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A COMPUTATIONAL METHOD FOR SOLVING A CLASS OF COUPLED CONDUCTIVE–RADIATIVE HEAT TRANSFER PROBLEMS

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Abstract—The P_N method, also called the spherical harmonics method, is used along with Hermite cubic splines to define an iterative technique for solving a class of nonlinear problems in radiative transfer. Anisotropic scattering and specularly and diffusely reflecting boundaries are allowed for the steady-state, combined-mode, conductive–radiative, heat transfer problem considered. Computational aspects of the technique are discussed, and the method is used to establish the reported numerical results.

1. INTRODUCTION

In a paper published in 1983, Benassi et al[1] used a computationally stable version of the P_N method[2] to compute the partial heat fluxes for a class of classical radiative transfer problems. In a more recent work Siewert and Thomas[3] reported a concise result for the particular solution required for the P_N method when the equation of transfer contains an inhomogeneous source term. Here we use these two previous works[1,3] and Hermite cubic splines to solve the steady-state problem in combined-mode (conduction and radiation) heat transfer that has been formulated by Özişik.[4] As Özişik[4] has reviewed carefully the numerous works that have contributed to this field of study, we do not repeat a review here.

We consider the equation of transfer written[4] as

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau, \mu')\, d\mu' + (1 - \varpi) \frac{\sigma n^2}{\pi} T^4(\tau), \tag{1}$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$, and the boundary conditions

$$I(0, \mu) = \varepsilon_1 \frac{\sigma n^2}{\pi} T_1^4 + \rho_1^s I(0, -\mu) + 2\rho_1^d \int_0^1 I(0, -\mu')\mu'\, d\mu' \tag{2a}$$

and

$$I(\tau_0, -\mu) = \varepsilon_2 \frac{\sigma n^2}{\pi} T_2^4 + \rho_2^s I(\tau_0, \mu) + 2\rho_2^d \int_0^1 I(\tau_0, \mu')\mu'\, d\mu' \tag{2b}$$

for $\mu \in [0, 1]$. Here $\tau \in [0, \tau_0]$ is the optical variable, μ is the direction cosine measured from the positive τ axis and ϖ is the albedo for single scattering. In addition, we have assumed that the scattering law $p(\Theta)$ can be represented by a finite Legendre expansion in terms of the cosine of the scattering angle Θ, i.e.

$$p(\Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos \Theta) \tag{3}$$

where $\beta_0 = 1$ and $|\beta_l| < 2l + l$ for $l \geqslant 1$. In regard to the boundary conditions, we note that ρ_α^s and ρ_α^d, for $\alpha = 1$ and 2, are coefficients for specular and diffuse reflection and that $\varepsilon_\alpha = 1 - \rho_\alpha^s - \rho_\alpha^d$, $\alpha = 1$ and 2, are the emissivities for the two surfaces. In addition n is the index of refraction and σ is the Stefan–Boltzmann constant.

The nonlinear aspect of this problem comes from the fact that the temperature distribution $T(\tau)$ appearing in Eq. (1) must satisfy the heat conduction equation[4]

$$k\beta \frac{\mathrm{d}^2}{\mathrm{d}\tau^2} T(\tau) = \frac{\mathrm{d}}{\mathrm{d}\tau} q_r(\tau) \tag{4}$$

subject to the boundary conditions $T(0) = T_1$ and $T(\tau_0) = T_2$, where T_1 and T_2 are the temperatures that also appear in Eqs. (2). In addition, k is the thermal conductivity of the medium, β is the extinction coefficient and $q_r(\tau)$ is the radiative heat flux, i.e.

$$q_r(\tau) = 2\pi \int_{-1}^{1} I(\tau, \mu')\mu' \, \mathrm{d}\mu'. \tag{5}$$

Our general approach to the solution of the given problem is the same as that of Lii and Özişik[5] and that of Thomas,[6] viz. we assume an initial temperature distribution $T(\tau)$, solve the radiation problem to get the radiative heat flux $q_r(\tau)$ and use that result in the conduction equation which we subsequently solve to get a new temperature distribution. We then repeat this procedure and consider that we have the solution if there appears to be convergence for the desired quantities.

