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A SPHERICAL-HARMONICS METHOD FOR MULTI-GROUP OR NON-GRAY RADIATION TRANSPORT

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Abstract—The spherical-harmonics method, also called the P_N method, is used to develop solutions to a class of multi-group or non-gray radiation transport problems. The multi-group model considered allows an anisotropic scattering law and transfer from any group to any group. In addition to a spherical-harmonics solution for the case of a homogeneous radiative-transfer equation, a particular solution for the P_N method is derived for the case of multi-group radiative transfer in a homogeneous plane-parallel medium that contains group sources that vary with position and direction. Computational aspects of the developed solutions are discussed, and numerical results for a test case are reported.

INTRODUCTION

We consider here the multi-group or non-gray radiation transport equation written as

$$\mu \frac{\partial}{\partial z} \boldsymbol{\Psi}(z, \mu) + \mathbf{S}\boldsymbol{\Psi}(z, \mu) = \frac{1}{2} \sum_{l=0}^{L} P_l(\mu)\mathbf{T}_l \int_{-1}^{1} P_l(\mu')\boldsymbol{\Psi}(z, \mu')\,\mathrm{d}\mu' + \mathbf{E}(z, \mu) \tag{1}$$

for $z \in (0, z_0)$ and $\mu \in [-1, 1]$. Here the Legendre polynomials are denoted by $P_l(\mu)$, and the transfer matrices $\mathbf{T}_l$ are such that particle transfer (by, say, scattering and/or fission) between and within all energy groups is allowed. In addition, the elements $\psi_1(z, \mu), \psi_2(z, \mu), \ldots, \psi_M(z, \mu)$ of the M-vector $\boldsymbol{\Psi}(z, \mu)$ are the group angular fluxes or intensities, the elements $s_1, s_2, \ldots, s_M$ of the diagonal $\mathbf{S}$ matrix are the group total cross sections, z is the position variable measured in cm and μ is the direction cosine, with respect to the positive z axis, that defines the direction of motion. Finally, we use $\mathbf{E}(z, \mu)$ in Eq. (1) to represent an inhomogeneous (specified) source that could describe, for example, spontaneous emission or could be present in the equation because of a mathematical decomposition, as Chandrasekhar[1] did, of some previously formulated problem.

Along with Eq. (1), we consider here boundary conditions of the form

$$\boldsymbol{\Psi}(0, \mu) = \mathbf{F}_1(\mu) \tag{2a}$$

and

$$\boldsymbol{\Psi}(z_0, -\mu) = \mathbf{F}_2(\mu) \tag{2b}$$

for $\mu \in [0, 1]$. Here $\mathbf{F}_1(\mu)$ and $\mathbf{F}_2(\mu)$ are considered given.

In order to use dimensionless units we introduce an optical variable $\tau = zs_{\min}$ and an optical thickness $\tau_0 = z_0 s_{\min}$, where $s_{\min}$ is the minimum of the set $\{s_i\}$, and rewrite Eqs. (1) and (2) as

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{\Psi}(\tau, \mu) + \boldsymbol{\Sigma}\boldsymbol{\Psi}(\tau, \mu) = \frac{1}{2} \sum_{l=0}^{L} P_l(\mu)\mathbf{C}_l \int_{-1}^{1} P_l(\mu')\boldsymbol{\Psi}(\tau, \mu')\,\mathrm{d}\mu' + \mathbf{Q}(\tau, \mu), \tag{3}$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$, and

$$\boldsymbol{\Psi}(0, \mu) = \mathbf{F}_1(\mu) \tag{4a}$$

and

$$\boldsymbol{\Psi}(\tau_0, -\mu) = \mathbf{F}_2(\mu), \tag{4b}$$

for $\mu \in [0, 1]$. Here the diagonal matrix $\boldsymbol{\Sigma}$ has entries $\sigma_i = s_i/s_{\min}$, the dimensionless transfer matrices are defined by $\mathbf{C}_l = \mathbf{T}_l/s_{\min}$ and $\mathbf{Q}(\tau, \mu) = \mathbf{E}(\tau, \mu)/s_{\min}$.

THE HOMOGENEOUS EQUATION

Following our previous work, for example Refs. 2–5, with the spherical-harmonics method, we express our approximate P_N solution to the homogeneous version of Eq. (3) as

$$\boldsymbol{\Psi}_c(\tau,\mu)=\sum_{l=0}^{N}\frac{2l+1}{2}P_l(\mu)\sum_{j=1}^{J}[A_j\,e^{-\tau/\xi_j}+(-1)^l B_j\,e^{-(\tau_0-\tau)/\xi_j}]\mathbf{G}_l(\xi_j)\mathbf{N}(\xi_j) \tag{5}$$

where the constants A_j and B_j are to be fixed by the boundary conditions. We consider N to be odd, and so the spectrum is given by $\xi=\xi_j, j=1,2,\ldots,J=M(N+1)/2$. Here the $M\times M$ matrix of polynomials $\mathbf{G}_l(\xi)$ are the result of a multi-group extension of the Chandrasekhar polynomials,[1] and the ξ_j denote the J zeros of det $\mathbf{G}_{N+1}(\xi)$ that lie in the right half-plane. Finally the vector $\mathbf{N}(\xi_j)$ is used to denote a null-vector of $\mathbf{G}_{N+1}(\xi_j)$. To be specific, we note from Ref. 6 that the matrix version of the Chandrasekhar polynomials required here can be defined by the starting value

