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ON COUPLED CONDUCTIVE–RADIATIVE HEAT TRANSFER PROBLEMS IN A CYLINDER

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Abstract—An integral transformation technique and the spherical harmonics method are used, along with Hermite cubic splines, to define an iterative technique for solving a class of nonlinear radiative transfer problems in a solid cylinder of infinite length. Computational aspects of the technique are discussed, and the method is used to establish numerical results for several test problems.

INTRODUCTION

In a paper[1] published in 1984 we used an integral transformation technique that was developed by Mitsis[2] in order to reduce a class of radiation transport problems formulated in cylindrical geometry to more easily solved "pseudo" problems. In Ref. 1 the inhomogeneous source term in the equation of transfer was assumed to be given, and so now we generalize that work to allow coupling between radiative and conductive modes of heat transfer. More specifically, we use the integral transformation technique[1,2] along with the spherical harmonics method[3–5] and Hermite cubic splines to solve a cylindrical version of the steady-state problem in combined-mode (conduction and radiation) heat transfer that has been formulated by Özışık.[6] In regard to other semi-analytical work on this class of problems, we note that Thynell and Özışık have reported[7] a derivation of the integral form of the equation of transfer and that the same two authors later published[8] some numerical results relevant to the albedo problem.[1] Özışık and Thynell[8] also gave some numerical results for the case of a prescribed inhomogeneous source term that is quadratic in the optical variable. In regard to combined-mode, conduction and radiation, heat transfer in a solid cylinder, we note that Tsai and Özışık[9] have reported a study of the effects of some of the physical parameters on the transient temperature and heat flux distributions in a solid cylinder. Here we attempt to provide, for the steady-state case, benchmark quality results for the temperature distribution and the relevant heat fluxes in a solid cylinder of infinite length.

As this work is the fifth in a series of papers based on slabs,[5,10] spheres,[11,12] and now cylinders, our presentation here will be brief.

We consider the equation of transfer written as

$$\left[(1-\mu^2)^{1/2}\left(\cos\phi\frac{\partial}{\partial r}-\frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\right)+1\right]I(r,\mu,\phi)$$
$$=\frac{\varpi}{\pi}\int_0^1\int_0^\pi I(r,\mu',\phi')\,\mathrm{d}\phi'\,\mathrm{d}\mu'+(1-\varpi)\frac{\sigma n^2}{\pi}T^4(r) \quad (1)$$

for $r\in(0,R)$, $\mu\in[0,1]$ and $\phi\in(0,\pi)$. We seek a solution to Eq. (1) subject to the boundary condition

$$I(R,\mu,\phi)=\epsilon\frac{\sigma n^2}{\pi}T^4+\frac{4\rho}{\pi}\int_0^1\int_0^{\pi/2}I(R,\mu',\phi')(1-\mu'^2)^{1/2}\cos\phi'\,\mathrm{d}\phi'\,\mathrm{d}\mu' \quad (2)$$

for $\mu\in[0,1]$ and $\phi\in[\pi/2,\pi]$. Here $r\in[0,R]$ is the optical variable, θ is the polar angle, $\mu=\cos\theta$, ϕ is the azimuthal angle and ϖ is the albedo for single scattering.

Note that in writing Eqs. (1) and (2) as we have, we have taken into account the facts that

$$I(r, -\eta, \phi) = I(r, \eta, \phi), \quad \eta \in [0, 1], \tag{3a}$$

and

$$I(r, \mu, 2\pi - \alpha) = I(r, \mu, \alpha), \quad \alpha \in [0, \pi]. \tag{3b}$$

In regard to the boundary condition, we note that ρ is the coefficient for diffuse reflection and that ϵ is the emissivity of the surface. In addition n is the index of refraction and σ is the Stefan–Boltzmann constant.

