J. Quant. Spectrosc. Radiat. Transfer Vol. 46, No. 2, pp. 63–72, 1991
Printed in Great Britain.
0022-4073/91 $3.00 + 0.00

ON COUPLED CONDUCTIVE–RADIATIVE HEAT-TRANSFER PROBLEMS IN A SPHERE

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(Received 11 December 1990)

Abstract—A sphere-to-plane transformation technique and the spherical harmonics method are used, along with Hermite cubic splines, to define an iterative technique for solving a class of nonlinear radiative transfer problems in a sphere. Computational aspects of the technique are discussed, and the method is used to establish numerical results for several test problems.

1. INTRODUCTION

In a paper published in 1975, Wu and Siewert[1] generalized a transformation technique reported by Mitsis[2] in order to reduce a class of radiation transport problems formulated in spherical geometry to more easily solved "pseudo" problems that have plane symmetry. Although the paper by Wu and Siewert[1] has been essentially overlooked by researchers in the radiative heat-transfer field, the transformation technique was used by Siewert and Grandjean[3] in 1979 to solve two problems, formulated in terms of neutron transport theory, that are of interest in the field of radiative transfer.[4]

It is clear that the technique of transforming from spherical problems to plane problems, as discussed by Wu and Siewert,[1] is not sufficiently general to solve all problems with spherical symmetry; however, problems in a solid sphere, with a diffusely reflecting surface, that are based on isotropic scattering can be solved in this manner for an arbitrary inhomogeneous source term and for an arbitrary distribution of radiation incident on the boundary.

In order to demonstrate the merits of the sphere-to-plane transformation technique,[1] we use the technique along with the spherical harmonics method[5–7] and Hermite cubic splines to solve a spherical version of the steady-state problem in combined-mode (conduction and radiation) heat transfer that has been formulated by Özışık.[4] As Özışık[4] has reviewed carefully the numerous works that have contributed to this field of study, we do not repeat a review here; however, we do note that the present work draws heavily from our recent solution[8] for the plane geometry case. Also, we note that Thynell and Özışık,[9] Tsai and Özışık,[10] Jia et al[11] and Thynell[12] have reported numerical results for coupled problems with spherical symmetry. The work of Jia et al[11] addresses simultaneous radiation and conduction between concentric spheres, and so the problems considered in the work are clearly outside the class of problems solved in Ref. 1. The paper[12] by Thynell includes an effect of anisotropic scattering in the equation of transfer, and so the heat transfer problems in a solid sphere that were solved in the work are also outside the class of problems solved in Ref. 1.

We consider the equation of transfer written as

$$\mu \frac{\partial}{\partial r} I(r,\mu) + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu} I(r,\mu) + I(r,\mu) = \frac{\varpi}{2}\int_{-1}^{1} I(r,\mu')\,\mathrm{d}\mu' + (1-\varpi)\frac{\sigma n^2}{\pi} T^4(r), \tag{1}$$

for $r \in (0, R)$ and $\mu \in [-1, 1]$. We seek a solution to Eq. (1) subject to the boundary condition

$$I(R,-\mu) = \epsilon \frac{\sigma n^2}{\pi} T^4 + 2\rho \int_0^1 I(R,\ \mu')\mu'\,\mathrm{d}\mu' \tag{2}$$

for $\mu \in [0, 1]$. Here $r \in [0, R]$ is the optical variable, μ is the direction cosine measured from the r axis and ϖ is the albedo for single scattering. In regard to the boundary condition, we note that ρ is the coefficient for diffuse reflection and that ϵ is the emissivity of the surface. In addition n is the index of refraction and σ is the Stefan–Boltzmann constant.