$$\mathbf{G}_0(\xi)=\mathbf{I} \tag{6}$$

and the three-term recursion formula

$$\xi\mathbf{h}_l\mathbf{G}_l(\xi)=(l+1)\mathbf{G}_{l+1}(\xi)+l\mathbf{G}_{l-1}(\xi) \tag{7}$$

for $l=0,1,\ldots$. Here $\mathbf{I}$ denotes the $M\times M$ identity matrix and

$$\mathbf{h}_l=(2l+1)\boldsymbol{\Sigma}-\boldsymbol{C}_l \tag{8a}$$

for $l=0,1,\ldots,L$ and

$$\mathbf{h}_l=(2l+1)\boldsymbol{\Sigma} \tag{8b}$$

for $l>L$. It should be noted that in Ref. 6, Siewert and Thomas reported a method for computing the discrete spectrum for the formally exact method of elementary solutions,[7] and while the method used in Ref. 6 became (in high order) a very good technique for computing the required discrete spectrum, the method can also define (at every order N of the approximation) exactly the spectrum we require here. To pursue this point, we consider ξ to be on the spectrum, let

$$\mathbf{T}_l(\xi)=\mathbf{G}_l(\xi)\mathbf{N}(\xi) \tag{9}$$

for $l=0,1,\ldots,N+1$ and multiply Eq. (7) by $\mathbf{N}(\xi)$ to obtain

$$\xi\mathbf{h}_l\mathbf{T}_l(\xi)=(l+1)\mathbf{T}_{l+1}(\xi)+l\mathbf{T}_{l-1}(\xi) \tag{10}$$

for $l=0,1,\ldots,N$. Following Ref. 6, we continue to consider N to be odd and eliminate the odd-order $\mathbf{T}$ vectors from Eq. (10) to obtain, for $l=0,2,4,\ldots,N-1$,

$$\mathbf{X}_l\mathbf{T}_{l-2}(\xi)+\mathbf{Y}_l\mathbf{T}_l(\xi)+\mathbf{Z}_l\mathbf{T}_{l+2}(\xi)=\xi^2\mathbf{T}_l(\xi) \tag{11}$$

where

$$\mathbf{X}_l=l(l-1)\mathbf{h}_l^{-1}\mathbf{h}_{l-1}^{-1}, \tag{12a}$$

$$\mathbf{Y}_l=l^2\mathbf{h}_l^{-1}\mathbf{h}_{l-1}^{-1}+(l+1)^2\mathbf{h}_l^{-1}\mathbf{h}_{l+1}^{-1} \tag{12b}$$

and

$$\mathbf{Z}_l=(l+1)(l+2)\mathbf{h}_l^{-1}\mathbf{h}_{l+1}^{-1}. \tag{12c}$$

Equation (11) and the truncation condition $\mathbf{T}_{N+1}(\xi)=\mathbf{0}$ can now be expressed as the eigenvalue problem

$$\mathbf{AX}=\xi^2\mathbf{X} \tag{13}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{Y}_0 & \mathbf{Z}_0 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{X}_2 & \mathbf{Y}_2 & \mathbf{Z}_2 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_4 & \mathbf{Y}_4 & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{Y}_{N-5} & \mathbf{Z}_{N-5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{X}_{N-3} & \mathbf{Y}_{N-3} & \mathbf{Z}_{N-3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{X}_{N-1} & \mathbf{Y}_{N-1} \end{bmatrix}. \tag{14}$$

Here the $\mathbf{A}$ matrix is $M(N+1)/2$ square and the $\mathbf{X}$ vector has entries $\mathbf{T}_0(\xi)$, $\mathbf{T}_2(\xi), \dots, \mathbf{T}_{N-1}(\xi)$. It thus is clear that the $J = M(N+1)/2$ eigenvalues of $\mathbf{A}$ are the squares of the $J \pm$ pairs of zeros of $\det \mathbf{G}_{N+1}(\xi)$.

We can now use, for example, the linear-algebra package EISPACK[8] to compute the eigenvalues and eigenvectors of the $\mathbf{A}$ matrix. Of course the eigenvectors of $\mathbf{A}$ contain only the even-order $\mathbf{T}$ vectors, i.e., $\mathbf{T}_0(\xi), \mathbf{T}_2(\xi), \dots, \mathbf{T}_{N-1}(\xi)$. However, given the even-order $\mathbf{T}$ vectors, we can immediately find the odd-order $\mathbf{T}$ vectors, i.e., $\mathbf{T}_1(\xi), \mathbf{T}_3(\xi), \dots, \mathbf{T}_N(\xi)$, from Eq. (10). With the eigenvalue spectrum and the $\mathbf{T}$ vectors so established, we rewrite our spherical-harmonics solution to the homogeneous version of Eq. (3) as

$$\boldsymbol{\Psi}_c(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} [A_j e^{-\tau/\xi_j} + (-1)^l B_j e^{-(\tau_0 - \tau)/\xi_j}] \mathbf{T}_l(\xi_j) \tag{15}$$

with only the constants $\{A_j, B_j\}$ left to be determined from the boundary conditions.