Having stated the general approach to be used here, we note that there are two major issues that should be addressed. First of all, for this problem there are, to the best of our knowledge, no existence or uniqueness theorems that state the conditions for which there is a solution and, if the solution exists, when it is unique. In addition and specific to our method of solution, we do not have proof that the method converges to the desired results. Anticipating that these two matters will be addressed in later works, we proceed to develop our solution and to report some numerical results.

2. BASIC DEVELOPMENT

To follow a tradition in the heat transfer literature,[4] we normalize the problem by introducing a convenient reference temperature T_r and by using

$$I(\tau, \mu) = \left(\frac{\sigma n^2}{\pi} T_r^4\right) I^*(\tau, \mu), \tag{6}$$

$$q_r(\tau) = \left(\frac{\sigma n^2}{\pi} T_r^4\right) q_r^*(\tau) \tag{7}$$

and

$$T(\tau) = T_r \Theta(\tau) \tag{8}$$

to rewrite our problem as

$$\mu \frac{\partial}{\partial \tau} I^*(\tau, \mu) + I^*(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I^*(\tau, \mu') \, \mathrm{d}\mu' + (1 - \varpi)\Theta^4(\tau), \tag{9}$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$,

$$I^*(0, \mu) = \varepsilon_1 \Theta_1^4 + \rho_1^s I^*(0, -\mu) + 2\rho_1^d \int_0^1 I^*(0, -\mu')\mu' \, \mathrm{d}\mu' \tag{10a}$$

and

$$I^*(\tau_0, -\mu) = \varepsilon_2 \Theta_2^4 + \rho_2^s I^*(\tau_0, \mu) + 2\rho_2^d \int_0^1 I^*(\tau_0, \mu')\mu' \, \mathrm{d}\mu' \tag{10b}$$

for $\mu \in [0, 1]$. In addition

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \Theta(\tau) = \frac{1}{4\pi N_c} \frac{\mathrm{d}}{\mathrm{d}\tau} q_r^*(\tau), \tag{11}$$

with

$$\Theta(0) = \Theta_1 = \frac{T_1}{T_r}, \tag{12a}$$

$$\Theta(\tau_0) = \Theta_2 = \frac{T_2}{T_r} \tag{12b}$$

and

$$q_r^*(\tau) = 2\pi \int_{-1}^{1} I^*(\tau, \mu')\mu' \, d\mu'. \tag{13}$$

Here

$$N_c = \frac{k\beta}{4\sigma n^2 T_r^3} \tag{14}$$

is called the conduction-to-radiation parameter.[4]

Having formulated the problem in a convenient way, we follow Refs. 1 and 2 and express our P_N approximation to $I^*(\tau, \mu)$, for N odd, in the form

$$I^*(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \{A_j \exp(-\tau/\xi_j) + (-1)^l B_j \exp[-(\tau_0 - \tau)/\xi_j]\} g_l(\xi_j) + I_p^*(\tau, \mu) \tag{15}$$

where $I_p^*(\tau, \mu)$ denotes a particular solution of Eq. (9) corresponding to the inhomogeneous source term

$$S(\tau) = (1 - \varpi)\Theta^4(\tau). \tag{16}$$

Here the Chandrasekhar polynominals[7] are denoted by $g_l(\xi)$ and the coefficients A_j and B_j are constants that are to be fixed by the boundary conditions. The P_N eigenvalues are $\xi_j, j = 1, 2, \dots, J = (N+1)/2$. In addition, we follow Ref. 3 and express the particular solution as

$$I_p^*(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \frac{C_j}{\xi_j} \left\{ \int_0^{\tau} S(x) \exp[-(\tau - x)/\xi_j] \, dx + (-1)^l \int_{\tau}^{\tau_0} S(x) \exp[-(x - \tau)/\xi_j] \, dx \right\} g_l(\xi_j) \tag{17}$$

