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PARTICULAR SOLUTIONS FOR THE RADIATIVE TRANSFER EQUATION

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Abstract—Particular solutions for both the (formally exact) method of elementary solutions and the P_N method are derived for the case of monochromatic radiative transfer in a homogeneous plane-parallel medium which contains a source that varies with position and direction.

INTRODUCTION

To analyze the radiation intensity and the partial heat fluxes in a homogeneous plane-parallel medium for the case when there is a source of radiation, we consider the equation of transfer[1,2]

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(\mu') I(\tau, \mu')\, d\mu' + S(\tau, \mu) \tag{1}$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$. Here, ϖ is the albedo for single scattering ($\varpi < 1$), the elements β_l are the coefficients in a Legendre expansion of the scattering law, and τ_0 is the optical thickness of the layer. In order to include cases for which the uncollided intensity is separated out from the total intensity, as Chandrasekhar[1] did, we allow the source term $S(\tau, \mu)$ in Eq. (1) to be a function of μ. For the case of classical heat transfer in an emitting medium,[2,3] the source term is simply

$$S(\tau, \mu) = (1 - \varpi) \frac{n^2 \sigma}{\pi} T^4(\tau), \tag{2}$$

where n is the index of refraction, σ is the Stefan–Boltzmann constant, and $T(\tau)$ is the local temperature in the medium.

We consider the source term $S(\tau, \mu)$ given, and so we write the solution to Eq. (1) as

$$I(\tau, \mu) = I_c(\tau, \mu) + I_p(\tau, \mu), \tag{3}$$

where the complementary part $I_c(\tau, \mu)$ satisfies the homogeneous version of Eq. (1) and $I_p(\tau, \mu)$ is a particular solution that corresponds to the inhomogeneous source term $S(\tau, \mu)$. In this work we derive a particular solution for both the method of elementary solutions[4–6] and the P_N method.[7–9] Since we seek only particular solutions, we need not specify any boundary conditions that constrain the solution given by Eq. (3).

We consider first the method of elementary solutions, for which we express the complementary part of the complete solution as

$$I_c(\tau, \mu) = \int_\sigma \{A(\xi)\phi(\xi, \mu)\exp(-\tau/\xi) + B(\xi)\phi(-\xi, \mu)\exp[-(\tau_0 - \tau)/\xi]\}\, d\xi, \tag{4}$$

where the symbol σ implies that we must integrate over the continuum, $\xi = \nu \in (0, 1)$, and sum over the discrete spectrum that lies in the right half-plane, $\xi = \nu_\alpha$, $\alpha = 1, 2, \ldots, \aleph$. The $A(\xi)$ and $B(\xi)$

are expansion coefficients to be determined from the boundary conditions and the $\phi(\pm\xi, \mu)$ are the so-called singular eigenfunctions.[4-6] These singular eigenfunctions satisfy the equation

$$(\xi - \mu)\phi(\xi, \mu) = \frac{\varpi}{2}\sum_{l=0}^{L} \beta_l P_l(\mu) g_l(\xi), \tag{5}$$

where

$$g_l(\xi) = \int_{-1}^{1} P_l(\mu)\phi(\xi, \mu)\,\mathrm{d}\mu. \tag{6}$$

The Chandrasekhar polynomials[1] $g_l(\xi)$ can be defined, in general, by the three-term recursion formula

$$h_l \xi g_l(\xi) = (l + 1)g_{l+1}(\xi) + l g_{l-1}(\xi), \quad l = 1, 2, \ldots, \tag{7}$$

with $g_0(\xi) = 1$ and $g_1(\xi) = \xi h_0$. Here $h_l = 2l + 1 - \varpi\beta_l$, $0 \leqslant l \leqslant L$, and $h_l = 2l + 1$ for $l > L$.

For the P_N method, we consider N to be odd and write the complementary part of the solution as

$$I_c(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \{A_j \exp(-\tau/\xi_j) + (-1)^l B_j \exp[-(\tau_0 - \tau)/\xi_j]\} g_l(\xi_j), \tag{8}$$

where A_j and B_j are to be fixed by the boundary conditions and where the spectrum is given by $\xi = \xi_j$, $j = 1, 2, \ldots, J = (N + 1)/2$. Here, the ξ_j denote the J zeros of $g_{N+1}(\xi)$ that lie in the right half-plane.