A PARTICULAR SOLUTION

Giving attention now to the problem of finding a particular solution of Eq. (3) appropriate to the spherical-harmonics method, we note first of all that Roux, Smith, and Todd[9] used the method of variation of parameters to reduce the job of finding a particular solution for an isotropic source term for the one-group or gray model to the need to solve a system of linear algebraic equations, which subsequently was solved analytically by Siewert and Thomas.[5] Following the paper by Siewert and Thomas,[5] McCormick and Siewert[10] were able to generalize the results of Ref. 5 to find a particular solution for the formally exact method of elementary solutions and for the spherical harmonics method for the case of an angularly- and spatially-dependent inhomogeneous source term.

To begin we express the desired particular solution as

$$\boldsymbol{\Psi}_p(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} [A_j(\tau) e^{-\tau/\xi_j} + (-1)^l B_j(\tau) e^{-(\tau_0 - \tau)/\xi_j}] \mathbf{T}_l(\xi_j) \tag{16}$$

where the functions $A_j(\tau)$ and $B_j(\tau)$ are to be found. Substituting Eq. (16) into Eq. (3), multiplying the resulting equation by $P_\beta(\mu)$, for $\beta = 0, 1, \dots, N$, and integrating over μ, we find

$$\sum_{j=1}^{J} \xi_j [A_j'(\tau) e^{-\tau/\xi_j} - (-1)^\beta B_j'(\tau) e^{-(\tau_0 - \tau)/\xi_j}] \mathbf{h}_\beta \mathbf{T}_\beta(\xi_j) = 2\mathbf{Q}_\beta(\tau) \tag{17}$$

where

$$\mathbf{Q}_\beta(\tau) = \frac{2\beta+1}{2} \int_{-1}^{1} \mathbf{Q}(\tau, \mu) P_\beta(\mu) \, d\mu \tag{18}$$

and where the symbol prime is used to denote differentiation with respect to τ. At this point we let

$$X_j(\tau) = \xi_j [A_j'(\tau) e^{-\tau/\xi_j} - B_j'(\tau) e^{-(\tau_0 - \tau)/\xi_j}] \tag{19a}$$

and

$$Y_j(\tau) = \xi_j [A_j'(\tau) e^{-\tau/\xi_j} + B_j'(\tau) e^{-(\tau_0 - \tau)/\xi_j}] \tag{19b}$$

and rewrite Eq. (17) as

$$\sum_{j=1}^{J} X_j(\tau)\mathbf{T}_{2k-2}(\xi_j) = 2\mathbf{h}_{2k-2}^{-1}\mathbf{Q}_{2k-2}(\tau) \tag{20a}$$

and

$$\sum_{j=1}^{J} Y_j(\tau)\mathbf{T}_{2k-1}(\xi_j) = 2\mathbf{h}_{2k-1}^{-1}\mathbf{Q}_{2k-1}(\tau) \tag{20b}$$

for $k = 1, 2, \ldots, (N+1)/2$. In order to express Eqs. (20) in matrix form we introduce

$$\mathbf{T}_e = \begin{bmatrix} \mathbf{T}_0(\xi_1) & \mathbf{T}_0(\xi_2) & \mathbf{T}_0(\xi_3) & \ldots & \mathbf{T}_0(\xi_J) \\ \mathbf{T}_2(\xi_1) & \mathbf{T}_2(\xi_2) & \mathbf{T}_2(\xi_3) & \ldots & \mathbf{T}_2(\xi_J) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_{N-1}(\xi_1) & \mathbf{T}_{N-1}(\xi_2) & \mathbf{T}_{N-1}(\xi_3) & \ldots & \mathbf{T}_{N-1}(\xi_J) \end{bmatrix} \tag{21a}$$

and

$$\mathbf{T}_o = \begin{bmatrix} \mathbf{T}_1(\xi_1) & \mathbf{T}_1(\xi_2) & \mathbf{T}_1(\xi_3) & \ldots & \mathbf{T}_1(\xi_J) \\ \mathbf{T}_3(\xi_1) & \mathbf{T}_3(\xi_2) & \mathbf{T}_3(\xi_3) & \ldots & \mathbf{T}_3(\xi_J) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{T}_N(\xi_1) & \mathbf{T}_N(\xi_2) & \mathbf{T}_N(\xi_3) & \ldots & \mathbf{T}_N(\xi_J) \end{bmatrix} \tag{21b}$$

so that we can write

$$\mathbf{T}_e\mathbf{X}(\tau) = \mathbf{U}(\tau) \quad \text{and} \quad \mathbf{T}_o\mathbf{Y}(\tau) = \mathbf{V}(\tau). \tag{22a, b}$$

Here

$$\mathbf{U}(\tau) = 2\begin{bmatrix} \mathbf{h}_0^{-1}\mathbf{Q}_0(\tau) \\ \mathbf{h}_2^{-1}\mathbf{Q}_2(\tau) \\ \vdots \\ \mathbf{h}_{N-1}^{-1}\mathbf{Q}_{N-1}(\tau) \end{bmatrix} \quad \text{and} \quad \mathbf{V}(\tau) = 2\begin{bmatrix} \mathbf{h}_1^{-1}\mathbf{Q}_1(\tau) \\ \mathbf{h}_3^{-1}\mathbf{Q}_3(\tau) \\ \vdots \\ \mathbf{h}_N^{-1}\mathbf{Q}_N(\tau) \end{bmatrix} \tag{23a, b}$$

and also

$$\mathbf{X}(\tau) = \begin{bmatrix} X_1(\tau) \\ X_2(\tau) \\ \vdots \\ X_J(\tau) \end{bmatrix} \quad \text{and} \quad \mathbf{Y}(\tau) = \begin{bmatrix} Y_1(\tau) \\ Y_2(\tau) \\ \vdots \\ Y_J(\tau) \end{bmatrix}. \tag{24a, b}$$