where the constants $C_j, j = 1, 2, \dots, J$, are given by

$$C_j = \left(\sum_{k=1}^{J} g_{2k-2}^2(\xi_j) h_{2k-2} \right)^{-1}, \quad j = 1, 2, \dots, J, \tag{18}$$

with

$$h_l = 2l + 1 - \varpi\beta_l, \quad l = 0, 1, \dots, L, \tag{19a}$$

and

$$h_l = 2l + 1, \quad l > L. \tag{19b}$$

Following Lii and Özişik,[5] we consider for the moment that $q_r^*(\tau)$ is known and we express the solution to Eq. (11), subject to the boundary conditions given by Eqs. (12), as

$$\Theta(\tau) = \Theta_1 + \frac{\tau}{\tau_0}(\Theta_2 - \Theta_1) + \frac{1}{4\pi N_c}\left(\int_0^{\tau} q_r^*(x) \, dx - \frac{\tau}{\tau_0} \int_0^{\tau_0} q_r^*(x) \, dx \right). \tag{20}$$

In addition, we can integrate Eqs. (15) and (17) to find

$$q_r^*(\tau) = 2\pi(1 - \varpi) \sum_{j=1}^{J} \langle \xi_j \{A_j \exp(-\tau/\xi_j) - B_j \exp[-(\tau_0 - \tau)/\xi_j]\} + C_j[U_j(\tau) - V_j(\tau)] \rangle \tag{21}$$

where

$$U_j(\tau) = \int_0^{\tau} S(x) \exp[-(\tau - x)/\xi_j] \, dx \tag{22a}$$

and

$$V_j(\tau) = \int_{\tau}^{\tau_0} S(x)\exp[-(x-\tau)/\xi_j]\,dx \tag{22b}$$

with $S(\tau)$ as defined by Eq. (16). We can now substitute Eq. (21) into Eq. (20) to find

$$\Theta(\tau) = H_1 + \frac{\tau}{\tau_0} H_2 - \frac{(1-\varpi)}{2N_c} \sum_{j=1}^{J} \langle \xi_j^2 \{A_j \exp(-\tau/\xi_j) + B_j \exp[-(\tau_0-\tau)/\xi_j]\} + C_j\xi_j[U_j(\tau) + V_j(\tau)]\rangle \tag{23}$$

where

$$H_1 = \Theta_1 + \frac{(1-\varpi)}{2N_c} \sum_{j=1}^{J} \{\xi_j^2[A_j + B_j\exp(-\tau_0/\xi_j)] + C_j\xi_j V_j(0)\} \tag{24}$$

and

$$H_2 = \Theta_2 - \Theta_1 - \frac{(1-\varpi)}{2N_c} \sum_{j=1}^{J} \{\xi_j^2(A_j - B_j)[1-\exp(-\tau_0/\xi_j)] + C_j\xi_j[V_j(0) - U_j(\tau_0)]\}. \tag{25}$$

It is clear that we can, at least in principle, now proceed in the following iterative manner. We start with an initial normalized temperature distribution obtained, for example, by ignoring the integral term in Eq. (20); next we use the initial normalized temperature distribution to define, by way of Eq. (16), the source term $S(\tau)$ and subsequently the functions $U_j(\tau)$ and $V_j(\tau)$. Since the particular solution, as given by Eq. (17), has at this point become known, we can substitute Eq. (15) into the boundary conditions given by Eqs. (10) and use a projection technique[1,2] to define a set of linear algebraic equations that can be solved to yield the required constants A_j and $B_j, j = 1, 2, \ldots, J$. These constants and the previously defined $U_j(\tau)$ and $V_j(\tau)$ can now be used in Eq. (23) to give the next normalized temperature iterate.