The nonlinear aspect of this problem comes from the fact that the temperature distribution $T(r)$ that appears in Eq. (1) must satisfy the heat-conduction equation[4]

$$\beta \frac{\mathrm{d}}{\mathrm{d}r}\left\{r^2\left[k\beta \frac{\mathrm{d}}{\mathrm{d}r} T(r) - q_r(r)\right]\right\} + r^2 h = 0 \tag{3}$$

subject to the boundary conditions $T(R) = T$, where T is the temperature that also appears in Eq. (2), and $T'(r) = 0$, for $r = 0$. In addition, k is the thermal conductivity of the medium, β is the extinction coefficient, the constant h is used to denote prescribed heat generation in the medium and $q_r(r)$ is the radiative heat flux, i.e.

$$q_r(r) = 2\pi \int_{-1}^{1} I(r, \mu)\mu \, \mathrm{d}\mu. \tag{4}$$

Our general approach to the solution of the given problem is the same as that of Lii and Özışık[13] and that of our previous paper,[8] viz. we assume an initial temperature distribution $T(r)$, solve the radiation problem to get the radiative heat flux $q_r(r)$ and use that result in the conduction equation which subsequently we solve to get a new temperature distribution. We then repeat this procedure and consider that we have the solution if there appears to be convergence for the desired quantities.

Having stated the general approach to be used here, we note that, as was also mentioned in Ref. 8 for the plane case, there are two major issues that should be addressed. First of all, for this problem there are, to the best of our knowledge, no existence or uniqueness theorems that state the conditions for which there is a solution and, if the solution exists, when it is unique. In addition and specific to our method of solution, we do not have proof that the method converges to the desired results. Anticipating that these two matters will be addressed in later works, we proceed to develop our solution and to report some numerical results.

2. BASIC FORMULATION

To follow a tradition in the heat transfer literature,[4] we normalize the problem by introducing a convenient reference temperature T_r and by using

$$I(r, \mu) = \left(\frac{\sigma n^2}{\pi} T_r^4\right) I^*(r, \mu), \tag{5}$$

$$q_r(r) = \left(\frac{\sigma n^2}{\pi} T_r^4\right) q_r^*(r) \tag{6}$$

and

$$T(r) = T_r \Theta(r) \tag{7}$$

to rewrite our radiation problem as

$$\mu \frac{\partial}{\partial r} I^*(r, \mu) + \frac{1 - \mu^2}{r} \frac{\partial}{\partial \mu} I^*(r, \mu) + I^*(r, \mu) = \frac{\varpi}{2} \int_{-1}^{1} I^*(r, \mu') \, \mathrm{d}\mu' + (1 - \varpi)\Theta^4(r), \tag{8}$$

for $r \in (0, R)$ and $\mu \in [-1, 1]$, and

$$I^*(R, -\mu) = \epsilon \Theta^4 + 2\rho \int_0^1 I^*(R, \mu')\mu' \, \mathrm{d}\mu' \tag{9}$$

for $\mu \in [0, 1]$. In addition

$$r^2 \frac{\mathrm{d}^2}{\mathrm{d}r^2} \Theta(r) + 2r \frac{\mathrm{d}}{\mathrm{d}r} \Theta(r) = \frac{1}{4\pi N_c} \frac{\mathrm{d}}{\mathrm{d}r} [r^2 q_r^*(r)] - r^2 H \tag{10}$$

with

$$\Theta(R) = \Theta = \frac{T}{T_r}, \quad \frac{d}{dr}\Theta(r)\bigg|_{r=0} = 0 \tag{11a, b}$$

and

$$q_r^*(r) = 2\pi \int_{-1}^{1} I^*(r, \mu)\mu \, d\mu. \tag{12}$$

Here

$$N_c = \frac{k\beta}{4\sigma n^2 T_r^3} \tag{13}$$

is called the conduction-to-radiation parameter.[4] In addition

$$H = [k\beta^2 T_r]^{-1} h \tag{14}$$

is the normalized, and presumed given, constant that represents heat generation in the medium that is independent of the radiation intensity.