The nonlinear aspect of this problem comes from the fact that the temperature distribution $T(r)$ that appears in Eq. (1) must satisfy the heat-conduction equation[6]

$$\beta \frac{\mathrm{d}}{\mathrm{d}r}\left\{r\left[k\beta \frac{\mathrm{d}}{\mathrm{d}r} T(r) - q_r(r)\right]\right\} + rh = 0 \tag{4}$$

subject to the boundary conditions $T(R) = T$, where T is the temperature that also appears in Eq. (2), and $T'(r) = 0$, for $r = 0$. In addition, k is the thermal conductivity of the medium, β is the extinction coefficient, the constant h is used to denote prescribed heat generation in the medium and $q_r(r)$ is the radiative heat flux, i.e.

$$q_r(r) = 4\int_0^1\int_0^\pi I(r, \mu, \phi)(1 - \mu^2)^{1/2}\cos\phi \,\mathrm{d}\phi\,\mathrm{d}\mu. \tag{5}$$

As in Refs. 10 and 12, we do not address the existence and uniqueness aspects of the formulated problem, and so we proceed to define our iterative solution and to compute some numerical results for a selection of test problems.

BASIC FORMULATION

To follow a tradition in the heat transfer literature,[6] we normalize the problem by introducing a convenient reference temperature T_r and using

$$I(r, \mu, \phi) = \left(\frac{\sigma n^2}{\pi} T_r^4\right) I^*(r, \mu, \phi), \tag{6}$$

$$q_r(r) = \left(\frac{\sigma n^2}{\pi} T_r^4\right) q_r^*(r) \tag{7}$$

and

$$T(r) = T_r \Theta(r) \tag{8}$$

to rewrite our radiation problem as

$$\left[(1 - \mu^2)^{1/2}\left(\cos\phi \frac{\partial}{\partial r} - \frac{1}{r}\sin\phi \frac{\partial}{\partial \phi}\right) + 1\right] I^*(r, \mu, \phi)$$
$$= \frac{\varpi}{\pi}\int_0^1\int_0^\pi I^*(r, \mu', \phi')\,\mathrm{d}\phi'\,\mathrm{d}\mu' + (1 - \varpi)\Theta^4(r) \tag{9}$$

for $r \in (0, R)$, $\mu \in [0, 1]$ and $\phi \in (0, \pi)$ and

$$I^*(R, \mu, \phi) = \epsilon\Theta^4 + \frac{4\rho}{\pi}\int_0^1\int_0^{\pi/2} I^*(R, \mu', \phi')(1 - \mu'^2)^{1/2}\cos\phi'\,\mathrm{d}\phi'\,\mathrm{d}\mu' \tag{10}$$

for $\mu \in [0, 1]$ and $\phi \in [\pi/2, \pi]$. In addition

$$r\frac{\mathrm{d}^2}{\mathrm{d}r^2}\Theta(r) + \frac{\mathrm{d}}{\mathrm{d}r}\Theta(r) = \frac{1}{4\pi N_c}\frac{\mathrm{d}}{\mathrm{d}r}[rq_r^*(r)] - rH \tag{11}$$

with

$$\Theta(R) = \Theta = \frac{T}{T_r} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}r}\Theta(r)|_{r=0} = 0. \tag{12a and b}$$

Here

$$q_r^*(r) = 4\int_0^1 \int_0^\pi I^*(r,\mu,\phi)(1-\mu^2)^{1/2}\cos\phi \, d\phi \, d\mu. \tag{13}$$

In addition

$$N_c = \frac{k\beta}{4\sigma n^2 T_r^3} \tag{14}$$

is called the conduction-to-radiation parameter,[6] and

$$H = [k\beta^2 T_r]^{-1} h \tag{15}$$

is the normalized, and presumed given, constant that represents heat generation in the medium that is independent of the radiation intensity.