As an alternative to formulating our eigenvalue problem as Eq. (13), we can eliminate the even-order **T** vectors from Eq. (10) to find

$$\mathbf{B}\mathbf{Y} = \xi^2\mathbf{Y} \tag{25}$$

where

$$\mathbf{B} = \begin{bmatrix} \mathbf{Y}_1 & \mathbf{Z}_1 & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{X}_3 & \mathbf{Y}_3 & \mathbf{Z}_3 & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_5 & \mathbf{Y}_5 & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{Y}_{N-4} & \mathbf{Z}_{N-4} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{X}_{N-2} & \mathbf{Y}_{N-2} & \mathbf{Z}_{N-2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{X}_N & \mathbf{Y}'_N \end{bmatrix}. \tag{26}$$

Here

$$\mathbf{Y}'_N = N^2 \mathbf{h}_N^{-1} \mathbf{h}_{N-1}^{-1} \tag{27}$$

and the $\mathbf{Y}$ vector has elements $\mathbf{T}_1(\xi), \mathbf{T}_3(\xi), \ldots, \mathbf{T}_N(\xi)$.

At this point we are ready to solve Eqs. (22) to find $\mathbf{X}(\tau)$ and $\mathbf{Y}(\tau)$; however, lacking a required proof, we must now assume that the matrices $\mathbf{A}$ and $\mathbf{B}$ are not defective so as to ensure that $\mathbf{T}_e$ and $\mathbf{T}_o$ are invertible. Following this assumption, we can write

$$\mathbf{X}(\tau) = \mathbf{T}_e^{-1}\mathbf{U}(\tau) \tag{28a}$$

and

$$\mathbf{Y}(\tau) = \mathbf{T}_o^{-1}\mathbf{V}(\tau). \tag{28b}$$

We can now use Eqs. (28) in Eqs. (19) to find

$$\mathbf{A}'(\tau) = \tfrac{1}{2}\mathbf{D}\, e^{\tau \mathbf{D}}[\mathbf{T}_o^{-1}\mathbf{V}(\tau) + \mathbf{T}_e^{-1}\mathbf{U}(\tau)] \tag{29a}$$

and

$$\mathbf{B}'(\tau) = \tfrac{1}{2}\mathbf{D}\, e^{(\tau_0 - \tau)\mathbf{D}}[\mathbf{T}_o^{-1}\mathbf{V}(\tau) - \mathbf{T}_e^{-1}\mathbf{U}(\tau)] \tag{29b}$$

where

$$\mathbf{A}(\tau) = \begin{bmatrix} A_1(\tau) \\ A_2(\tau) \\ \vdots \\ A_J(\tau) \end{bmatrix} \quad \text{and} \quad \mathbf{B}(\tau) = \begin{bmatrix} B_1(\tau) \\ B_2(\tau) \\ \vdots \\ B_J(\tau) \end{bmatrix}. \tag{30a, b}$$

In addition, we have introduced

$$\mathbf{D} = \operatorname{diag}\left\{\frac{1}{\xi_1}, \frac{1}{\xi_2}, \ldots, \frac{1}{\xi_J}\right\}. \tag{31}$$

Of course, we can integrate Eqs. (29) to find the final results required in Eq. (16), viz.

$$e^{-\tau \mathbf{D}}\mathbf{A}(\tau) = \tfrac{1}{2}\mathbf{D}\int_0^{\tau} e^{-(\tau - x)\mathbf{D}}[\mathbf{T}_e^{-1}\mathbf{U}(x) + \mathbf{T}_o^{-1}\mathbf{V}(x)]\, dx \tag{32a}$$

and

$$e^{-(\tau_0 - \tau)\mathbf{D}}\mathbf{B}(\tau) = \tfrac{1}{2}\mathbf{D}\int_{\tau}^{\tau_0} e^{-(x - \tau)\mathbf{D}}[\mathbf{T}_e^{-1}\mathbf{U}(x) - \mathbf{T}_o^{-1}\mathbf{V}(x)]\, dx. \tag{32b}$$

In order to obtain the results given by Eqs. (32) for the functions $\{A_j(\tau)\}$ and $\{B_j(\tau)\}$ required in Eq. (16), we had only to make the assumption that the matrices $\mathbf{A}$ and $\mathbf{B}$ are not defective. Now, following those assumptions, we can obtain somewhat more explicit results. To this end, we first let $\mathbf{T}_e(\xi_j)$ and $\mathbf{T}_o(\xi_j)$ denote the j-th columns of the matrices $\mathbf{T}_e$ and $\mathbf{T}_o$ respectively, so that we can write

$$\mathbf{A}\mathbf{T}_e(\xi_j) = \xi_j^2 \mathbf{T}_e(\xi_j) \tag{33a}$$

and

$$\mathbf{B}\mathbf{T}_o(\xi_j) = \xi_j^2 \mathbf{T}_o(\xi_j). \tag{33b}$$

As $\mathbf{A}$ and $\mathbf{B}$ are considered to have complete sets of eigenvectors, we let $\tilde{\mathbf{T}}_e^*(\xi_j)$ and $\tilde{\mathbf{T}}_o^*(\xi_j)$, where the tilde is used to denote the transpose operation, represent the left eigenvectors of $\mathbf{A}$ and $\mathbf{B}$, i.e.