In order to make direct use of the sphere-to-plane transformation discussed in Ref. 1, we find it convenient to subdivide our given problem into two related problems. The first of the sub-problems is the so-called albedo problem; we seek $\phi(r, \mu)$ such that

$$\mu \frac{\partial}{\partial r}\phi(r, \mu) + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\phi(r, \mu) + \phi(r, \mu) = \frac{\varpi}{2}\int_{-1}^{1} \phi(r, \mu')\, d\mu', \tag{15}$$

for $r \in (0, R)$ and $\mu \in [-1, 1]$, and

$$\phi(R, -\mu) = 1 \tag{16}$$

for $\mu \in [0, 1]$. In addition we seek $\psi(r, \mu)$ such that

$$\mu \frac{\partial}{\partial r}\psi(r, \mu) + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu}\psi(r, \mu) + \psi(r, \mu) = \frac{\varpi}{2}\int_{-1}^{1} \psi(r, \mu')\, d\mu' + F(r), \tag{17}$$

for $r \in (0, R)$ and $\mu \in [-1, 1]$, and

$$\psi(R, -\mu) = 0 \tag{18}$$

for $\mu \in [0, 1]$. Here $F(r)$ is the inhomogeneous source term, i.e.

$$F(r) = (1 - \varpi)\Theta^4(r). \tag{19}$$

We thus find that the desired solution for the radiation intensity can be expressed as

$$I^*(r, \mu) = \psi(r, \mu) + \lambda\phi(r, \mu) \tag{20}$$

if the constant λ is defined by

$$\lambda = (1 - \rho A^*)^{-1}[\epsilon\Theta^4 + 2\rho\psi_1(R)] \tag{21}$$

where the albedo[3] is

$$A^* = 2\int_{0}^{1} \phi(R, \mu)\mu \, d\mu, \tag{22}$$

and where, in general,

$$\psi_\alpha(r) = \int_{-1}^{1} \mu^\alpha \psi(r, \mu)\, d\mu \tag{23a}$$

and

$$\phi_\alpha(r) = \int_{-1}^{1} \mu^\alpha \phi(r, \mu)\, d\mu. \tag{23b}$$

Once the two sub-problems are solved and the constant λ has been computed, then the radiative heat flux is given by

$$q_r^*(r) = 2\pi[\psi_1(r) + \lambda\phi_1(r)]. \tag{24}$$

If we consider for the moment that the radiative heat flux is known, then we can solve Eq. (10) to find

$$\Theta(r) = \Theta + \frac{1}{6}(R^2 - r^2)H - \frac{1}{4\pi N_c}\int_r^R q_r^*(x)\,\mathrm{d}x \tag{25}$$

or, after we use Eq. (24),

$$\Theta(r) = \Theta + \frac{1}{6}(R^2 - r^2)H - \frac{1}{2N_c}\int_r^R [\psi_1(x) + \lambda\phi_1(x)]\,\mathrm{d}x. \tag{26}$$

It is clear that the albedo problem (the ϕ problem) is independent of the temperature distribution $\Theta(r)$ and thus needs to be solved only once. On the other hand, there is coupling betweeen $\psi(r, \mu)$ and the temperature distribution $\Theta(r)$ that is evident from Eqs. (17), (19) and (26).

We focus our attention now on the albedo problem and note from Refs. 1 and 3 that the moments $\phi_0(r)$ and $\phi_1(r)$ can be expressed as

$$\phi_0(r) = \frac{1}{r}\Phi_0(r) \tag{27a}$$

and

$$\phi_1(r) = \frac{1}{r}\Phi_1(r) + \frac{1}{r^2}\Phi_2(r) \tag{27b}$$

where, in general,

$$\Phi_\alpha(r) = \int_{-1}^{1} \Phi(r, \mu)\mu^\alpha\,\mathrm{d}\mu. \tag{28}$$

Here $\Phi(r, \mu)$ is a solution of a pseudo problem, i.e., $\Phi(r, \mu)$ must satisfy

$$\mu\frac{\partial}{\partial r}\Phi(r, \mu) + \Phi(r, \mu) = \frac{\varpi}{2}\int_{-1}^{1} \Phi(r, \mu')\,\mathrm{d}\mu', \tag{29}$$

for $r \in (-R, R)$ and $\mu \in [-1, 1]$. In addition we require that $\Phi(r, \mu) = -\Phi(-r, -\mu)$ and that

$$\Phi(R, -\mu) = R + \mu \tag{30}$$

for $\mu \in [0, 1]$. We note from Ref. 3 that the albedo A^* can be expressed in terms of $\Phi_1(r)$ and $\Phi_2(r)$, viz.

