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A PARTICULAR SOLUTION FOR THE P_N METHOD IN SPHERICAL GEOMETRY

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Abstract—A particular solution that can be used with the P_N method for the solution of radiative heat-transfer problems with spherical symmetry is reported.

INTRODUCTION

In a recently published paper,[1] we reported a particular solution basic to the P_N method in plane geometry. Here we establish an analogous particular solution that is relevant to radiative heat-transfer problems that have spherical symmetry.

We consider radiative heat-transfer problems defined by the equation of transfer[2,3]

$$\mu \frac{\partial}{\partial r} I(r, \mu) + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} I(r, \mu) + I(r, \mu) = \frac{\varpi}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(r, \mu')\, d\mu' + S(r), \tag{1}$$

for $r \in (R_1, R_2)$ and $\mu \in (-1, 1)$, and the boundary conditions

$$I(R_1, \mu) = \epsilon_1 \frac{\sigma n^2}{\pi} T_1^4 + \rho_1^s I(R_1, -\mu) + 2\rho_1^d \int_0^1 I(R_1, -\mu')\mu'\, d\mu' \tag{2a}$$

and

$$I(R_2, -\mu) = \epsilon_2 \frac{\sigma n^2}{\pi} T_2^4 + \rho_2^s I(R_2, \mu) + 2\rho_2^d \int_0^1 I(R_2, \mu')\mu'\, d\mu' \tag{2b}$$

for $\mu \in [0, 1]$. Here

$$S(r) = (1 - \varpi) \frac{\sigma n^2}{\pi} T^4(r) \tag{3}$$

is the inhomogeneous source term, $r \in [R_1, R_2]$ is the optical variable, μ is the direction cosine measured from the r axis, the coefficients β_l define the scattering law and ϖ is the albedo for single scattering. In regard to the boundary conditions, we note that T_1 and T_2 refer to the two boundary temperatures, ρ_α^s and ρ_α^d, $\alpha = 1$ and 2, are the coefficients for specular and diffuse reflection and ϵ_1 and ϵ_2 are the emissivities. In addition n is the index of refraction and σ is the Stefan–Boltzmann constant. We consider here that the temperature distribution $T(r)$ is specified and that $T(R_1) = T_1$ and $T(R_2) = T_2$.

Following Davison[4] and Aronson,[5,6] we express our spherical harmonics or P_N solution, for N odd, as

$$I(r, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} [A_j k_l(r/\xi_j) + (-1)^l B_j i_l(r/\xi_j)] g_l(\xi_j) + I_p(r, \mu) \tag{4}$$

where $I_p(r, \mu)$ denotes a particular solution of Eq. (1) corresponding to the inhomogeneous source term $S(r)$. Here, as discussed in Refs. 7, 8 and 9, the Chandrasekhar polynomials are denoted by

$g_l(\xi_j)$ and the P_N eigenvalues are given by ξ_j, $j = 1, 2, \ldots, J = (N+1)/2$. The modified spherical Bessel functions of the first and third kind[10] are denoted by

$$i_l(z) = \left(\frac{\pi}{2z}\right)^{1/2} I_{l+1/2}(z) \tag{5a}$$

and

$$k_l(z) = \left(\frac{\pi}{2z}\right)^{1/2} K_{l+1/2}(z). \tag{5b}$$

The arbitrary constants $\{A_j\}$ and $\{B_j\}$ appearing in Eq. (4) are to be determined so that the boundary conditions given as Eqs. (2) can be satisfied in some approximate manner.

We consider it important to restate here an observation made by Aronson[5,6] in regard to using the solution given in Eq. (4), without the particular solution, to solve a homogeneous version of Eq. (1): the use of the Mark or Marshak boundary conditions leads to a system of linear algebraic equations that becomes more and more poorly conditioned as the order of the approximation is increased. We have confirmed Aronson's observation, and we have not succeeded in finding a stable way of computing the required constants $\{A_j\}$ and $\{B_j\}$ for all orders of the approximation. It is for this reason that, to date, we have had no confidence in using the P_N method to try to obtain benchmark quality results for general radiative transfer problems with spherical symmetry.

Despite the mentioned difficulty with the P_N method for problems with spherical symmetry, we proceed to develop a particular solution that is appropriate for a general source term $S(r)$. This particular solution can, of course, be used with the P_N method for sufficiently low orders of the approximation to solve some practical problems in radiative transfer. Our particular solution will also be available in the event we eventually find a numerically stable way to find the constants $\{A_j\}$ and $\{B_j\}$ for all orders of the approximation.

A PARTICULAR SOLUTION

As in Ref. 1, we use the method of variation of parameters to find the particular solution we require. We thus propose

$$I_p(r, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} [U_j(r) k_l(r/\xi_j) + (-1)^l V_j(r) i_l(r/\xi_j)] g_l(\xi_j) \tag{6}$$

where $U_j(r)$ and $V_j(r)$, $j = 1, 2, \ldots, J$, are to be found. If we substitute Eq. (6) into Eq. (1), multiply the resulting equation by $P_\beta(\mu)$, for $\beta = 0, 1, \ldots, N$, use the three-term recursion relation for the Legendre polynomials,

$$(2l+1)\mu P_l(\mu) = (l+1)P_{l+1}(\mu) + lP_{l-1}(\mu), \tag{7}$$

and integrate over μ from -1 to 1, we find

$$\sum_{j=1}^{J} [\beta F_{j,\beta-1}(r) + (\beta+1)F_{j,\beta+1}(r)] = 2\delta_{\beta,0} S(r), \quad \beta = 0, 1, \ldots, N, \tag{8}$$

where

$$F_{j,\beta}(r) = [U_j'(r) k_\beta(r/\xi_j) + (-1)^\beta V_j'(r) i_\beta(r/\xi_j)] g_\beta(\xi_j). \tag{9}$$

Here we use the superscript prime to denote differentiation with respect to r.

