IMPROVEMENTS IN THE F_N METHOD FOR RADIATIVE TRANSFER CALCULATIONS IN CLOUDS

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1. INTRODUCTION

In 1985 we applied the F_N method to solve a collection of radiative transfer problems (including one for a cloud) posed by the Radiation Commission of the International Association of Meteorology and Atmospheric Physics (Lenoble, 1977). Among the methods that were used to solve these test problems, the F_N method has proved to be one of the most successful, as demonstrated by the quality of the results published in the literature (Garcia and Siewert, 1985).

The version of the F_N method that we used in 1985 incorporated several computational improvements not available in previous versions of the method (Devaux and Siewert, 1980; Devaux, Siewert and Yuan, 1982); however, about three years ago, while trying to solve a problem for a cloud, with very strong scattering anisotropy, illuminated by an azimuthally dependent incident distribution, we detected numerical instabilities that precluded the application of the method to this class of problems. Since then we have focused our efforts on improving the numerical aspects of the method in order to obtain an accurate tool for solving highly anisotropic, azimuthally dependent, radiative transfer problems.

The purpose of this paper is to review the F_N method for computing the radiance and the flux in a cloud and to report our latest results on computational methods that we hope will end up in extending the range of applicability of the method to more demanding calculations.

2. STATEMENT OF THE PROBLEM

We let $I(\tau,\mu,\varphi)$ denote the intensity (radiance) of the radiation field and utilize the equation of transfer for a plane-parallel medium (Chandrasekhar, 1950) to model our cloud. We write

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu,\varphi)+I(\tau,\mu,\varphi) = \frac{\varpi}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}p(\cos\Theta)I(\tau,\mu',\varphi')\mathrm{d}\varphi'\mathrm{d}\mu' \quad (1)$$

where $\tau\in[0,\tau_0]$ is the optical variable, $\mu\in[-1,1]$ and $\varphi\in[0,2\pi]$ are respectively the cosine of the polar angle (as measured from the *positive* τ axis) and the azimuthal angle which describe the direction of propagation of the radiation and ϖ is the albedo for single scattering. In addition, the phase function $p(\cos\Theta)$ is represented by a finite Legendre expansion in terms of the cosine of the scattering angle Θ, *i.e.*

$$p(\cos\Theta)=\sum_{l=0}^{L}\beta_l P_l(\cos\Theta), \quad (2)$$

where the coefficients $\beta_0 = 1$ and $|\beta_l| < 2l+1$, $l\geq 1$, are computed from the phase function for a single particle (Mie scattering) averaged over the size distribution of the cloud particles. We point out that to describe cloud phase functions in an adequate manner a large number of terms is usually required in Eq. (2). This is due to the strong scattering anisotropy that is characteristic of clouds; for example, an expansion with $L = 299$ was used in our previous work (Garcia and Siewert, 1985) to represent the Cloud C1 phase function defined in the IAMAP report (Lenoble, 1977).

We assume that the cloud is illuminated uniformly by a radiation beam with direction specified by (μ_0,φ_0) and so we seek a solution to Eq. (1) that satisfies the boundary conditions, for $\mu>0$ and $\varphi\in[0,2\pi]$,

$$I(0,\mu,\varphi)=\pi\delta(\mu-\mu_0)\delta(\varphi-\varphi_0) \quad (3a)$$

and

$$I(\tau_0,-\mu,\varphi)=0. \quad (3b)$$

A convenient way of treating the azimuthal dependence of the problem is to use a Fourier decomposition to reduce the original problem to a series of azimuthally independent problems (Chandrasekhar, 1950; Garcia and Siewert, 1985). Here we can use the decomposition

$$I(\tau,\mu,\varphi)=R(\mu,\varphi)\delta(\mu-\mu_0)e^{-\tau/\mu} + \sum_{m=0}^{L}I^m(\tau,\mu)\cos m(\varphi-\varphi_0), \quad (4)$$