Now that the complementary parts of the complete solution for the method of elementary solutions and for the P_N method are defined, we proceed to develop the desired particular solutions using the technique of variation of parameters.[9]

PARTICULAR SOLUTIONS

For the method of elementary solutions, we propose

$$I_p(\tau, \mu) = \int_\sigma \{U(\tau, \xi)\phi(\xi, \mu)\exp(-\tau/\xi) + V(\tau, \xi)\phi(-\xi, \mu)\exp[-(\tau_0 - \tau)/\xi]\}\,\mathrm{d}\xi, \tag{9}$$

where $U(\tau, \xi)$ and $V(\tau, \xi)$ are to be found. We can substitute Eq. (9) into Eq. (1) to find

$$\int_\sigma \mu\left\{\exp(-\tau/\xi)\frac{\partial}{\partial\tau}U(\tau, \xi)\phi(\xi, \mu) + \exp[-(\tau_0 - \tau)/\xi]\frac{\partial}{\partial\tau}V(\tau, \xi)\phi(-\xi, \mu)\right\}\mathrm{d}\xi = S(\tau, \mu). \tag{10}$$

After using the full-range orthogonality relations for the singular eigenfunctions,[4-6] we obtain

$$U(\tau, \xi)\exp(-\tau/\xi) = \frac{1}{N(\xi)}\int_0^\tau S_*(x, \xi)\exp[-(\tau - x)/\xi]\,\mathrm{d}x \tag{11a}$$

and

$$V(\tau, \xi)\exp[-(\tau_0 - \tau)/\xi] = \frac{1}{N(\xi)}\int_\tau^{\tau_0} S_*(x, -\xi)\exp[-(x - \tau)/\xi]\,\mathrm{d}x, \tag{11b}$$

where $N(\xi)$ is the full-range normalization factor[4-6] and

$$S_*(\tau, \xi) = \int_{-1}^{1} S(\tau, \mu)\phi(\xi, \mu)\,\mathrm{d}\mu. \tag{12}$$

We note that if the source $S(\tau, \mu)$ can be expressed as a finite sum of Legendre polynomials, i.e.

$$S(\tau, \mu) = \sum_{\alpha=0}^{M} S_\alpha(\tau) P_\alpha(\mu), \tag{13}$$

then Eqs. (11) can be written as

$$U(\tau,\xi)\exp(-\tau/\xi)=\frac{1}{N(\xi)}\sum_{\alpha=0}^{M}g_\alpha(\xi)\int_0^\tau S_\alpha(x)\exp[-(\tau-x)/\xi]\,\mathrm{d}x \tag{14a}$$

and

$$V(\tau,\xi)\exp[-(\tau_0-\tau)/\xi]=\frac{1}{N(\xi)}\sum_{\alpha=0}^{M}(-1)^\alpha g_\alpha(\xi)\int_\tau^{\tau_0} S_\alpha(x)\exp[-(x-\tau)/\xi]\,\mathrm{d}x. \tag{14b}$$

For the P_N method, we examine Ref. 9 and Eqs. (14) and propose the form

$$I_p(\tau,\mu)=\sum_{l=0}^{N}\frac{2l+1}{2}P_l(\mu)\sum_{j=1}^{J}\frac{C_j}{\xi_j}\sum_{\alpha=0}^{K}g_\alpha(\xi_j)\left\{\int_0^\tau S_\alpha(x)\exp[-(\tau-x)/\xi_j]\,\mathrm{d}x\right.$$
$$\left.+(-1)^{l+\alpha}\int_\tau^{\tau_0}S_\alpha(x)\exp[-(x-\tau)/\xi_j]\,\mathrm{d}x\right\}g_l(\xi_j), \tag{15}$$

where $K=\min\{M,N\}$ and where the constants $\{C_j\}$ are to be found. If we substitute Eq. (15) into Eq. (1), multiply the resulting equation by $P_\beta(\mu)$, for $\beta=0,1,\ldots,N$, and integrate over μ from -1 to 1 we find that Eq. (15) will provide the desired particular solution if the constants $\{C_j\}$ satisfy

$$h_\alpha\sum_{j=1}^{J}C_j[1+(-1)^{\alpha+\beta}]g_\alpha(\xi_j)g_\beta(\xi_j)=2\delta_{\alpha,\beta} \tag{16}$$

for $\beta=0,1,\ldots,N$ and $\alpha=0,1,\ldots,K$. Considering α to be even or odd, we rewrite the last equation as

$$h_\alpha\sum_{j=1}^{J}C_jg_\alpha(\xi_j)g_{2k-2}(\xi_j)=\delta_{2k-2,\alpha},\quad \alpha\text{ even}, \tag{17a}$$

and

$$h_\alpha\sum_{j=1}^{J}C_jg_\alpha(\xi_j)g_{2k-1}(\xi_j)=\delta_{2k-1,\alpha},\quad \alpha\text{ odd}, \tag{17b}$$

for $k=1,2,\ldots,J$ and $\alpha=0,1,2,\ldots,K$. We note that the set of equations given by Eq. (17a) with $\alpha=0$ is the one solved by Siewert and Thomas[9] who found

$$C_j=\left(\sum_{k=1}^{J}h_{2k-2}g_{2k-2}^2(\xi_j)\right)^{-1},\quad j=1,2,\ldots,J. \tag{18}$$