Before turning to the numerical methods that we use to implement the iterative technique, we define some additional physical quantities we wish to compute. Noting the definitions[4] of the conductive, radiative and total heat fluxes, viz.

$$q_c(\tau) = -k\beta \frac{d}{d\tau} T(\tau), \tag{26a}$$

$$q_r(\tau) = 2\pi \int_{-1}^{1} I(\tau,\mu)\mu\,d\mu \tag{26b}$$

and

$$q(\tau) = q_c(\tau) + q_r(\tau), \tag{26c}$$

we can use Eqs. (21) and (23) to write

$$\frac{q_c(\tau)}{k\beta T_r} = -\frac{H_2}{\tau_0} - \frac{1}{4\pi N_c} q_r^*(\tau), \tag{27a}$$

$$\frac{q_r(\tau)}{k\beta T_r} = \frac{1}{4\pi N_c} q_r^*(\tau) \tag{27b}$$

and

$$\frac{q(\tau)}{k\beta T_r} = -\frac{H_2}{\tau_0} \tag{27c}$$

where H_2 is given by Eq. (25) and $q_r^*(\tau)$ is given by Eq. (21).

3. NUMERICAL METHODS

We first note that the P_N eigenvalues $\{\xi_j\}$ can be computed accurately and efficiently as described in Refs. 1 and 2. In addition Garcia and Siewert[8] have recently reported very precise methods for computing the Chandrasekhar polynominals $\{g_l(\xi_j)\}$ for both the P_N method and the F_N method.[9]

In order to find the constants $\{A_j\}$ and $\{B_j\}$ required in Eq. (15) to define the solution $I^*(\tau, \mu)$ at each step in the iteration procedure, we substitute Eq. (15), with $I_p^*(\tau, \mu)$ given by Eq. (17), into Eqs. (10) and use the Marshak projection scheme[1] to obtain, for $\alpha = 0, 1, \ldots, (N-1)/2$, the system of linear algebraic equations

$$\sum_{j=1}^{J}\sum_{l=0}^{N}\frac{2l+1}{2}\{[1-(-1)^l\rho_1^s]S_{\alpha,l}-2(-1)^l\rho_1^d S_{0,l}S_{\alpha,0}\}[A_j+(-1)^l B_j\exp(-\tau_0/\xi_j)]g_l(\xi_j)=R_{1,\alpha} \quad (28a)$$

and

$$\sum_{j=1}^{J}\sum_{l=0}^{N}\frac{2l+1}{2}\{[1-(-1)^l\rho_2^s]S_{\alpha,l}-2(-1)^l\rho_2^d S_{0,l}S_{\alpha,0}\}[B_j+(-1)^l A_j\exp(-\tau_0/\xi_j)]g_l(\xi_j)=R_{2,\alpha}. \quad (28b)$$

Here

$$R_{1,\alpha}=\varepsilon_1\Theta_1^4 S_{\alpha,0}+\sum_{j=1}^{J}\sum_{l=0}^{N}\frac{2l+1}{2}\{2\rho_1^d S_{0,l}S_{\alpha,0}+[\rho_1^s-(-1)^l]S_{\alpha,l}\}\frac{C_j}{\xi_j}V_j(0)g_l(\xi_j) \quad (29a)$$

and

$$R_{2,\alpha}=\varepsilon_2\Theta_2^4 S_{\alpha,0}+\sum_{j=1}^{J}\sum_{l=0}^{N}\frac{2l+1}{2}\{2\rho_2^d S_{0,l}S_{\alpha,0}+[\rho_2^s-(-1)^l]S_{\alpha,l}\}\frac{C_j}{\xi_j}U_j(\tau_0)g_l(\xi_j) \quad (29b)$$

where as discussed in Ref. 1

$$S_{\alpha,l}=\int_0^1 P_{2\alpha+1}(\mu)P_l(\mu)\,d\mu. \quad (30)$$

To start our iterative solution we use the initial normalized temperature distribution

$$\Theta_0(\tau)=\Theta_1+(\Theta_2-\Theta_1)\frac{\tau}{\tau_0} \quad (31)$$

and Eqs. (16) and (22) to define the initial values of the functions $U_j(\tau)$ and $V_j(\tau)$. We next solve Eqs. (28) to find the first estimates of the constants $\{A_j\}$ and $\{B_j\}$, and these results are then used in Eq. (23) to define the new normalized temperature distribution. Although we can use Eq. (23) as it is written, we prefer, in order to save some computation time, to use Hermite cubic splines to interpolate Eq. (23).

