditions

$$I(0, \mu) = \varepsilon_1 \frac{\sigma}{\pi} T_1^4 + \rho_1^s I(0, -\mu) + 2\rho_1^d \int_0^1 I(0, -\mu')\mu'\,d\mu', \quad \mu > 0, \tag{2a}$$

and

$$I(\tau_0, -\mu) = \varepsilon_2 \frac{\sigma}{\pi} T_2^4 + \rho_2^s I(\tau_0, \mu) + 2\rho_2^d \int_0^1 I(\tau_0, \mu')\mu'\,d\mu', \quad \mu > 0, \tag{2b}$$

where T_1 and T_2 refer to the two boundary temperatures, ρ_α^s and ρ_α^d, $\alpha = 1$ and 2, are the coefficients for specular and diffuse reflection and ε_1 and ε_2 are the emissivities. We consider here that the temperature distribution in the medium $T(\tau)$ is specified.

As in Ref. 1 we express our P_N approximation, for N odd, to $I(\tau, \mu)$ in the form[3]

$$I(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \{A_j \exp(-\tau/\xi_j) + (-1)^l B_j \exp[-(\tau_0 - \tau)/\xi_j]\} g_l(\xi_j) + I_p(\tau, \mu) \tag{3}$$

where $I_p(\tau, \mu)$ denotes a particular solution of Eq. (1) corresponding to the inhomogeneous source term

$$S(\tau) = (1 - \varpi) \frac{\sigma}{\pi} T^4(\tau). \tag{4}$$

Here, as discussed in Refs. 1 and 3, the Chandrasekhar polynomials are denoted by $g_l(\xi)$, the P_N eigenvalues are $\xi_j, j = 1, 2, \ldots, J = (N+1)/2$ and the coefficients A_j and B_j are constants that are to be fixed by the boundary conditions.

A PARTICULAR SOLUTION

We note that in 1975, Roux et al[4] used the method of variation of parameters to find a particular solution appropriate to a discrete ordinates solution of Eq. (1), and recently one of the authors (JRT), in a work[5] that reports a computational solution to a combined conductive and radiative heat transfer problem, used the same method to find a particular solution basic to a P_N solution of Eq. (1). Here we report a variation of Thomas' particular solution, and we provide an explicit solution to the encountered system of linear algebraic equations. We also make available a couple of expressions we consider necessary to complete (for the case of an arbitrary inhomogeneous source term) the work reported in Ref. 1.

We now note Eq. (3) and, as suggested[4] by the method of variation of parameters, propose

$$I_p(\tau,\mu)=\sum_{l=0}^{N}\frac{2l+1}{2}P_l(\mu)\sum_{j=1}^{J}\{U_j(\tau)\exp(-\tau/\xi_j)+(-1)^l V_j(\tau)\exp[-(\tau_0-\tau)/\xi_j]\}g_l(\xi_j) \tag{5}$$

where $U_j(\tau)$ and $V_j(\tau)$, $j=1,2,\ldots,J$, are to be found. If we substitute Eq. (5) into Eq. (1), multiply the resulting equation by $P_\beta(\mu)$, for $\beta=0,1,\ldots,N$, and integrate over μ from -1 to 1, we can solve the resulting system of first order differential equations to find $U_j(\tau)$ and $V_j(\tau)$, $j=1,2,\ldots,J$, and subsequently to find

$$I_p(\tau,\mu)=\sum_{l=0}^{N}\frac{2l+1}{2}P_l(\mu)\sum_{j=1}^{J}\frac{C_j}{\xi_j}\left\{\int_0^\tau S(x)\exp[-(\tau-x)/\xi_j]\,dx+(-1)^l\times\int_\tau^{\tau_0} S(x)\exp[-(x-\tau)/\xi_j]\,dx\right\}g_l(\xi_j) \tag{6}$$

where the constants C_j, $j=1,2,\ldots,J$, are the solutions to the system of linear algebraic equations

$$(1-\varpi)\sum_{j=1}^{J}g_{2k-2}(\xi_j)C_j=\delta_{k,1},\quad k=1,2,\ldots,J. \tag{7}$$

We note that both Roux et al[4] and Thomas[5] used numerical linear algebra techniques to solve systems of equations somewhat similar to Eq. (7); however, by noting some basic properties of the eigenvalues $\{\xi_j\}$ and the Chandrasekhar polynomials $\{g_l(\xi_j)\}$, we find that we can solve Eq. (7) analytically to find explicit results for the required constants $\{C_j\}$. In order to develop the required solution of Eq. (7), we first let $\mathbf{G}_j$ be a column vector with components $g_{2k-2}(\xi_j)$, for $k=1,2,\ldots,J$, and note[1,3] the eigenvalue problem

$$\mathbf{A}\mathbf{G}_j=\xi_j^2\mathbf{G}_j. \tag{8}$$

Here the tridiagonal matrix $\mathbf{A}$ has super-diagonal, diagonal and sub-diagonal elements given respectively by

$$A_{k,k+1}=\frac{(2k-1)(2k)}{h_{2k-2}h_{2k-1}},\quad k=1,2,\ldots,J-1, \tag{9a}$$