$$\tilde{\mathbf{T}}_e^*(\xi_j)\mathbf{A} = \xi_j^2 \tilde{\mathbf{T}}_e^*(\xi_j) \tag{34a}$$

and

$$\tilde{\mathbf{T}}_o^*(\xi_j)\mathbf{B} = \xi_j^2 \tilde{\mathbf{T}}_o^*(\xi_j), \tag{34b}$$

that are orthogonal to the corresponding right eigenvectors. We thus can write

$$\tilde{\mathbf{T}}_e^*(\xi_i)\mathbf{T}_e(\xi_j) = N_e(\xi_i)\delta_{i,j} \tag{35a}$$

and

$$\tilde{\mathbf{T}}_o^*(\xi_i)\mathbf{T}_o(\xi_j) = N_o(\xi_i)\delta_{i,j}. \tag{35b}$$

If we now take the transpose of Eqs. (34) and note Eqs. (10), (14) and (26), we can conclude that

$$\mathbf{T}_e^*(\xi_j) = \begin{bmatrix} \tilde{\mathbf{h}}_0\mathbf{T}_0^\dagger(\xi_j) \\ \tilde{\mathbf{h}}_2\mathbf{T}_2^\dagger(\xi_j) \\ \vdots \\ \tilde{\mathbf{h}}_{N-1}\mathbf{T}_{N-1}^\dagger(\xi_j) \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{T}}_o^*(\xi_j) = \begin{bmatrix} \tilde{\mathbf{h}}_1\mathbf{T}_1^\dagger(\xi_j) \\ \tilde{\mathbf{h}}_3\mathbf{T}_3^\dagger(\xi_j) \\ \vdots \\ \tilde{\mathbf{h}}_N\mathbf{T}_N^\dagger(\xi_j) \end{bmatrix} \tag{36a, b}$$

where the vectors $\mathbf{T}_l^\dagger(\xi_j)$ satisfy a three-term recursion formula similar to Eq. (10), viz

$$\xi\tilde{\mathbf{h}}\mathbf{T}_l^\dagger(\xi) = (l+1)\mathbf{T}_{l+1}^\dagger(\xi) + l\mathbf{T}_{l-1}^\dagger(\xi) \tag{37}$$

for $l = 0, 1, \ldots, N$. These vectors also have the truncation condition $\mathbf{T}_{N+1}^\dagger(\xi_j) = \mathbf{0}$ for $j = 1, 2, \ldots, J$.

Considering now Eqs. (35) and (36), we conclude that we can write

$$N_e(\xi_j) = \sum_{k=1}^{K} \tilde{\mathbf{T}}_{2k-2}^\dagger(\xi_j)\mathbf{h}_{2k-2}\mathbf{T}_{2k-2}(\xi_j) \tag{38a}$$

and

$$N_o(\xi_j) = \sum_{k=1}^{K} \tilde{\mathbf{T}}_{2k-1}^\dagger(\xi_j)\mathbf{h}_{2k-1}\mathbf{T}_{2k-1}(\xi_j) \tag{38b}$$

where $K = (N+1)/2$. In fact, we can also deduce from Eqs. (10) and (37) that $N_e(\xi_j) = N_o(\xi_j)$. We can now find from Eqs. (32) the explicit results

$$A_j(\tau)\,\mathrm{e}^{-\tau/\xi_j} = \frac{C_j}{\xi_j}\sum_{\alpha=0}^{N} \tilde{\mathbf{T}}_\alpha^\dagger(\xi_j)\int_0^\tau \mathbf{Q}_\alpha(x)\,\mathrm{e}^{-(\tau-x)/\xi_j}\,\mathrm{d}x \tag{39a}$$

and

$$B_j(\tau)\,\mathrm{e}^{-(\tau_0-\tau)/\xi_j} = \frac{C_j}{\xi_j}\sum_{\alpha=0}^{N} (-1)^\alpha\tilde{\mathbf{T}}_\alpha^\dagger(\xi_j)\int_\tau^{\tau_0} \mathbf{Q}_\alpha(x)\,\mathrm{e}^{-(x-\tau)/\xi_j}\,\mathrm{d}x \tag{39b}$$

where

$$C_j = \left(\sum_{k=1}^{K} \tilde{\mathbf{T}}_{2k-2}^\dagger(\xi_j)\mathbf{h}_{2k-2}\mathbf{T}_{2k-2}(\xi_j)\right)^{-1}, \quad j = 1, 2, \ldots, J. \tag{40}$$