To define our Hermite cubic splines, we first of all take the $M+1$ knots to be

$$\zeta_\alpha=\left(\frac{\alpha}{M}\right)^2 \quad (32)$$

for $\alpha = 0, 1, \ldots, M$. Since for the Hermite cubic splines there are two basis functions associated with each knot we write

$$\Theta(\tau)=\sum_{\alpha=0}^{\mathcal{N}}a_\alpha\phi_\alpha(\tau/\tau_0) \quad (33)$$

where $\mathcal{N} = 2M + 1$ and

$$\phi_{2\beta}(\mu)=\Phi_\beta(\mu) \quad \text{and} \quad \phi_{2\beta+1}(\mu)=\Psi_\beta(\mu), \quad (34a \text{ and } b)$$

for $\beta = 0, 1, \ldots, M$. To define the Hermite basis functions $\Phi_\alpha(\mu)$ and $\Psi_\alpha(\mu)$ we let $d_\alpha = \zeta_\alpha - \zeta_{\alpha-1}$, follow the notation of a previous paper,[10] make use of the representations given by Schultz[11] and write

$$\Phi_0(\mu)=\begin{cases}3(\zeta_1-\mu)^2/d_1^2-2(\zeta_1-\mu)^3/d_1^3, & \text{for } \mu\in[0,\zeta_1],\\ 0, & \text{otherwise,}\end{cases} \quad (35a)$$

$$\Phi_\alpha(\mu)=\begin{cases}3(\mu-\zeta_{\alpha-1})^2/d_\alpha^2-2(\mu-\zeta_{\alpha-1})^3/d_\alpha^3, & \text{for } \mu\in[\zeta_{\alpha-1},\zeta_\alpha],\\ 3(\zeta_{\alpha+1}-\mu)^2/d_{\alpha+1}^2-2(\zeta_{\alpha+1}-\mu)^3/d_{\alpha+1}^3, & \text{for } \mu\in[\zeta_\alpha,\zeta_{\alpha+1}],\\ 0, & \text{otherwise,}\end{cases} \quad (35b)$$

for $\alpha = 1, 2, \ldots M-1$,

$$\Phi_M(\mu) = \begin{cases} 3(\mu - \zeta_{M-1})^2/d_M^2 - 2(\mu - \zeta_{M-1})^3/d_M^3, & \text{for } \mu \in [\zeta_{M-1}, 1], \\ 0, & \text{otherwise}, \end{cases} \tag{35c}$$

$$\Psi_0(\mu) = \begin{cases} \mu(\zeta_1 - \mu)^2/d_1^2, & \text{for } \mu \in [0, \zeta_1], \\ 0, & \text{otherwise}, \end{cases} \tag{36a}$$

$$\Psi_\alpha(\mu) = \begin{cases} (\mu - \zeta_\alpha)(\mu - \zeta_{\alpha-1})^2/d_\alpha^2, & \text{for } \mu \in [\zeta_{\alpha-1}, \zeta_\alpha], \\ (\mu - \zeta_\alpha)(\zeta_{\alpha+1} - \mu)^2/d_{\alpha-1}^2, & \text{for } \mu \in [\zeta_\alpha, \zeta_{\alpha+1}], \\ 0, & \text{otherwise}, \end{cases} \tag{36b}$$

for $\alpha = 1, 2, \ldots, M-1$, and

$$\Psi_M(\mu) = \begin{cases} (\mu - 1)(\mu - \zeta_{M-1})^2/d_M^2, & \text{for } \mu \in [\zeta_{M-1}, 1], \\ 0, & \text{otherwise}. \end{cases} \tag{36c}$$