We find it convenient to express the radiation intensity as

$$I^*(r,\mu,\phi) = \psi(r,\mu,\phi) + \lambda f(r,\mu,\phi) \tag{16}$$

where, first of all, $f(r,\mu,\phi)$ is the solution of the albedo problem considered in Ref. 1, i.e.

$$\left[(1-\mu^2)^{1/2}\left(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\right) + 1\right] f(r,\mu,\phi) = \frac{\varpi}{\pi}\int_0^1\int_0^\pi f(r,\mu',\phi')\, d\phi'\, d\mu' \tag{17a}$$

for $r \in (0, R)$, $\mu \in [0, 1]$ and $\phi \in (0, \pi)$ and

$$f(R,\mu,\phi) = 1, \quad \mu \in [0,1] \quad \text{and} \quad \phi \in [\pi/2, \pi]. \tag{17b}$$

We also seek $\psi(r,\mu,\phi)$ such that

$$\left[(1-\mu^2)^{1/2}\left(\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\sin\phi\frac{\partial}{\partial\phi}\right) + 1\right]\psi(r,\mu,\phi) = \frac{\varpi}{\pi}\int_0^1\int_0^\pi \psi(r,\mu',\phi')\, d\phi'\, d\mu' + (1-\varpi)\Theta^4(r) \tag{18a}$$

for $r \in (0, R)$, $\mu \in [0, 1]$ and $\phi \in (0, \pi)$ and

$$\psi(R,\mu,\phi) = 0, \quad \mu \in [0,1] \quad \text{and} \quad \phi \in [\pi/2, \pi]. \tag{18b}$$

It follows from Eqs. (9), (10), (16), (17) and (18) that the constant λ must be defined as

$$\lambda = (1-\rho A^*)^{-1}\left[\epsilon\Theta^4 + \frac{\rho}{\pi}\psi_1(R)\right] \tag{19}$$

where, in general,

$$\psi_1(r) = 4\int_0^1\int_0^\pi \psi(r,\mu,\phi)(1-\mu^2)^{1/2}\cos\phi\, d\phi\, d\mu. \tag{20}$$

The albedo A^* used in Eq. (19) was defined in Ref. 1 as

$$A^* = \frac{4}{\pi}\int_0^1\int_0^{\pi/2} f(R,\mu,\phi)(1-\mu^2)^{1/2}\cos\phi\, d\phi\, d\mu. \tag{21}$$

We note from Ref. 1 that A^* can also be expressed as

$$A^* = 1 + \frac{1}{\pi} f_1(R) \tag{22}$$

where, in general,

$$f_1(r) = 4\int_0^1\int_0^\pi f(r,\mu,\phi)(1-\mu^2)^{1/2}\cos\phi\, d\phi\, d\mu. \tag{23}$$

Once the two sub-problems have been solved we can compute the desired radiative heat flux from

$$q_r^*(r) = \psi_1(r) + \lambda f_1(r). \tag{24}$$

In addition, if we consider the radiative heat flux to be known, we can solve Eq. (11) and use Eqs. (12) to find

$$\Theta(r) = \Theta + \tfrac{1}{4}(R^2 - r^2)H - \frac{1}{4\pi N_c}\int_r^R q_r^*(x)\,\mathrm{d}x. \tag{25}$$

We note that the albedo problem is independent of the temperature distribution $\Theta(r)$ and thus does not require an iterative-type solution. On the other hand, there clearly is coupling, that is evident from Eqs. (18a), (20), (24) and (25), between the ψ problem and the temperature distribution $\Theta(r)$.

Focusing our attention now on the albedo problem, we conclude from Eq. (17a) that

$$f_1(r) = -\frac{1}{r}(1-\varpi)\int_0^r x f_0(x)\,\mathrm{d}x \tag{26}$$

where

$$f_0(r) = 4\int_0^1\int_0^\pi f(r,\mu,\phi)\,\mathrm{d}\phi\,\mathrm{d}\mu. \tag{27}$$

Referring to Ref. 1, we observe that $f_0(r)$ can be expressed as

$$f_0(r) = 4\pi\int_0^1 F(r,\mu)\,\mathrm{d}\mu \tag{28}$$

where $F(r,\mu)$ is a solution of the "pseudo problem" defined by

$$\left[\mu^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - 1\right]F(r,\mu) + \varpi\int_0^1 F(r,\mu)\,\mathrm{d}\mu = 0, \tag{29a}$$

for $r \in (0, R)$ and $\mu \in [0, 1]$, and

$$F(R,\mu) + \Upsilon(R,\mu)\frac{\partial}{\partial r}F(r,\mu)|_{r=R} = 1, \quad \mu \in [0, 1]. \tag{29b}$$

Here

$$\Upsilon(R,\mu) = \mu\frac{K_0(R/\mu)}{K_1(R/\mu)} \tag{30}$$

where $K_0(z)$ and $K_1(z)$ are modified Bessel functions.[13]

In the following section of this paper we develop our spherical harmonics solution for $F(r, \mu)$, and so now we consider the ψ problem.