where, for $\mu>0$,

$$R(\mu,\varphi)=\pi\delta(\varphi-\varphi_0) - \tfrac{1}{2}\csc\tfrac{1}{2}(\varphi-\varphi_0)\sin\tfrac{1}{2}(2L+1)(\varphi-\varphi_0) \quad (5a)$$

and

$$R(-\mu,\varphi)=0, \quad (5b)$$

and the addition theorem for the Legendre polynomials, to show that the problem stated by Eqs. (1) and (3) can be reduced to the problem of solving, for $m=0,1,\dots,L$,

$$\mu\frac{\partial}{\partial\tau}I^m(\tau,\mu)+I^m(\tau,\mu) = \frac{\varpi}{2}\sum_{l=m}^{L}\frac{(l-m)!}{(l+m)!}\beta_l P_l^m(\mu)\int_{-1}^{1}P_l^m(\mu')I^m(\tau,\mu')\mathrm{d}\mu' \quad (6)$$

subject to the boundary conditions, for $\mu > 0$,

$$I^m(0,\mu) = \tfrac{1}{2}(2-\delta_{0,m})\delta(\mu-\mu_0) \tag{7a}$$

and

$$I^m(\tau_0,-\mu) = 0. \tag{7b}$$

It is clear that once we solve the problem stated by Eqs. (6) and (7) for $m = 0, 1, \ldots, L$ we obtain the Fourier components $\{I^m(\tau,\mu)\}$ and consequently the radiance $I(\tau,\mu,\varphi)$, as expressed by Eq. (4). Finally, expressions for the partial and net fluxes can be obtained by substituting Eq. (4) into the definitions

$$q_{\pm}(\tau) = \int_0^1 \int_0^{2\pi} \mu I(\tau, \pm\mu, \varphi)\mathrm{d}\varphi\mathrm{d}\mu \tag{8}$$

and

$$q(\tau) = \int_{-1}^1 \int_0^{2\pi} \mu I(\tau, \mu, \varphi)\mathrm{d}\varphi\mathrm{d}\mu \tag{9}$$

and evaluating the required integrals.

3. THE F_N METHOD

The F_N method was introduced in the field of radiative transfer by Siewert (1978). It has also been used extensively in other fields such as neutron transport theory, rarefied gas dynamics, kinetic theory and neutral particle transport in plasmas (see the survey by Garcia, 1985). Here we only sketch the essential ideas of the method as applied to our problem; the complete description of the method can be found in the literature (Garcia and Siewert, 1985; Garcia, 1985).

We can summarize the procedure followed by the method in the steps:

a. Deduce a set of singular integral equations and constraints for the unknown exit distributions $I^m(0,-\mu)$ and $I^m(\tau_0,\mu)$, $\mu > 0$, by means of an integral transform applied to Eq. (6);
b. Approximate the unknown exit distributions with a polynomial basis orthogonal in $[0,1]$;
c. Use a collocation scheme to match the number of equations to the number of unknowns;
d. Solve the resulting system of linear algebraic equations for the unknown coefficients of the approximation introduced in step b;
e. Compute the exit distributions, and
f. Repeat the procedure to compute internal distributions.

Highly accurate results obtained with the F_N method for the five test problems posed in the IAMAP report (Lenoble, 1977) have been published (Garcia and Siewert, 1985). By way of comparing our F_N solution with a solution developed with another widely used approximate method for solving the equation of transfer — the spherical harmonics (SH) method (Benassi, Garcia, Karp and Siewert, 1984) — we concluded that, although conceptually more complex and somewhat more difficult to implement, the F_N method is more efficient than the SH method in the sense that, for a given level of accuracy, it requires a lower order of approximation and consequently less computer time than the SH method.

4. RECENT IMPROVEMENTS IN THE METHOD

In this section, we discuss the results of our ongoing work on computational aspects of the F_N method. As stated in the Introduction, we have undertaken the task of developing a numerically stable method for radiative transfer calculations in clouds described by realistic phase functions.

a. Determination of the Discrete Spectrum

The first computational difficulty we encountered when we applied the method to more challenging problems was the determination of the so-called discrete spectrum, *i.e.* the discrete eigenvalues related to Eq. (6). In the context of the F_N method, the discrete eigenvalues belong to the set of points that define the collocation scheme referred to in Section 3. They are also required by other methods for solving the equation of transfer, as for example the method of elementary solutions (Case and Zweifel, 1967; Kuščer and McCormick, 1991), also known as the singular eigenfunction expansion method or Case's method.