It follows from the definitions of the Hermite cubic splines that the coefficients in Eq. (33) are given by

$$a_{2\alpha} = \Theta(\tau)\big|_{\tau = \zeta_\alpha \tau_0} \tag{37a}$$

and

$$a_{2\alpha+1} = \tau_0 \frac{\mathrm{d}}{\mathrm{d}\tau} \Theta(\tau)\big|_{\tau = \zeta_\alpha \tau_0} \tag{37b}$$

for $\alpha = 0, 1, \ldots, M$. Since we have Eq. (23) that defines the normalized temperature and

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \Theta(\tau) = \frac{H_2}{\tau_0} + \frac{1}{4\pi N_c} q_r^*(\tau), \tag{38}$$

where $q_r^*(\tau)$ and H_2 are given by Eqs. (21) and (25), we can use Eqs. (37) to compute the constants $\{a_\alpha\}$ required in our spline representation of the normalized temperature distribution.

We note that we are now left only with the task of evaluating the functions $U_j(\tau)$ and $V_j(\tau)$ defined by Eqs. (22). Since we are using a spline representation of the temperature distribution we could, in fact, evaluate the integrals in Eqs. (22) analytically; however, for the current version of our algorithm we use a standard Gauss quadrature scheme and evaluate the integrals by numerical integration.

Before turning to the next section in which we report some numerical results, we record a few remarks. First of all, as an alternative to using the Hermite splines to represent the temperature distribution, we have also carried out some calculations where we represented the inhomogeneous source term, as given by Eq. (16), by the Hermite splines. Of course, if we intend to evaluate the integrals in Eqs. (22) analytically, then using splines for $S(\tau)$ rather than $\Theta(\tau)$ would make that task easier. For the few problems we considered, we did not see any real difference, from a numerical point-of-view, between these two usages of the splines.

In regard to the (outer) iterations between the equation of transfer and the heat conduction equation, we note that we have added an inner iteration step to improve the convergence of the method. Thus at each step in the outer iteration process we solve Eq. (23) iteratively, since the functions $U_j(\tau)$ and $V_j(\tau)$ depend on $\Theta(\tau)$, to find a new temperature $\Theta(\tau)$.

For some difficult problems we have also tried to use, as have Jia and Yener,[12] relaxation techniques[13] to keep the calculation from diverging.

4. NUMERICAL RESULTS

To have a specific scattering law for testing our solution technique, and to avoid having to provide a table of the scattering law coefficients $\{\beta_l\}$, we use here the binomial scattering law[14]

$$p(\cos\Theta) = \frac{L+1}{2^L}(1 + \cos\Theta)^L \tag{39}$$

Table 1. Physical data for different problems.

Problem	ϵ_1	ϵ_2	ρ_1^s	ρ_2^s	ρ_1^d	ρ_2^d	Θ_1	Θ_2	ϖ	τ_0	N_c	L
1	1.0	1.0	0.0	0.0	0.0	0.0	1.0	0.0	0.9	1.0	0.05	0
2	1.0	1.0	0.0	0.0	0.0	0.0	1.0	0.5	0.9	1.0	0.05	0
3	0.7	0.6	0.1	0.3	0.2	0.1	1.0	0.5	0.9	3.0	0.05	0
4	1.0	1.0	0.0	0.0	0.0	0.0	1.0	0.5	0.95	1.0	0.05	299
5	0.6	0.4	0.1	0.2	0.3	0.4	1.0	0.5	0.95	1.0	0.05	299
6	0.8	0.8	0.1	0.1	0.1	0.1	1.0	0.5	0.99	3.0	0.05	299

which can be represented exactly with $L + 1$ Legendre coefficients that can be computed with $\beta_0 = 1$ and[15]

$$\beta_l = \left(\frac{2l+1}{2l-1}\right)\left(\frac{L+1-l}{L+1+l}\right)\beta_{l-1}. \tag{40}$$

As we wish to make available some numerical results that have been obtained with the methods discussed here, we consider the six test problems defined in Table 1. Problem 1 was originally defined and solved by Lii and Özişik,[5] and the normalized temperature distribution for problem 2 is given in graphical form in Ref. 4. Problem 3 is a version of problem 2 that allows for reflecting boundaries. As problems 1, 2 and 3 are for an isotropic scattering model, we have elected to use the binomial scattering law with $L = 299$ for problems 4, 5 and 6.