$$A^* = 1 + \frac{2}{R^2}[R\Phi_1(R) + \Phi_2(R)]. \tag{31}$$

In the following section of this paper we develop our spherical harmonics solution for $\Phi(r, \mu)$, and so now we consider the ψ problem. Again we follow Refs. 1 and 3 to deduce that the moments $\psi_0(r)$ and $\psi_1(r)$ can be expressed as

$$\psi_0(r) = \frac{1}{r}\Psi_0(r) \tag{32a}$$

and

$$\psi_1(r) = \frac{1}{r}\Psi_1(r) + \frac{1}{r^2}\Psi_2(r) \tag{32b}$$

where, in general,

$$\Psi_\alpha(r) = \int_{-1}^{1} \Psi(r, \mu)\mu^\alpha\,\mathrm{d}\mu. \tag{33}$$

Here $\Psi(r, \mu)$ is a solution of a second pseudo problem, viz. $\Psi(r, \mu)$ must satisfy

$$\mu\frac{\partial}{\partial r}\Psi(r, \mu) + \Psi(r, \mu) = \frac{\varpi}{2}\int_{-1}^{1} \Psi(r, \mu')\,\mathrm{d}\mu' + S(r), \tag{34}$$

for $r \in (-R, R)$ and $\mu \in [-1, 1]$. Here the source term is

$$S(r) = (1 - \varpi)r\Theta^4(|r|). \tag{35}$$

In addition we require that $\Psi(r, \mu) = -\Psi(-r, -\mu)$ and that

$$\Psi(R, -\mu) = 0 \tag{36}$$

for $\mu \in [0, 1]$.

Clearly once we have solved the albedo problem to find A^* and $\phi_1(r)$, we can iterate between Eqs. (21), (26), (32b) and (34) to find the temperature distribution $\Theta(r)$ and the conductive, radiative and total heat fluxes, viz.

$$q_c(r) = -k\beta \frac{\mathrm{d}}{\mathrm{d}r} T(r), \tag{37a}$$

$$q_r(r) = 2\pi \int_{-1}^{1} I(r, \mu)\mu \, \mathrm{d}\mu \tag{37b}$$

and

$$q(r) = q_c(r) + q_r(r). \tag{37c}$$

Using Eq. (26), we write Eqs. (37) as

$$\frac{q_c(r)}{k\beta T_r} = \frac{r}{3} H - \frac{1}{4\pi N_c} q_r^*(r), \tag{38a}$$

$$\frac{q_r(r)}{k\beta T_r} = \frac{1}{4\pi N_c} q_r^*(r) \tag{38b}$$

and

$$\frac{q(r)}{k\beta T_r} = \frac{r}{3} H. \tag{38c}$$

At this point we are ready to develop our spherical harmonics solution to the two pseudo problems.

3. A SPHERICAL HARMONICS SOLUTION OF THE ALBEDO PROBLEM

Since the P_N eigenvalues $\{\xi_j\}$ can be computed accurately and efficiently as described in Refs. 5 and 14, and since a recent paper by Garcia and Siewert[15] has reported very precise methods for computing the Chandrasekhar polynomials $\{g_l(\xi_j)\}$ for both the P_N method and the F_N method,[16] it is only a minor exercise to solve the pseudo problem that defines $\Phi(r, \mu)$. We consider N to be odd, let $J = (N + 1)/2$ and write

$$\Phi(r, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} D_j \{\exp[-(R+r)/\xi_j] - (-1)^l \exp[-(R-r)/\xi_j]\} g_l(\xi_j). \tag{39}$$