To demonstrate that the particular solution given by Eq. (15) is correct we must now show that the $\{C_j\}$ given by Eq. (18) provide a solution to Eqs. (17) for all appropriate α and k. To develop the required proof we start with an orthogonality relation for the Chandrasekhar polynomials that was derived by Inönü,[10] viz.

$$h_\alpha\int_\sigma[1+(-1)^{\alpha+\beta}]g_\alpha(\xi)g_\beta(\xi)\frac{\xi}{N(\xi)}\,\mathrm{d}\xi=2\delta_{\alpha,\beta} \tag{19}$$

for $\alpha,\beta=0,1,2,\ldots$. We choose to rewrite the last equation as

$$h_\alpha\int_\sigma g_\alpha(\xi)g_{2k-2}(\xi)\frac{\xi}{N(\xi)}\,\mathrm{d}\xi=\delta_{2k-2,\alpha},\quad \alpha\text{ even}, \tag{20a}$$

and

$$h_\alpha\int_\sigma g_\alpha(\xi)g_{2k-1}(\xi)\frac{\xi}{N(\xi)}\,\mathrm{d}\xi=\delta_{2k-1,\alpha},\quad \alpha\text{ odd}, \tag{20b}$$

for $\alpha, k = 0, 1, 2, \ldots$. We combine Eqs. (17) and (20) to obtain

$$\int_\sigma g_\alpha(\xi) g_{2k-2}(\xi) \frac{\xi}{N(\xi)} \mathrm{d}\xi = \sum_{j=1}^{J} C_j g_\alpha(\xi_j) g_{2k-2}(\xi_j), \quad \alpha \text{ even}, \tag{21a}$$

and

$$\int_\sigma g_\alpha(\xi) g_{2k-1}(\xi) \frac{\xi}{N(\xi)} \mathrm{d}\xi = \sum_{j=1}^{J} C_j g_\alpha(\xi_j) g_{2k-1}(\xi_j), \quad \alpha \text{ odd}. \tag{21b}$$

Another result proved by Inönü[10] is that an arbitrary polynomial $R_l(\xi)$ of degree $l \leqslant 2N + 1$ can be integrated exactly in the following sense:

$$\int_\sigma [R_l(\xi) + R_l(-\xi)] \frac{\xi}{N(\xi)} \mathrm{d}\xi = \sum_{j=1}^{J} F_j [R_l(\xi_j) + R_l(-\xi_j)], \tag{22}$$

where

$$F_j = 2\left(\sum_{l=0}^{N} h_l g_l^2(\xi_j)\right)^{-1}, \quad j = 1, 2, \ldots, J. \tag{23}$$

Since, in Eq. (22), we can use for $[R_l(\xi) + R_l(-\xi)]$ either the $g_\alpha(\xi) g_{2k-1}(\xi)$ or the $g_\alpha(\xi) g_{2k-2}(\xi)$ of Eqs. (21) it remains to relate the constants $\{C_j\}$ to the constants $\{F_j\}$. We first rewrite Eq. (23) as

$$F_j = 2\left\{\sum_{k=1}^{J} [h_{2k-2} g_{2k-2}^2(\xi_j) + h_{2k-1} g_{2k-1}^2(\xi_j)]\right\}^{-1} \tag{24}$$

and observe that $C_j = F_j$ provided we can prove the (apparently new) identity

$$\sum_{k=1}^{J} h_{2k-2} g_{2k-2}^2(\xi_j) = \sum_{k=1}^{J} h_{2k-1} g_{2k-1}^2(\xi_j). \tag{25}$$

This identity can be shown to be true by using Eq. (7) for $l = 2k - 2$ and $l = 2k - 1$ to obtain

$$\sum_{k=1}^{J} [h_{2k-2} g_{2k-2}^2(\xi_j) - h_{2k-1} g_{2k-1}^2(\xi_j)] = -\frac{1}{\xi} \sum_{k=1}^{J} [2k g_{2k}(\xi_j) g_{2k-1}(\xi_j) - (2k-2) g_{2k-2}(\xi_j) g_{2k-3}(\xi_j)]. \tag{26}$$

Since the right-hand side of Eq. (26) vanishes when we use the fact that $g_{N+1}(\xi_j) = 0$, the identity given by Eq. (25) is established, and so we conclude that the particular solution given by Eq. (15) is correct.

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