Finally, we rewrite the desired particular solution as

$$\boldsymbol{\Psi}_p(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \frac{C_j}{\xi_j}[U_j(\tau) + (-1)^l V_j(\tau)]\mathbf{T}_l(\xi_j) \tag{41}$$

where

$$U_j(\tau) = \int_0^\tau u_j(x)\,\mathrm{e}^{-(\tau-x)/\xi_j}\,\mathrm{d}x \tag{42a}$$

and

$$V_j(\tau) = \int_\tau^{\tau_0} v_j(x)\,\mathrm{e}^{-(x-\tau)/\xi_j}\,\mathrm{d}x \tag{42b}$$

with

$$u_j(x) = \sum_{\alpha=0}^{N} \tilde{\mathbf{T}}_\alpha^\dagger(\xi_j)\mathbf{Q}_\alpha(x) \tag{43a}$$

and

$$v_j(x) = \sum_{\alpha=0}^{N} (-1)^\alpha\tilde{\mathbf{T}}_\alpha^\dagger(\xi_j)\mathbf{Q}_\alpha(x). \tag{43b}$$

AN APPLICATION

Having established in previous sections our spherical-harmonics solutions for the homogeneous and inhomogeneous versions of the equation of transfer, we are now ready to solve the general problem defined by Eqs. (3) and (4). We write

$$\mathbf{\Psi}(\tau,\mu)=\mathbf{\Psi}_p(\tau,\mu)+\sum_{l=0}^{N}\frac{2l+1}{2}P_l(\mu)\sum_{j=1}^{J}[A_j e^{-\tau/\xi_j}+(-1)^l B_j e^{-(\tau_0-\tau)/\xi_j}]\mathbf{T}_l(\xi_j) \tag{44}$$

where the constants A_j and B_j are to be found. Substituting Eq. (44) into Eqs. (4), we find

$$\sum_{l=0}^{N}\frac{2l+1}{2}P_l(\mu)\sum_{j=1}^{J}[A_j+(-1)^l B_j e^{-\tau_0/\xi_j}]\mathbf{T}_l(\xi_j)=\mathbf{F}_1(\mu)-\mathbf{\Psi}_p(0,\mu) \tag{45a}$$

and

$$\sum_{l=0}^{N}\frac{2l+1}{2}P_l(\mu)\sum_{j=1}^{J}[B_j+(-1)^l A_j e^{-\tau_0/\xi_j}]\mathbf{T}_l(\xi_j)=\mathbf{F}_2(\mu)-\mathbf{\Psi}_p(\tau_0,-\mu) \tag{45b}$$

for $\mu\in[0,1]$. As we can satisfy only approximate versions of Eqs. (45), we choose here to use the Marshak and the Mark approximations to the boundary conditions.[11]

For the Marshak approximation, we multiply Eqs. (45) by $P_{2\alpha+1}(\mu)$ and integrate over μ from 0 to 1 to obtain the system of linear algebraic equations

$$\sum_{l=0}^{N}\frac{2l+1}{2}S_{\alpha,l}\sum_{j=1}^{J}[A_j+(-1)^l B_j e^{-\tau_0/\xi_j}]\mathbf{T}_l(\xi_j)$$
$$=\int_0^1 P_{2\alpha+1}(\mu)\mathbf{F}_1(\mu)\,d\mu-\sum_{l=0}^{N}\frac{2l+1}{2}S_{\alpha,l}(-1)^l\sum_{j=1}^{J}\frac{C_j}{\xi_j}V_j(0)\mathbf{T}_l(\xi_j) \tag{46a}$$

and

$$\sum_{l=0}^{N}\frac{2l+1}{2}S_{\alpha,l}\sum_{j=1}^{J}[B_j+(-1)^l A_j e^{-\tau_0/\xi_j}]\mathbf{T}_l(\xi_j)$$
$$=\int_0^1 P_{2\alpha+1}(\mu)\mathbf{F}_2(\mu)\,d\mu-\sum_{l=0}^{N}\frac{2l+1}{2}S_{\alpha,l}(-1)^l\sum_{j=1}^{J}\frac{C_j}{\xi_j}U_j(\tau_0)\mathbf{T}_l(\xi_j) \tag{46b}$$

for $\alpha=0,1,\ldots,(N-1)/2$. Here the constants

$$S_{\alpha,l}=\int_0^1 P_{2\alpha+1}(\mu)P_l(\mu)\,d\mu \tag{47}$$

can be evaluated as mentioned in Ref. 3.

For the Mark approximation, we let μ_k, for $k=1,2,\ldots,K=(N+1)/2$, denote the K positive zeros of $P_{N+1}(\mu)$ and consider Eqs. (45) only at these points to obtain

$$\sum_{l=0}^{N}\frac{2l+1}{2}P_l(\mu_k)\sum_{j=1}^{J}[A_j+(-1)^l B_j e^{-\tau_0/\xi_j}]\mathbf{T}_l(\xi_j)=\mathbf{F}_1(\mu_k)-\mathbf{\Psi}_p(0,\mu_k) \tag{48a}$$

and

$$\sum_{l=0}^{N}\frac{2l+1}{2}P_l(\mu_k)\sum_{j=1}^{J}[B_j+(-1)^l A_j e^{-\tau_0/\xi_j}]\mathbf{T}_l(\xi_j)=\mathbf{F}_2(\mu_k)-\mathbf{\Psi}_p(\tau_0,-\mu_k) \tag{48b}$$

for $k=1,2,\ldots,K$.