From Eq. (18a) we deduce that

$$\psi_1(r) = \frac{1}{r}(1-\varpi)\int_0^r x[4\pi\Theta^4(x) - \psi_0(x)]\,\mathrm{d}x \tag{31}$$

where

$$\psi_0(r) = 4\int_0^1\int_0^\pi \psi(r,\mu,\phi)\,\mathrm{d}\phi\,\mathrm{d}\mu. \tag{32}$$

Again we refer to Ref. 1 and note that $\psi_0(r)$ can be expressed in terms of a solution of a pseudo problem. We therefore write[1]

$$\psi_0(r) = 4\pi\int_0^1 \Psi(r,\mu)\,\mathrm{d}\mu \tag{33}$$

where $\Psi(r,\mu)$ is defined as a solution of

$$\left[\mu^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) - 1\right]\Psi(r,\mu) + \varpi\int_0^1 \Psi(r,\mu)\,\mathrm{d}\mu = -(1-\varpi)\Theta^4(r), \tag{34a}$$

for $r \in (0, R)$ and $\mu \in [0, 1]$, and

$$\Psi(R,\mu) + \Upsilon(R,\mu)\frac{\partial}{\partial r}\Psi(r,\mu)|_{r=R} = 0, \quad \mu \in [0, 1]. \tag{34b}$$

Clearly once the albedo problem has been solved and we have used an iterative technique to establish $\psi_1(r)$ and $\Theta(r)$, we can compute the conductive, radiative and total heat fluxes, viz.

$$q_c(r) = -k\beta \frac{d}{dr} T(r), \tag{35a}$$

a

$$q_r(r) = 4 \int_0^1 \int_0^\pi I(r, \mu, \phi)(1-\mu^2)^{1/2} \cos\phi \, d\phi \, d\mu \tag{35b}$$

and

$$q(r) = q_c(r) + q_r(r). \tag{35c}$$

Using Eq. (25), we rewrite Eqs. (35) as

$$\frac{q_c(r)}{k\beta T_r} = \frac{r}{2} H - \frac{1}{4\pi N_c} q_r^*(r), \tag{36a}$$

$$\frac{q_r(r)}{k\beta T_r} = \frac{1}{4\pi N_c} q_r^*(r) \tag{36b}$$

and

$$\frac{q(r)}{k\beta T_r} = \frac{r}{2} H. \tag{36c}$$

We proceed now to develop a spherical harmonics solution that will allow us to compute $\Theta(r)$ and the desired heat fluxes.

A SPHERICAL HARMONICS SOLUTION OF THE ALBEDO PROBLEM

Since the P_N eigenvalues $\{\xi_j\}$ can be computed accurately and efficiently as described in Refs. 3 and 14, and since Garcia and Siewert[15] have described very precise methods for computing the Chandrasekhar polynomials $\{g_l(\xi_j)\}$ for both the P_N method and the F_N method,[16] it is only a minor exercise to solve the pseudo problem that defines $F(r, \mu)$. We consider N to be odd, let $J = (N+1)/2$ and write

$$F(r, \mu) = \sum_{l=0,2,4}^{N-1} (2l+1) P_l(\mu) \sum_{j=1}^{J} D_j \hat{I}_0(r/\xi_j) \, e^{-(R-r)/\xi_j} g_l(\xi_j) \tag{37}$$

where, in general,

$$\hat{I}_\nu(z) = I_\nu(z)\, e^{-z} \quad \text{and} \quad \hat{K}_\nu(z) = K_\nu(z)\, e^{z} \tag{38a and b}$$