In order to find the constants $\{D_j\}$ required in Eq. (39), we substitute Eq. (39) into Eq. (30) and use the Marshak projection scheme[6] to obtain, for $\alpha = 0, 1, \ldots, J-1$, the system of linear algebraic equations

$$\sum_{j=1}^{J} \sum_{l=0}^{N} \frac{2l+1}{2} S_{\alpha,l} g_l(\xi_j)[-1 + (-1)^l \exp(-2R/\xi_j)] D_j = R S_{\alpha,0} + S_{\alpha,1}, \tag{40}$$

where as discussed in Ref. 6

$$S_{\alpha,l} = \int_0^1 P_{2\alpha+1}(\mu) P_l(\mu) \, \mathrm{d}\mu. \tag{41}$$

Once we have solved the linear system given by Eq. (40), we can use Eqs. (27b), (31) and (39) to find the required results for the albedo problem, viz.

$$\phi_1(r) = \frac{(1-\varpi)}{r} \sum_{j=1}^{J} D_j \xi_j \left\{ \left(1 + \frac{\xi_j}{r}\right) \exp[-(R+r)/\xi_j] + \left(1 - \frac{\xi_j}{r}\right) \exp[-(R-r)/\xi_j] \right\} \tag{42}$$

and

$$A^* = 1 + 2\phi_1(R). \tag{43}$$

4. AN ITERATIVE SOLUTION OF THE SECOND PSEUDO PROBLEM

Considering now that A^* and $\phi_1(r)$ are available, we develop, in essentially the same way as we did in Ref. 8, an iterative solution of the second pseudo problem. We express our P_N approximation to $\Psi(r, \mu)$, for N odd and $J = (N+1)/2$, in the form

$$\Psi(r, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} A_j\{\exp[-(R+r)/\xi_j] - (-1)^l \exp[-(R-r)/\xi_j]\} g_l(\xi_j) + \Psi_p(r, \mu) \quad (44)$$

where $\Psi_p(r, \mu)$ denotes a particular solution of Eq. (34) corresponding to the inhomogeneous source term given by Eq. (35). Following Refs. 7 and 8, we express the particular solution as

$$\Psi_p(r, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \frac{C_j}{\xi_j} [U_j(r) + (-1)^l V_j(r)] g_l(\xi_j) \quad (45)$$

where

$$U_j(r) = \int_{-R}^{r} S(x) \exp[-(r-x)/\xi_j] \, dx \quad (46a)$$

and

$$V_j(r) = \int_{r}^{R} S(x) \exp[-(x-r)/\xi_j] \, dx \quad (46b)$$

and where the constants C_j, $j = 1, 2, \ldots, J$, are given by

$$C_j = \left[\sum_{k=1}^{J} g_{2k-2}^2(\xi_j) h_{2k-2} \right]^{-1}, \quad j = 1, 2, \ldots, J, \quad (47)$$

with $h_0 = 1 - \varpi$ and $h_l = 2l+1$, $l > 0$. We note from Eq. (35) that $S(-x) = -S(x)$ and so it follows from Eqs. (46) that $U_j(-r) = -V_j(r)$.

In order to find the constants $\{A_j\}$ required in Eq. (44), we substitute Eq. (44) into Eq. (36) and use the Marshak projection scheme[6] to obtain, for $\alpha = 0, 1, \ldots, J-1$, the system of linear algebraic equations

$$\sum_{j=1}^{J} \sum_{l=0}^{N} \frac{2l+1}{2} S_{\alpha,l} g_l(\xi_j) [-1 + (-1)^l \exp(-2R/\xi_j)] A_j = R_\alpha \quad (48)$$

where

$$R_\alpha = -\sum_{j=1}^{J} \sum_{l=0}^{N} \frac{2l+1}{2} (-1)^l S_{\alpha,l} \frac{C_j}{\xi_j} U_j(R) g_l(\xi_j). \quad (49)$$