Considering now that we have solved either Eqs. (46) or (48) to find the constants A_j and B_j, we can find the group fluxes and currents,

$$\mathbf{\Psi}_0(\tau)=\int_{-1}^{1}\mathbf{\Psi}(\tau,\mu)\,d\mu \tag{49a}$$

and

$$\mathbf{\Psi}_1(\tau)=\int_{-1}^{1}\mathbf{\Psi}(\tau,\mu)\mu\,d\mu, \tag{49b}$$

by integrating Eq. (44). We find

$$\boldsymbol{\Psi}_0(\tau) = \sum_{j=1}^{J} \{A_j \mathrm{e}^{-\tau/\xi_j} + B_j \mathrm{e}^{-(\tau_0-\tau)/\xi_j} + \frac{C_j}{\xi_j}[U_j(\tau) + V_j(\tau)]\}\mathbf{T}_0(\xi_j) \tag{50a}$$

and

$$\boldsymbol{\Psi}_1(\tau) = \sum_{j=1}^{J} \{A_j \mathrm{e}^{-\tau/\xi_j} - B_j \mathrm{e}^{-(\tau_0-\tau)/\xi_j} + \frac{C_j}{\xi_j}[U_j(\tau) - V_j(\tau)]\}\mathbf{T}_1(\xi_j). \tag{50b}$$

Of course, having found the constants A_j and B_j, we can also compute the the angular fluxes for each group. Rather than compute the angular fluxes from Eq. (44) we substitute Eq. (44) into the right-hand side of Eq. (3) and then integrate the resulting equation to find the group angular fluxes. In this manner we find

$$\boldsymbol{\Psi}(\tau, \mu) = \mathrm{e}^{-\boldsymbol{\Sigma}\tau/\mu}\mathbf{F}_1(\mu) + \frac{1}{\mu}\int_0^\tau \mathrm{e}^{-\boldsymbol{\Sigma}(\tau-x)/\mu}\mathbf{Q}(x, \mu)\,\mathrm{d}x + \boldsymbol{\Xi}(\tau, \mu)$$

$$+ \frac{1}{2}\sum_{l=0}^{N} P_l(\mu) \sum_{j=1}^{J} \{\xi_j[A_j\mathbf{C}(\tau:\mu\boldsymbol{\Sigma}^{-1}, \xi_j) + (-1)^l B_j \mathrm{e}^{-(\tau_0-\tau)/\xi_j}\mathbf{S}(\tau:\mu\boldsymbol{\Sigma}^{-1}, \xi_j)]\}\boldsymbol{\Sigma}^{-1}\mathbf{C}_l\mathbf{T}_l(\xi_j) \tag{51a}$$

and

$$\boldsymbol{\Psi}(\tau, -\mu) = \mathrm{e}^{-\boldsymbol{\Sigma}(\tau_0-\tau)/\mu}\mathbf{F}_2(\mu) + \frac{1}{\mu}\int_\tau^{\tau_0} \mathrm{e}^{-\boldsymbol{\Sigma}(x-\tau)/\mu}\mathbf{Q}(x, -\mu)\,\mathrm{d}x + \boldsymbol{\Xi}(\tau, -\mu)$$

$$+ \frac{1}{2}\sum_{l=0}^{N} P_l(\mu) \sum_{j=1}^{J} \{\xi_j[(-1)^l A_j \mathrm{e}^{-\tau/\xi_j}\mathbf{S}(\tau_0-\tau:\mu\boldsymbol{\Sigma}^{-1}, \xi_j) + B_j\mathbf{C}(\tau_0-\tau:\mu\boldsymbol{\Sigma}^{-1}, \xi_j)]\}\boldsymbol{\Sigma}^{-1}\mathbf{C}_l\mathbf{T}_l(\xi_j) \tag{51b}$$

for $\mu \in [0, 1]$. Here we have introduced the definitions

$$\mathbf{C}(a:\mu\boldsymbol{\Sigma}^{-1}, \xi_j) = \mathrm{diag}\{C(a, \mu/\sigma_1, \xi_j), C(a, \mu/\sigma_2, \xi_j), \ldots, C(a, \mu/\sigma_M, \xi_j)\} \tag{52a}$$

and

$$\mathbf{S}(a:\mu\boldsymbol{\Sigma}^{-1}, \xi_j) = \mathrm{diag}\{S(a, \mu/\sigma_1, \xi_j), S(a, \mu/\sigma_2, \xi_j), \ldots, S(a, \mu/\sigma_M, \xi_j)\} \tag{52b}$$

where

$$C(a:x, y) = \frac{\mathrm{e}^{-a/x} - \mathrm{e}^{-a/y}}{x - y} \tag{53a}$$

and

$$S(a:x, y) = \frac{1 - \mathrm{e}^{-a/x}\,\mathrm{e}^{-a/y}}{x + y}. \tag{53b}$$