and $I_\nu(z)$ and $K_\nu(z)$, $\nu = 0, 1, 2, \ldots$, are modified Bessel functions.[13] In order to find the constants $\{D_j\}$ required in Eq. (37), we substitute Eq. (37) into Eq. (29b) and use the Marshak projections scheme[4] to obtain, for $\alpha = 0, 1, \ldots, J-1$, the system of linear algebraic equations

$$\sum_{j=1}^{J} \sum_{l=0,2,4}^{N-1} (2l+1) D_j \left[S_{\alpha,l} \hat{I}_0(R/\xi_j) + \frac{1}{\xi_j} T_{\alpha,l}(R) \hat{I}_1(R/\xi_j) \right] g_l(\xi_j) = S_{\alpha,0}, \tag{39}$$

where as discussed in Ref. 4

$$S_{\alpha,l} = \int_0^1 P_{2\alpha+1}(\mu) P_l(\mu) \, d\mu. \tag{40}$$

In addition

$$T_{\alpha,l}(R) = \int_0^1 P_{2\alpha+1}(\mu) P_l(\mu) \Upsilon(R, \mu) \, d\mu. \tag{41}$$

Once we have solved the linear system given by Eq. (39), we can use Eqs. (22), (26) and (28) to find some results for the albedo problem, viz.

$$f_0(r) = 4\pi \sum_{j=1}^{J} D_j \hat{I}_0(r/\xi_j)\,\mathrm{e}^{-(R-r)/\xi_j}, \tag{42}$$

$$f_1(r) = -(1-\varpi)4\pi \sum_{j=1}^{J} D_j \xi_j \hat{I}_1(r/\xi_j)\,\mathrm{e}^{-(R-r)/\xi_j} \tag{43}$$

and

$$A^* = 1 + \frac{1}{\pi} f_1(R). \tag{44}$$

AN ITERATIVE SOLUTION OF THE SECOND PROBLEM

Considering now that A^* and $f_1(r)$ are available, we develop, in essentially the same way as we did in Refs. 10 and 12, an iterative solution of the second problem. We express our P_N approximation to $\Psi(r,\mu)$, for N odd and $J = (N+1)/2$, in the form

$$\Psi(r,\mu) = \sum_{l=0,2,4}^{N-1} (2l+1)P_l(\mu) \sum_{j=1}^{J} A_j \hat{I}_0(r/\xi_j)\,\mathrm{e}^{-(R-r)/\xi_j} g_l(\xi_j) + \Psi_p(r,\mu) \tag{45}$$

where $\Psi_p(r,\mu)$ denotes a particular solution of Eq. (34a). Following Refs. 5 and 11, we express the particular solution as

$$\Psi_p(r,\mu) = \sum_{l=0,2,4}^{N-1} (2l+1)P_l(\mu) \sum_{j=1}^{J} \frac{C_j}{\xi_j^2} [\hat{I}_0(r/\xi_j)V_j(r) + \hat{K}_0(r/\xi_j)U_j(r)] g_l(\xi_j) \tag{46}$$

where

$$U_j(r) = \int_0^r x S(x) \hat{I}_0(x/\xi_j)\,\mathrm{e}^{-(r-x)/\xi_j}\,\mathrm{d}x \tag{47a}$$

and

$$V_j(r) = \int_r^R x S(x) \hat{K}_0(x/\xi_j)\,\mathrm{e}^{-(x-r)/\xi_j}\,\mathrm{d}x \tag{47b}$$

and where the constants C_j, $j = 1, 2, \ldots, J$, are given by

$$C_j = \left(\sum_{k=1}^{J} g_{2k-2}^2(\xi_j) h_{2k-2} \right)^{-1}, \quad j = 1, 2, \ldots, J, \tag{48}$$

with $h_0 = 1 - \varpi$ and $h_l = 2l + 1$, $l > 0$. Here

$$S(r) = (1-\varpi)\Theta^4(r). \tag{49}$$

In order to find the constants $\{A_j\}$ required in Eq. (45), we substitute Eq. (45) into Eq. (34b) and again use the Marshak projection scheme to obtain, for $\alpha = 0, 1, \ldots, J-1$, the system of linear algebraic equations