Our converged results for the normalized temperature distribution and the normalized heat fluxes, defined from Eqs. (27) as

$$Q_c(\tau) = \frac{q_c(\tau)}{k\beta T_r} = -\frac{H_2}{\tau_0} - \frac{1}{4\pi N_c} q_r^*(\tau), \tag{41a}$$

$$Q_r(\tau) = \frac{q_r(\tau)}{k\beta T_r} = \frac{1}{4\pi N_c} q_r^*(\tau) \tag{41b}$$

and

$$Q(\tau) = \frac{q(\tau)}{k\beta T_r} = -\frac{H_2}{\tau_0}, \tag{41c}$$

Table 2. Normalized temperature distribution and heat fluxes for problem 1.

τ/τ_0	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0.00	1.0	8.37894(−1)	2.96126	3.79915
0.10	9.18027(−1)	8.08916(−1)	2.99024	3.79915
0.20	8.36956(−1)	8.17843(−1)	2.98131	3.79915
0.30	7.53557(−1)	8.53900(−1)	2.94525	3.79915
0.40	6.65558(−1)	9.08530(−1)	2.89062	3.79915
0.50	5.71475(−1)	9.74454(−1)	2.82470	3.79915
0.60	4.70505(−1)	1.04528	2.75387	3.79915
0.70	3.62437(−1)	1.11560	2.68356	3.79915
0.80	2.47544(−1)	1.18120	2.61795	3.79915
0.90	1.26449(−1)	1.23928	2.55987	3.79915
1.00	0.0	1.28795	2.51121	3.79915

Table 3. Normalized temperature distribution and heat fluxes for problem 2.

τ/τ_0	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0.00	1.0	4.76641(−1)	2.65452	3.13116
0.10	9.54270(−1)	4.41642(−1)	2.68952	3.13116
0.20	9.11003(−1)	4.26559(−1)	2.70460	3.13116
0.30	8.68433(−1)	4.27284(−1)	2.70388	3.13116
0.40	8.25127(−1)	4.40737(−1)	2.69042	3.13116
0.50	7.79940(−1)	4.64560(−1)	2.66660	3.13116
0.60	7.31936(−1)	4.96753(−1)	2.63441	3.13116
0.70	6.80375(−1)	5.35397(−1)	2.59576	3.13116
0.80	6.24709(−1)	5.78464(−1)	2.55270	3.13116
0.90	5.64610(−1)	6.23663(−1)	2.50750	3.13116
1.00	5.0(−1)	6.68074(−1)	2.46309	3.13116

Table 4. Normalized temperature distribution and heat fluxes for problem 3.

τ/τ_0	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0.00	1.0	2.00827(−1)	1.09931	1.30014
0.10	9.52498(−1)	1.26820(−1)	1.17332	1.30014
0.20	9.19507(−1)	9.79176(−2)	1.20222	1.30014
0.30	8.91715(−1)	8.97962(−2)	1.21034	1.30014
0.40	8.64462(−1)	9.35590(−2)	1.20658	1.30014
0.50	8.34612(−1)	1.07116(−1)	1.19302	1.30014
0.60	7.99059(−1)	1.32088(−1)	1.16805	1.30014
0.70	7.53816(−1)	1.72532(−1)	1.12761	1.30014
0.80	6.93470(−1)	2.33627(−1)	1.06651	1.30014
0.90	6.11200(−1)	3.18904(−1)	9.81236(−1)	1.30014
1.00	5.0(−1)	4.24849(−1)	8.75291(−1)	1.30014

Table 5. Normalized temperature distribution and heat fluxes for problem 4.