$$A_{k,k}=\frac{1}{h_{2k-2}}\left[\frac{(2k-1)^2}{h_{2k-1}}+\frac{4(k-1)^2}{h_{2k-3}}\right],\quad k=1,2,\ldots,J, \tag{9b}$$

and

$$A_{k+1,k}=\frac{(2k-1)(2k)}{h_{2k-1}h_{2k}},\quad k=1,2,\ldots,J-1, \tag{9c}$$

with

$$h_l=2l+1-\varpi\beta_l,\quad l=0,1,\ldots,L, \tag{10a}$$

and

$$h_l=2l+1,\quad l>L. \tag{10b}$$

If we let

$$\mathbf{S}=\text{diag}\{\sqrt{h_0},\sqrt{h_2},\ldots,\sqrt{h_{N-1}}\} \tag{11}$$

and $\mathbf{X}_j = n_j\mathbf{S}\mathbf{G}_j$, we can rewrite Eq. (8) as

$$\mathbf{B}\mathbf{X}_j = \xi_j^2\mathbf{X}_j \tag{12}$$

where

$$\mathbf{B} = \mathbf{S}\mathbf{A}\mathbf{S}^{-1} \tag{13}$$

is tridiagonal and symmetric. Noting that the ξ_j^2 are distinct,[6,7] we conclude that the eigenvectors $\mathbf{X}_j$ form an orthogonal set that we make orthonormal by taking

$$n_j = \left(\sum_{k=1}^{J} g_{2k-2}^2(\xi_j)h_{2k-2}\right)^{-1/2}, \quad j = 1, 2, \ldots, J. \tag{14}$$

It follows that if we let $\mathbf{G}$ and $\mathbf{X}$ be $J \times J$ matrices with columns $\mathbf{G}_j$ and $\mathbf{X}_j, j = 1, 2, \ldots, J$, and we let $\mathbf{N}$ be a $J \times J$ diagonal matrix with elements $n_j, j = 1, 2, \ldots, J$, then we can write, since $\mathbf{G} = \mathbf{S}^{-1}\mathbf{X}\mathbf{N}^{-1}$ and $\mathbf{X}^{-1} = \mathbf{X}^T$,

$$\mathbf{G}^{-1} = \mathbf{N}\mathbf{X}^T\mathbf{S}. \tag{15}$$

It follows that we can use Eq. (15) to solve Eq. (7) to find

$$C_j = \left(\sum_{k=1}^{J} g_{2k-2}^2(\xi_j)h_{2k-2}\right)^{-1}, \quad j = 1, 2, \ldots, J. \tag{16}$$

Equations (6) and (16) define our explicit form of the required particular solution.

We can, of course, use Eq. (4) and rewrite Eq. (6) as

$$I_p(\tau, \mu) = (1 - \varpi)\frac{\sigma}{\pi}\sum_{l=0}^{N}\frac{2l+1}{2}P_l(\mu)\sum_{j=1}^{J}\frac{C_j}{\xi_j}\left\{\int_0^\tau T^4(x)\exp[-(\tau - x)/\xi_j]\,dx + (-1)^l \times \int_\tau^{\tau_0} T^4(x)\exp[-(x - \tau)/\xi_j]\,dx\right\}g_l(\xi_j). \tag{17}$$

If we let

$$q_p(\tau) = 2\pi\int_{-1}^{1} I_p(\tau, \mu)\mu\,d\mu \tag{18}$$

and

$$q_p^{\pm}(\tau) = 2\pi\int_0^1 I_p(\tau, \pm\mu)\mu\,d\mu \tag{19}$$

denote the particular solution components of the net and partial heat fluxes, then we can integrate Eq. (17) to find

$$q_p(\tau) = 2\sigma(1 - \varpi)^2\sum_{j=1}^{J} C_j\left\{\int_0^\tau T^4(x)\exp[-(\tau - x)/\xi_j]\,dx - \int_\tau^{\tau_0} T^4(x)\exp[-(x - \tau)/\xi_j]\,dx\right\} \tag{20}$$

and

$$q_p^{\pm}(\tau) = \sigma(1 - \varpi)\sum_{l=0}^{N}(2l+1)(\pm 1)^l S_{0,l}\sum_{j=1}^{J}\frac{C_j}{\xi_j}\left\{\int_0^\tau T^4(x)\exp[-(\tau - x)/\xi_j]\,dx + (-1)^l \times \int_\tau^{\tau_0} T^4(x)\exp[-(x - \tau)/\xi_j]\,dx\right\}g_l(\xi_j) \tag{21}$$

where, as defined in Ref. 1,

$$S_{\alpha,l} = \int_0^1 P_{2\alpha+1}(\mu)P_l(\mu)\,d\mu. \tag{22}$$

To conclude this work we note that Refs. 1 and 3 have used a very convenient and accurate algorithm to compute the P_N eigenvalues and that Garcia and Siewert[8] have recently reported an especially accurate algorithm for computing the Chandrasekhar polynomials.

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