$$\sum_{j=1}^{J} \sum_{l=0,2,4}^{N-1} (2l+1) A_j \left[S_{\alpha,l} \hat{I}_0(R/\xi_j) + \frac{1}{\xi_j} T_{\alpha,l}(R) \hat{I}_1(R/\xi_j) \right] g_l(\xi_j) = R_\alpha, \tag{50}$$

where

$$R_\alpha = -\sum_{j=1}^{J} \sum_{l=0,2,4}^{N-1} (2l+1) \left(\frac{C_j}{\xi_j^2} \right) \left[S_{\alpha,l} \hat{K}_0(R/\xi_j) - \frac{1}{\xi_j} T_{\alpha,l}(R) \hat{K}_1(R/\xi_j) \right] U_j(R) g_l(\xi_j). \tag{51}$$

Once we have solved the linear system given by Eq. (50) to find the constants A_j we can use Eqs. (45) and (46) in Eqs. (31) and (33) to find

$$\psi_0(r) = 4\pi \sum_{j=1}^{J} \left\{ A_j \hat{I}_0(r/\xi_j)\,\mathrm{e}^{-(R-r)/\xi_j} + \frac{C_j}{\xi_j^2}[\hat{I}_0(r/\xi_j)V_j(r) + \hat{K}_0(r/\xi_j)U_j(r)] \right\} \tag{52}$$

and

$$\psi_1(r) = -4\pi(1-\varpi) \sum_{j=1}^{J} \left\{ A_j \xi_j \hat{I}_1(r/\xi_j)\,\mathrm{e}^{-(R-r)/\xi_j} + \frac{C_j}{\xi_j}[\hat{I}_1(r/\xi_j)V_j(r) - \hat{K}_1(r/\xi_j)U_j(r)] \right\}. \tag{53}$$

At this point we can substitute Eqs. (43) and (53) into Eq. (25) to find the next iterate of the temperature distribution. We thus find we can write

$$\Theta(r) = \Theta + \tfrac{1}{4}(R^2 - r^2)H - \frac{1-\varpi}{N_c}\Delta(r) \tag{54}$$

where

$$\Delta(r) = \sum_{j=1}^{J} (A_j + \lambda D_j)\xi_j^2[\hat{I}_0(r/\xi_j)\,\mathrm{e}^{-(R-r)/\xi_j} - \hat{I}_0(R/\xi_j)] + \sum_{j=1}^{J} C_j[\hat{K}_0(r/\xi_j)U_j(r) - \hat{K}_0(R/\xi_j)U_j(R) + \hat{I}_0(r/\xi_j)V_j(r)]. \tag{55}$$

To evaluate this solution, we proceed, as in Refs. 10 and 12, to solve a collection of test problems.

NUMERICAL METHODS AND RESULTS

Before reporting some numerical results for several test problems, we make note of some additional matters regarding the numerical solution of the second pseudo problem. First of all, although we can use Eq. (54) as it is written, we prefer, in order to save some computation time, to follow our work in Refs. 10 and 12 and to use Hermite cubic splines to interpolate that equation. It follows that since we are using a spline representation of the temperature distribution we could, in fact, evaluate the integrals in Eqs. (47) analytically; however, for the current version of our algorithm we use a standard Gauss quadrature scheme and evaluate the integrals by numerical integration.

In regard to the (outer) iterations between the ψ problem and the heat conduction equation, we note that we have added an inner iteration step to improve the convergence of the method. Thus at each step in the outer iteration process we solve Eq. (54) iteratively, since the functions $U_j(r)$ and $V_j(r)$ depend on $\Theta(r)$, to find a new temperature $\Theta(r)$.

Having encountered considerable difficulty in obtaining a converging computation for cases where the effects of radiation are very strong, we have used, for the first few iterations, a relaxation technique[17] to keep the computation from exploding. However, after completing a certain number of iterations with relaxation in place, we removed the relaxation procedure and completed the calculation to obtain our final results. For these cases we also found it helpful to start our computation with the initial temperature distribution $\Theta(r) = \Theta$ rather than the radiation-free result.