τ/τ_0	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0.00	1.0	5.06896(−1)	4.46433	4.97123
0.10	9.51076(−1)	4.74292(−1)	4.49694	4.97123
0.20	9.04648(−1)	4.56516(−1)	4.51471	4.97123
0.30	8.59367(−1)	4.50940(−1)	4.52029	4.97123
0.40	8.14121(−1)	4.55549(−1)	4.51568	4.97123
0.50	7.67971(−1)	4.68769(−1)	4.50246	4.97123
0.60	7.20124(−1)	4.89290(−1)	4.48194	4.97123
0.70	6.69908(−1)	5.15953(−1)	4.45528	4.97123
0.80	6.16765(−1)	5.47669(−1)	4.42356	4.97123
0.90	5.60242(−1)	5.83354(−1)	4.38788	4.97123
1.00	5.0(−1)	6.21826(−1)	4.34940	4.97123

are given in Tables 2–7. Having varied the order of the P_N approximation, the number of Hermite splines used and the number of Gauss points used to evaluate the $U_j(\tau)$ and $V_j(\tau)$ functions, we have some confidence that the results given in Tables 2–7 are correct to within one unit in the last digit given. In comparing our results for problem 1 with those of Lii and Özişik,[5] we found general agreement to within one digit in the fifth significant figure; however, we did find one result where we differed by three digits in the fourth significant figure.

To conclude this work we would like to record a few remarks concerning matters that are still unresolved. First of all as mentioned in Sec. 1, there are, to our knowledge, no existence and/or

Table 6. Normalized temperature distribution and heat fluxes for problem 5.

τ/τ_0	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0.00	1.0	4.73502(−1)	1.57059	2.04409
0.10	9.54143(−1)	4.46344(−1)	1.59774	2.04409
0.20	9.10237(−1)	4.34047(−1)	1.61004	2.04409
0.30	8.66919(−1)	4.34219(−1)	1.60987	2.04409
0.40	8.23037(−1)	4.45092(−1)	1.59900	2.04409
0.50	7.77590(−1)	4.65293(−1)	1.57880	2.04409
0.60	7.29706(−1)	4.93675(−1)	1.55041	2.04409
0.70	6.78617(−1)	5.29200(−1)	1.51489	2.04409
0.80	6.23661(−1)	5.70858(−1)	1.47323	2.04409
0.90	5.64275(−1)	6.17628(−1)	1.42646	2.04409
1.00	5.0(−1)	6.68457(−1)	1.37563	2.04409

Table 7. Normalized temperature distribution and heat fluxes for problem 6.

τ/τ_0	$\Theta(\tau)$	$Q_c(\tau)$	$Q_r(\tau)$	$Q(\tau)$
0.00	1.0	1.74205(−1)	3.01765	3.19186
0.10	9.50989(−1)	1.54176(−1)	3.03768	3.19186
0.20	9.06614(−1)	1.42965(−1)	3.04889	3.19186
0.30	8.64504(−1)	1.38831(−1)	3.05303	3.19186
0.40	8.22726(−1)	1.40597(−1)	3.05126	3.19186
0.50	7.79637(−1)	1.47456(−1)	3.04440	3.19186
0.60	7.33803(−1)	1.58809(−1)	3.03305	3.19186
0.70	6.83954(−1)	1.74142(−1)	3.01772	3.19186
0.80	6.28972(−1)	1.92936(−1)	2.99892	3.19186
0.90	5.67905(−1)	2.14600(−1)	2.97726	3.19186
1.00	5.0(−1)	2.38360(−1)	2.95350	3.19186

uniqueness theorems that apply directly to this problem, and of course it would be useful to know if this class of problems has been well formulated mathematically. Also as we have no proof that the straightforward iteration scheme we use actually converges, we can only conjecture that the results given in Tables 2–7 are actually correct. Finally we note that for the six problems considered here, we observed what appeared to be convergence toward the established temperature distribution; however we did encounter problems where the method failed to converge. In fact, for a fixed value of the coupling coefficient, say $N_c = 0.05$, we found, for example, that increasing the thickness τ_0 of the medium could cause a previously converging computation to diverge.

While it is clear that the numerical methods used in this work can be used to solve a broad class of combined-mode, radiation–conduction, heat transfer problems, we note that there are, in this class of problems, cases that we have not been able to solve. It is anticipated that more sophisticated iteration techniques will be investigated in future work.

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