Inspired by the easily obtained result for the case $N = 1$, we now propose

$$U_j'(r) = \frac{4r^2}{\pi\xi_j^3} C_j i_0(r/\xi_j) S(r) \tag{10a}$$

and

$$V_j'(r) = -\frac{4r^2}{\pi\xi_j^3} C_j k_0(r/\xi_j) S(r) \tag{10b}$$

where we seek to find constants $\{C_j\}$ so that Eqs. (10) will satisfy Eq. (8). If we substitute Eqs. (10) into Eq. (9) and impose, in general, the conditions

$$(1-\varpi)\sum_{j=1}^{J} g_{2k-2}(\xi_j)C_j = \delta_{k,1}, \quad k = 1, 2, \ldots, J, \tag{11}$$

then we readily conclude that Eq. (8) is satisfied for, say, $\beta = 0$, 1 and 2. Our task thus is to show that Eqs. (10) provide a solution to Eq. (8) for all appropriate β.

We now let

$$W_\beta(z) = \frac{2z^2}{\pi}[i_0(z)k_\beta(z) - (-1)^\beta k_0(z)i_\beta(z)] \tag{12}$$

so that we can, after using Eqs. (10), rewrite Eq. (8) as

$$\sum_{j=1}^{J} \frac{C_j}{\xi_j}[\beta W_{\beta-1}(r/\xi_j)g_{\beta-1}(\xi_j) + (\beta+1)W_{\beta+1}(r/\xi_j)g_{\beta+1}(\xi_j)] = \delta_{\beta,0} \tag{13}$$

for $\beta = 0, 1, \ldots, N$. Using the recursion formulas for the spherical Bessel functions,[10] we can deduce that

$$W_{k+1}(z) = \frac{2k+1}{z}W_k(z) + W_{k-1}(z) \tag{14}$$

which we can use with the starting values

$$W_0(z) = 0 \quad \text{and} \quad W_1(z) = 1 \tag{15a and b}$$

to deduce that the functions $W_k(z)$ are polynomials in $1/z$ of degree $k-1$ and that they are alternately even and odd functions. We now consider, for $k \geqslant 1$, the left-hand side of Eq. (13) written as

$$T_k(r) = \sum_{j=1}^{J} \frac{C_j}{\xi_j}[kW_{k-1}(r/\xi_j)g_{k-1}(\xi_j) + (k+1)W_{k+1}(r/\xi_j)g_{k+1}(\xi_j)]. \tag{16}$$

We can now use Eq. (14) and the recursion relation for the Chandrasekhar polynomials, i.e.,

$$h_k\xi g_k(\xi) = (k+1)g_{k+1}(\xi) + kg_{k-1}(\xi), \tag{17}$$

where

$$h_k = 2k + 1 - \varpi\beta_k, \tag{18}$$

to rewrite Eq. (16) as

$$T_k(r) = h_k\Delta_{k-1}(r) + \frac{(k+1)(2k+1)}{r}\Delta_k(r) \tag{19}$$

where

$$\Delta_k(r) = \sum_{j=1}^{J} C_j W_k(r/\xi_j)g_{k+1}(\xi_j). \tag{20}$$

We can use the facts that $W_k(r/\xi_j)$ and $g_k(\xi_j)$ are each alternately even or odd polynomials in ξ_j, along with Eq. (17) and the condition that $g_{N+1}(\xi_j) = 0$, for $j = 1, 2, \ldots, J$, to conclude, after we make use of Eq. (11), that

$$\Delta_k(r) = 0, \quad \text{for} \quad k = 0, 1, \ldots, N. \tag{21}$$

It thus follows that $T_k(r) = 0$, for $k = 1, 2, \ldots, N$, and so the justification of Eqs. (10) is complete.

We can, of course, integrate Eqs. (10) to obtain

$$U_j(r) = \frac{4C_j}{\pi\xi_j^3}\int_{R_1}^{r} x^2 i_0(x/\xi_j)S(x)\,\mathrm{d}x \tag{22a}$$

and

$$V_j(r) = \frac{4C_j}{\pi\xi_j^3}\int_{r}^{R_2} x^2 k_0(x/\xi_j)S(x)\,\mathrm{d}x \tag{22b}$$

that are required in Eq. (6).

To conclude this work we note that the system of linear algebraic equations given by Eq. (11) is exactly the same one that was encountered in Ref. 1 where the particular solution for the case of plane geometry was reported. As the linear system given by Eqs. (11) was solved analytically in Ref. 1, we can write the last ingredient of our particular solution as

$$C_j = \left(\sum_{k=1}^{J} g_{2k-2}^2(\xi_j)h_{2k-2}\right)^{-1}, \quad j = 1, 2, \ldots, J. \tag{23}$$

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