As we wish to make available some numerical results that have been obtained with the methods discussed here, we consider the six test problems defined in Table 1. Our converged results for the normalized temperature distribution and the normalized heat fluxes, defined from Eqs. (36) as

$$Q_c(r) = \frac{q_c(r)}{k\beta T_r} = \frac{r}{2}H - \frac{1}{4\pi N_c}q_r^*(r), \tag{56a}$$

$$Q_r(r) = \frac{q_r(r)}{k\beta T_r} = \frac{1}{4\pi N_c}q_r^*(r) \tag{56b}$$

and

$$Q(r) = \frac{q(r)}{k\beta T_r} = \frac{r}{2}H, \tag{56c}$$

Table 1. Physical data for different problems.

Problem	ϵ	ρ	Θ	ϖ	R	N_c	H
1	0.8	0.2	1.0	0.9	1.0	0.05	1.5
2	0.9	0.1	1.0	0.9	0.5	0.05	100
3	0.9	0.1	1.0	0.9	0.05	0.0005	4000
4	0.9	0.1	1.0	0.9	0.5	0.005	40
5	0.9	0.1	1.0	0.9	5.0	0.5	0.4
6	1.0	0.0	1.0	0.9	1.0	0.1	1.0

are given in Tables 2–7. Having varied the order of the P_N approximation, the number of Hermite splines used and the number of Gauss points used to evaluate the $U_j(r)$ and $V_j(r)$ functions, we have some confidence that the results given in Tables 2–7 are correct to within one unit in the last digit given.

To conclude this work we would like to record a few remarks concerning matters that are still unresolved. First of all, as for the problems solved in Refs. 10 and 12, there are, to our knowledge, no existence and/or uniqueness theorems that apply directly to this problem, and of course it would

Table 2. Normalized temperature distribution and heat fluxes for Problem 1.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	1.14135	0.0	0.0	0.00
0.10	1.14057	1.58317(−2)	5.91683(−2)	7.50(−2)
0.20	1.13815	3.28689(−2)	1.17131(−1)	1.50(−1)
0.30	1.13391	5.24174(−2)	1.72583(−1)	2.25(−1)
0.40	1.12753	7.59858(−2)	2.24014(−1)	3.00(−1)
0.50	1.11852	1.05392(−1)	2.69608(−1)	3.75(−1)
0.60	1.10618	1.42868(−1)	3.07132(−1)	4.50(−1)
0.70	1.08959	1.91149(−1)	3.33851(−1)	5.25(−1)
0.80	1.06748	2.53521(−1)	3.46479(−1)	6.00(−1)
0.90	1.03829	3.33766(−1)	3.41234(−1)	6.75(−1)
1.00	1.0	4.35937(−1)	3.14063(−1)	7.50(−1)

Table 3. Normalized temperature distribution and heat fluxes for Problem 2.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	2.68018	0.0	0.0	0.0
0.10	2.67697	1.31221(−1)	2.36878	2.50
0.20	2.66648	2.97875(−1)	4.70212	5.00
0.30	2.64585	5.45711(−1)	6.95429	7.50
0.40	2.60944	9.42846(−1)	9.05715	1.00(+1)
0.50	2.54736	1.59384	1.09062(+1)	1.25(+1)
0.60	2.44331	2.65103	1.23490(+1)	1.50(+1)
0.70	2.27228	4.30604	1.31940(+1)	1.75(+1)
0.80	1.99985	6.72698	1.32730(+1)	2.00(+1)
0.90	1.58658	9.91733	1.25827(+1)	2.25(+1)
1.00	1.0	1.35877(+1)	1.14123(+1)	2.50(+1)