Once we have solved Eq. (48) to find the constants A_j we can use Eq. (44) in Eq. (32b) to find

$$\psi_1(r) = \frac{(1-\varpi)}{r} \sum_{j=1}^{J} \left\langle \left(1 + \frac{\xi_j}{r}\right) \{A_j \xi_j \exp[-(R+r)/\xi_j] + C_j U_j(r)\} + \left(1 - \frac{\xi_j}{r}\right) \{A_j \xi_j \exp[-(R-r)/\xi_j] - C_j V_j(r)\} \right\rangle. \quad (50)$$

At this point we can substitute Eqs. (42) and (50) into Eq. (26) to obtain our spherical harmonics approximation to the normalized temperature distribution. We thus find, after we use Eqs. (46),

$$\Theta(r) = \Theta + \frac{1}{6}(R^2 - r^2) H - \frac{1-\varpi}{2N_c} \Delta(r) \quad (51)$$

where

$$\Delta(r) = \sum_{j=1}^{J} \{(\lambda D_j + A_j) \xi_j^2 [X_j(r) - X_j(R)] + C_j \xi_j [Y_j(r) - Y_j(R)]\}. \quad (52)$$

Here

$$X_j(r) = \frac{1}{r} \{\exp[-(R+r)/\xi_j] - \exp[-(R-r)/\xi_j]\} \quad (53a)$$

and

$$Y_j(r) = \frac{1}{r}\,[U_j(r) + V_j(r)]. \tag{53b}$$

It is clear that we can, at least in principle, now proceed in the following iterative manner. We start with an initial normalized temperature distribution obtained, for example, by ignoring the integral term in Eq. (25); next we use the initial normalized temperature distribution to define, by way of Eq. (35), the source term $S(r)$ and subsequently the functions $U_j(r)$ and $V_j(r)$. Following the defining of $U_j(r)$ and $V_j(r)$, we can use Eq. (46a) in Eq. (49) and solve the linear algebraic equations given by Eq. (48) to obtain the required constants A_j, $j = 1, 2, \ldots, J$. These constants and the previously defined $U_j(r)$ and $V_j(r)$ can now be used in Eq. (50) to give $\psi_1(R)$ and thus, after we use Eq. (21), λ. At this point Eq. (51) can be evaluated to give the next normalized temperature iterate.

5. NUMERICAL METHODS AND RESULTS

Before reporting some numerical results for several test problems, we make note of some additional matters regarding the numerical solution of the second pseudo problem. First of all, although we can use Eq. (51) as it is written, we prefer, in order to save some computation time, to follow our work in Ref. 8 and to use Hermite cubic splines to interpolate that equation. It follows that since we are using a spline representation of the temperature distribution we could, in fact, evaluate the integrals in Eqs. (46) analytically; however, for the current version of our algorithm we use a standard Gauss quadrature scheme and evaluate the integrals by numerical integration.

We note that, as in Ref. 8, we have, as an alternative to using the Hermite splines to represent the temperature distribution, also carried out some calculations in which we represented the inhomogeneous source term, as given by Eq. (35), by the Hermite splines. Of course, if we intend to evaluate the integrals in Eqs. (46) analytically, then using splines for $S(r)$ rather than $\Theta(r)$ would make that task easier. For the few problems we considered, we did not see any real difference, from a numerical point-of-view, between these two usages of the splines.

In regard to the (outer) iterations between the pseudo problem and the heat conduction equation, we note that we have added an inner iteration step to improve the convergence of the method. Thus at each step in the outer iteration process we solve Eq. (51) iteratively, since the functions $U_j(r)$ and $V_j(r)$ depend on $\Theta(r)$, to find a new temperature $\Theta(r)$.

Having encountered considerable difficulty in obtaining a converging computation for cases where the effects of radiation are very strong, we have used, for the first few iterations, a relaxation technique[17] to keep the computation from exploding. However, after completing a certain number of iterations with relaxation in place, we removed the relaxation procedure and completed the calculation to obtain our final results. For these cases we also found it helpful to start our computation with the initial temperature distribution $\Theta(r) = \Theta$ rather than the radiation-free result.