The discrete spectrum is composed of the zeros in the complex plane cut from -1 to 1 along the real axis of the dispersion function

$$\Lambda^m(z) = 1 + z\int_{-1}^1 \psi^m(\mu)\frac{\mathrm{d}\mu}{\mu - z} \tag{10}$$

where the characteristic function is

$$\psi^m(\mu) = \frac{\varpi}{2}(1-\mu^2)^{m/2}\sum_{l=m}^{L}\frac{(l-m)!}{(l+m)!}\beta_l g_l^m(\mu)P_l^m(\mu) \tag{11}$$

and the Chandrasekhar polynomials $g_l^m(\xi)$, with the starting value $g_m^m(\xi) = (2m-1)!!$, satisfy, for $l \geq m$,

$$(2l+1-\varpi\beta_l)\xi g_l^m(\xi) = (l-m+1)g_{l+1}^m(\xi) + (l+m)(1-\delta_{l,m})g_{l-1}^m(\xi). \tag{12}$$

For years, the argument principle (Ahlfors, 1953) was considered the standard tool for computing the number of zeros of the dispersion function defined by Eq. (10). However, during an implementation of the argument principle to compute the number of zeros of Eq. (10) for all m for a problem defined with the $L = 299$ phase function mentioned in Section 2, we found that argument-principle calculations suffer from severe numerical limitations when applied to problems with strong scattering anisotropy and azimuthal dependence (Garcia and

Siewert, 1989). We then went on to propose an alternative method based on Sturm sequences (Wilkinson, 1965) that we found to be very stable and efficient. In addition, we also proposed the use of a bisection procedure based on a Sturm sequence in order to compute estimates for zeros of Eq. (10) with magnitudes just greater than 1 and to refine estimates of all the zeros of Eq. (10). Our procedure was found to be much more effective than previously used procedures based on Newton's method, mainly because it avoids the need of computing the dispersion function and its derivative accurately at each iteration step. This, as discussed in detail in our work (Garcia and Siewert, 1989), can be a terribly difficult, if not impossible, job.

b. Computation of the g-polynomials

The second computational difficulty that we resolved recently has to do with the computation of the Chandrasekhar (or g-) polynomials. In the past, it was thought that these polynomials could be computed in an accurate manner by using Eq. (12) in either the forward or backward direction. This technique is good for problems with a moderate degree of scattering anisotropy, but it may fail when the scattering is highly anisotropic.

Fortunately, we were able to devise an effective method for computing the Chandrasekhar polynomials on the real axis (Garcia and Siewert, 1990). As a matter of fact, we worked with a normalized version of the Chandrasekhar polynomials which is defined in a slightly different way than the definition we use in this paper. However, for our purposes here, this distinction is not important. Our method consists in using forward recursion of Eq. (12) when $|\xi| \in [0,1]$, a combination of backward and forward recursion of Eq. (12) plus the solution of a linear system when ξ is a discrete eigenvalue and a Darboux formula when ξ is not a discrete eigenvalue and $|\xi| \notin [0,1]$. A numerical study confirmed that the proposed method is capable of computing the Chandrasekhar polynomials accurately for problems with highly anisotropic phase functions (Garcia and Siewert, 1990).

c. Computation of the $T^m_{\alpha,l}$ Integrals

We recall from Section 3 that the procedure followed by the F_N method makes it possible to reduce the original problem [Eqs. (6) and (7) of Section 2] to the problem of solving one linear system (or two, if internal radiances and fluxes are desired). In order to compute the matrix elements of these linear systems (Garcia and Siewert, 1985), the integrals

$$T^m_{\alpha,l} = \int_0^1 \mu(1-\mu^2)^{m/2} P_\alpha(2\mu - 1) P^m_l(\mu) \mathrm{d}\mu \qquad (13)$$

must be evaluated for $m = 0, 1, \ldots, L$, $l = m, m+1, \ldots, L$ and $\alpha = 0, 1, \ldots, N$, where N is the order of the F_N approximation.