Table 4. Normalized temperature distribution and heat fluxes for Problem 3.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	2.04106	0.0	0.0	0.0
0.10	2.03634	1.90642	8.09358	1.0(+1)
0.20	2.02159	4.05550	1.59445(+1)	2.0(+1)
0.30	1.99494	6.71132	2.32887(+1)	3.0(+1)
0.40	1.95313	1.01762(+1)	2.98238(+1)	4.0(+1)
0.50	1.89126	1.47952(+1)	3.52048(+1)	5.0(+1)
0.60	1.80264	2.09346(+1)	3.90654(+1)	6.0(+1)
0.70	1.67883	2.89189(+1)	4.10811(+1)	7.0(+1)
0.80	1.51008	3.89136(+1)	4.10864(+1)	8.0(+1)
0.90	1.28655	5.07816(+1)	3.92184(+1)	9.0(+1)
1.00	1.00000	6.40005(+1)	3.59995(+1)	1.0(+2)

Table 5. Normalized temperature distribution and heat fluxes for Problem 4.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	1.32936	0.0	0.0	0.0
0.10	1.32896	1.61753(−2)	9.83825(−1)	1.0
0.20	1.32767	3.70481(−2)	1.96295	2.0
0.30	1.32508	6.91398(−2)	2.93086	3.0
0.40	1.32039	1.23314(−1)	3.87669	4.0
0.50	1.31206	2.18860(−1)	4.78114	5.0
0.60	1.29725	3.90260(−1)	5.60974	6.0
0.70	1.27079	6.97497(−1)	6.30250	7.0
0.80	1.22364	1.23819	6.76181	8.0
0.90	1.14079	2.15037	6.84963	9.0
1.00	1.0	3.57497	6.42503	1.0(+1)

Table 6. Normalized temperature distribution and heat fluxes for Problem 5.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	1.59143	0.0	0.0	0.0
0.10	1.58908	9.45763(−3)	9.05424(−2)	1.0(−1)
0.20	1.58188	1.94624(−2)	1.80538(−1)	2.0(−1)
0.30	1.56940	3.07732(−2)	2.69227(−1)	3.0(−1)
0.40	1.55068	4.46777(−2)	3.55322(−1)	4.0(−1)
0.50	1.52391	6.35493(−2)	4.36451(−1)	5.0(−1)
0.60	1.48559	9.18078(−2)	5.08192(−1)	6.0(−1)
0.70	1.42925	1.37339(−1)	5.62661(−1)	7.0(−1)
0.80	1.34329	2.12767(−1)	5.87233(−1)	8.0(−1)
0.90	1.20882	3.33971(−1)	5.66029(−1)	9.0(−1)
1.00	1.0	5.10041(−1)	4.89959(−1)	1.0

Table 7. Normalized temperature distribution and heat fluxes for Problem 6.

r/R	$\Theta(r)$	$Q_c(r)$	$Q_r(r)$	$Q(r)$
0.00	1.13784	0.0	0.0	0.0
0.10	1.13682	2.05282(−2)	2.94718(−2)	5.0(−2)
0.20	1.13371	4.18525(−2)	5.81475(−2)	1.0(−1)
0.30	1.12839	6.47934(−2)	8.52066(−2)	1.5(−1)
0.40	1.12067	9.02177(−2)	1.09782(−1)	2.0(−1)
0.50	1.11024	1.19058(−1)	1.30942(−1)	2.5(−1)
0.60	1.09671	1.52323(−1)	1.47677(−1)	3.0(−1)
0.70	1.07959	1.91102(−1)	1.58898(−1)	3.5(−1)
0.80	1.05827	2.36547(−1)	1.63453(−1)	4.0(−1)
0.90	1.03202	2.89839(−1)	1.60161(−1)	4.5(−1)
1.00	1.0	3.52111(−1)	1.47889(−1)	5.0(−1)

be useful to know if this class of problems has been well formulated mathematically. Also, as we have no proof that the straightforward iteration scheme we use converges, we can only conjecture that the results given in Tables 2–7 are correct. Finally we note that for the six problems considered here, we observed what appeared to be convergence toward the established temperature distribution; however we did encounter problems where the method failed to converge.

While it is clear that the numerical methods used in this work can be used to solve some combined mode, radiation-conduction, heat transfer problems in a cylinder, we note that there are, in this class of problems, cases that we have not been able to solve. It is anticipated that more sophisticated iteration techniques will be investigated in future work.

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