As we wish to make available some numerical results that have been obtained with the methods discussed here, we consider the six test problems defined in Table 1. Our converged results for the normalized temperature distribution and the normalized heat fluxes, defined from Eqs. (38) as

$$Q_c(r) = \frac{q_c(r)}{k\beta T_r} = \frac{r}{3}H - \frac{1}{4\pi N_c}q_r^*(r), \tag{54a}$$

$$Q_r(r) = \frac{q_r(r)}{k\beta T_r} = \frac{1}{4\pi N_c}q_r^*(r) \tag{54b}$$

and

$$Q(r) = \frac{q(r)}{k\beta T_r} = \frac{r}{3}H, \tag{54c}$$

are given in Tables 2–7. Having varied the order of the P_N approximation, the number of Hermite splines used and the number of Gauss points used to evaluate the $U_j(r)$ and $V_j(r)$ functions, we have some confidence that the results given in Tables 2–7 are correct to within one unit in the last digit given.

To conclude this work we would like to record a few remarks concerning matters that are still unresolved. First of all as mentioned in Sec. 1, there are, to our knowledge, no existence and/or uniqueness theorems that apply directly to this problem, and of course it would be useful to know if this class of problems has been well formulated mathematically. Also as we have no proof that the straightforward iteration scheme we use actually converges, we can only conjecture that the results given in Tables 2–7 are actually correct. Finally we note that for the six problems considered here, we observed what appeared to be convergence toward the established temperature distribution; however, we did encounter problems for which the method failed to converge.

While it is clear that the numerical methods used is this work can be used to solve some combined mode, radiation–conduction, heat-transfer problems in a sphere, we note that there are, in this class of problems, cases that we have not been able to solve. It is anticipated that more sophisticated iteration techniques will be investigated in future work.

Table 1. Physical data for different problems.

Problem	ϵ	ρ	Θ	ϖ	R	N_c	H
1	0.8	0.2	1.0	0.9	1.0	0.05	1.5
2	0.9	0.1	1.0	0.9	0.5	0.05	100
3	0.9	0.1	1.0	0.9	0.05	0.0005	4000
4	0.9	0.1	1.0	0.9	0.5	0.005	40
5	0.9	0.1	1.0	0.9	5.0	0.5	0.4
6	1.0	0.0	1.0	0.9	5.0	0.1	1.0

Table 2. Normalized temperature distribution and heat fluxes for Problem 1.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	1.12138	0.0	0.0	0.0
0.10	1.12062	1.52369(−2)	3.47631(−2)	5.0(−2)
0.20	1.11830	3.13948(−2)	6.86052(−2)	1.0(−1)
0.30	1.11428	4.94546(−2)	1.00545(−1)	1.5(−1)
0.40	1.10831	7.05177(−2)	1.29482(−1)	2.0(−1)
0.50	1.10003	9.58660(−2)	1.54134(−1)	2.5(−1)
0.60	1.08894	1.27020(−1)	1.72980(−1)	3.0(−1)
0.70	1.07438	1.65783(−1)	1.84217(−1)	3.5(−1)
0.80	1.05546	2.14270(−1)	1.85730(−1)	4.0(−1)
0.90	1.03112	2.74873(−1)	1.75127(−1)	4.5(−1)
1.00	1.0	3.50169(−1)	1.49831(−1)	5.0(−1)

Table 3. Normalized temperature distribution and heat fluxes for Problem 2.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	2.62513	0.0	0.0	0.0
0.10	2.62062	1.83096(−1)	1.48357	1.66667
0.20	2.60618	4.04520(−1)	2.92881	3.33333
0.30	2.57877	7.10759(−1)	4.28924	5.00000
0.40	2.53265	1.16486	5.50181	6.66667
0.50	2.45839	1.85365	6.47968	8.33333
0.60	2.34153	2.88826	7.11174	1.00000(+1)
0.70	2.16183	4.38524	7.28143	1.16667(+1)
0.80	1.89410	6.41222	6.92111	1.33333(+1)
0.90	1.51276	8.90332	6.09668	1.50000(+1)
1.00	1.0	1.16178(+1)	5.04885	1.66667(+1)