A recursive scheme for computing the required $T^m_{\alpha,l}$ was reported by Devaux, Siewert and Yuan (1982). Later this scheme was found to be unstable as $m \to \infty$ and an alternative recursive scheme was proposed (Garcia and Siewert, 1985). More recently, however, we have found that, although slightly superior to the previous scheme, the new scheme is also unstable as $m \to \infty$.

Our basic approach to overcome this problem was to look for ways of reducing the dimensionality of the calculation. We note that the problem is defined in a tridimensional space (the α-l-m space) and that the above mentioned recursive schemes (Devaux, Siewert and Yuan, 1982; Garcia and Siewert, 1985) work in bidimensional spaces (α-l and l-m respectively). In our study (Garcia and Siewert, to appear), we were able to deduce a 7-term recursion relation that works in the α-space, thereby allowing the entire calculation to be reduced to several one-dimensional calculations. Our final formula to compute the required $T^m_{\alpha,l}$ can be written as

$$A^m_{\alpha,l} T^m_{\alpha-3,l} + B^m_{\alpha,l} T^m_{\alpha-2,l} + C^m_{\alpha,l} T^m_{\alpha-1,l} + D^m_{\alpha,l} T^m_{\alpha,l} + E^m_{\alpha,l} T^m_{\alpha+1,l} + F^m_{\alpha,l} T^m_{\alpha+2,l} + G^m_{\alpha,l} T^m_{\alpha+3,l} = 0 \qquad (14)$$

where the coefficients $A^m_{\alpha,l}, B^m_{\alpha,l}, \ldots, G^m_{\alpha,l}$ are rational numbers that depend only on α, l and m.

We have concluded from the numerical implementation of our scheme that, although Eq. (14) can be used in the backward direction without loss of accuracy for almost the entire α-range, quite often the $T^m_{\alpha,l}$ integrals saturate and roundoff errors start to propagate when $\alpha \to 0$. When this is detected, we stop using Eq. (14) in the backward direction and switch to forward recursion to complete the calculation.

d. Computation of the A_α and B_α Functions

We are presently involved in evaluating the performance of existing methods (Garcia and Siewert, 1985) for computing the basic functions $A_\alpha(\xi)$ and $B_\alpha(\xi)$, for $\alpha = 0, 1, \ldots, N$, for problems with highly anisotropic phase functions. In the F_N method, these functions are needed to compute the coefficient matrices of the linear systems referred to in Section 3.

We have found that the two-term inhomogeneous recursion formula for $A_\alpha(\xi)$ [Eq. (71) of Garcia and Siewert, 1985] is not sufficiently accurate for the $L = 299$ *cloud* phase function, the reason being that the scheme used for computing the inhomogeneous term $V_\alpha(\xi)$ [Eqs. (69) and (70) of Garcia and Siewert, 1985] is subject to the propagation of roundoff errors. In order to overcome this problem, we are considering the alternative of using a three-term recursion formula for $A_\alpha(\xi)$ [Eq. (66) of Garcia and Siewert, 1985] in the backward direction; however the calculation of the starting values for backward recursion of the three-term formula will require the evaluation of the characteristic function $\psi^m(\mu)$ in an accurate manner, which can be a difficult problem *per se*.

5. CONCLUSIONS

In the past few years, we have been able to introduce significant improvements in the computational aspects of the F_N method for problems with highly anisotropic phase functions. New methods for determining the discrete spectrum, for computing the Chandrasekhar polynomials and for computing the $T^m_{\alpha,l}$ integrals were discussed in this paper. We are confident that in the near future we will be able to resolve the standing problem of computing the $A_\alpha(\xi)$ and $B_\alpha(\xi)$ functions in an accurate manner. Once this difficulty is resolved, we will be ready to apply the F_N method with confidence in the solution of general radiative transfer problems for clouds.

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