Table 4. Normalized temperature distribution and heat fluxes for Problem 3.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	1.95215	0.0	0.0	0.0
0.10	1.94668	2.20633	4.46034	6.66667
0.20	1.92976	4.60828	8.72506	1.33333(+1)
0.30	1.89993	7.40913	1.25909(+1)	2.00000(+1)
0.40	1.85465	1.08231(+1)	1.58436(+1)	2.66667(+1)
0.50	1.79031	1.50690(+1)	1.82644(+1)	3.33333(+1)
0.60	1.70224	2.03477(+1)	1.96523(+1)	4.00000(+1)
0.70	1.58487	2.67992(+1)	1.98674(+1)	4.66667(+1)
0.80	1.43225	3.44416(+1)	1.88917(+1)	5.33333(+1)
0.90	1.23873	4.31123(+1)	1.68877(+1)	6.00000(+1)
1.00	1.0	5.24539(+1)	1.42127(+1)	6.66667(+1)

Table 5. Normalized temperature distribution and heat fluxes for Problem 4.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	1.32004	0.0	0.0	0.0
0.10	1.31953	2.10701(−2)	6.45597(−1)	6.66667(−1)
0.20	1.31785	4.73914(−2)	1.28594	1.33333
0.30	1.31459	8.59090(−2)	1.91409	2.00000
0.40	1.30888	1.47485(−1)	2.51918	2.66667
0.50	1.29916	2.50259(−1)	3.08307	3.33333
0.60	1.28267	4.24775(−1)	3.57523	4.00000
0.70	1.25467	7.21014(−1)	3.94565	4.66667
0.80	1.20728	1.21515	4.11818	5.33333
0.90	1.12819	2.00727	3.99273	6.00000
1.00	1.0	3.19008	3.47659	6.66667

Table 6. Normalized temperature distribution and heat fluxes for Problem 5.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	1.49251	0.0	0.0	0.0
0.10	1.49049	8.14823(−3)	5.85184(−2)	6.66667(−2)
0.20	1.48427	1.68525(−2)	1.16481(−1)	1.33333(−1)
0.30	1.47342	2.68523(−2)	1.73148(−1)	2.00000(−1)
0.40	1.45701	3.93193(−2)	2.27347(−1)	2.66667(−1)
0.50	1.43336	5.62435(−2)	2.77090(−1)	3.33333(−1)
0.60	1.39947	8.10213(−2)	3.18979(−1)	4.00000(−1)
0.70	1.35012	1.19207(−1)	3.47460(−1)	4.66667(−1)
0.80	1.27667	1.79019(−1)	3.54314(−1)	5.33333(−1)
0.90	1.16585	2.70291(−1)	3.29709(−1)	6.00000(−1)
1.00	1.0	3.99335(−1)	2.67331(−1)	6.66667(−1)

Table 7. Normalized temperature distribution and heat fluxes for Problem 6.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	1.35750	0.0	0.0	0.0
0.10	1.35626	4.95826(−3)	1.61708(−1)	1.66667(−1)
0.20	1.35253	1.00184(−2)	3.23315(−1)	3.33333(−1)
0.30	1.34621	1.53099(−2)	4.84690(−1)	5.00000(−1)
0.40	1.33714	2.10549(−2)	6.45612(−1)	6.66667(−1)
0.50	1.32499	2.77786(−2)	8.05555(−1)	8.33333(−1)
0.60	1.30896	3.70412(−2)	9.62959(−1)	1.00000
0.70	1.28673	5.40039(−2)	1.11266	1.16667
0.80	1.25097	9.59937(−2)	1.23734	1.33333
0.90	1.17804	2.16897(−1)	1.28310	1.50000
1.00	1.0	5.43128(−1)	1.12354	1.66667

Acknowledgements—One of the authors (CES) wishes to express his appreciation to M. N. Özışık for several helpful discussions concerning this work and to S. T. Thynell who provided some test results from his solution to this class of problems. This work was supported in part by the U.S. National Science Foundation.

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