In addition

$$\boldsymbol{\Xi}(\tau, \mu) = \frac{1}{2}\sum_{l=0}^{N} P_l(\mu) \sum_{j=1}^{J} C_j\Bigg\{\int_0^\tau u_j(x)\mathbf{C}(\tau - x:\mu\boldsymbol{\Sigma}^{-1}, \xi_j)\,\mathrm{d}x$$

$$+ (-1)^l\left[V_j(\tau)\mathbf{S}(\tau:\mu\boldsymbol{\Sigma}^{-1}, \xi_j) + \int_0^\tau v_j(x)\,\mathrm{e}^{-\boldsymbol{\Sigma}(\tau-x)/\mu}\mathbf{S}(x:\mu\boldsymbol{\Sigma}^{-1}, \xi_j)\,\mathrm{d}x\right]\Bigg\}\boldsymbol{\Sigma}^{-1}\mathbf{C}_l\mathbf{T}_l(\xi_j) \tag{54a}$$

and

$$\boldsymbol{\Xi}(\tau, -\mu) = \frac{1}{2}\sum_{l=0}^{N} P_l(\mu) \sum_{j=1}^{J} C_j\Bigg\{\int_\tau^{\tau_0} v_j(x)\mathbf{C}(x - \tau:\mu\boldsymbol{\Sigma}^{-1}, \xi_j)\,\mathrm{d}x + (-1)^l\Bigg[U_j(\tau)\mathbf{S}(\tau_0 - \tau:\mu\boldsymbol{\Sigma}^{-1}, \xi_j)$$

$$+ \int_\tau^{\tau_0} u_j(x)\,\mathrm{e}^{-\boldsymbol{\Sigma}(x-\tau)/\mu}\mathbf{S}(\tau_0 - x:\mu\boldsymbol{\Sigma}^{-1}, \xi_j)\,\mathrm{d}x\Bigg]\Bigg\}\boldsymbol{\Sigma}^{-1}\mathbf{C}_l\mathbf{T}_l(\xi_j) \tag{54b}$$

for $\mu \in [0, 1]$.

AN ALTERNATIVE SOLUTION

In the previous section we developed what we consider to be a very straightforward solution to the basic multi-group or non-gray problem defined by Eqs. (3) and (4). Here, as an attempt to improve the accuracy of the solution, we follow Chandrasekhar[1] and decompose the solution into scattered and unscattered components. To this end, we first write

$$\boldsymbol{\Psi}(\tau, \mu) = \boldsymbol{\Psi}_0(\tau, \mu) + \boldsymbol{\Phi}(\tau, \mu) \tag{55}$$

where $\boldsymbol{\Psi}_0(\tau, \mu)$ satisfies

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{\Psi}_0(\tau, \mu) + \boldsymbol{\Sigma} \boldsymbol{\Psi}_0(\tau, \mu) = \mathbf{Q}(\tau, \mu) \tag{56}$$

subject to the boundary conditions

$$\boldsymbol{\Psi}_0(0, \mu) = \mathbf{F}_1(\mu) \tag{57a}$$

and

$$\boldsymbol{\Psi}_0(\tau_0, -\mu) = \mathbf{F}_2(\mu) \tag{57b}$$

for $\mu \in [0, 1]$. It follows that $\boldsymbol{\Phi}(\tau, \mu)$ must satisfy

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{\Phi}(\tau, \mu) + \boldsymbol{\Sigma} \boldsymbol{\Phi}(\tau, \mu) = \frac{1}{2} \sum_{l=0}^{L} P_l(\mu) \mathbf{C}_l \int_{-1}^{1} P_l(\mu') \boldsymbol{\Phi}(\tau, \mu') \, d\mu' + \mathbf{Q}_0(\tau, \mu) \tag{58}$$

subject to

$$\boldsymbol{\Phi}(0, \mu) = \mathbf{0} \tag{59a}$$

and

$$\boldsymbol{\Phi}(\tau_0, -\mu) = \mathbf{0} \tag{59b}$$

for $\mu \in [0, 1]$. Here

$$\mathbf{Q}_0(\tau, \mu) = \frac{1}{2} \sum_{l=0}^{L} P_l(\mu) \mathbf{C}_l \int_{-1}^{1} P_l(\mu') \boldsymbol{\Psi}_0(\tau, \mu') \, d\mu'. \tag{60}$$

We can readily solve Eqs. (56) and (57) to obtain

$$\boldsymbol{\Psi}_0(\tau, \mu) = e^{-\boldsymbol{\Sigma}\tau/\mu} \mathbf{F}_1(\mu) + \frac{1}{\mu} \int_0^{\tau} e^{-\boldsymbol{\Sigma}(\tau - x)/\mu} \mathbf{Q}(x, \mu) \, dx \tag{61a}$$

and

$$\boldsymbol{\Psi}_0(\tau, -\mu) = e^{-\boldsymbol{\Sigma}(\tau_0 - \tau)/\mu} \mathbf{F}_2(\mu) + \frac{1}{\mu} \int_{\tau}^{\tau_0} e^{-\boldsymbol{\Sigma}(x - \tau)/\mu} \mathbf{Q}(x, -\mu) \, dx \tag{61b}$$

for $\mu \in [0, 1]$, and so we can now consider $\mathbf{Q}_0(\tau, \mu)$ in Eq. (58) as known. Since Eqs. (58) and (59) define a problem in the class of problems defined by Eqs. (3) and (4), we can use our foregoing development to establish our spherical-harmonics solution for $\boldsymbol{\Phi}(\tau, \mu)$.

NUMERICAL METHODS AND RESULTS

As our basic formulation is complete, we are now ready to discuss the numerical methods we use to implement the solution. First of all, we use the driver problem RG from the EISPACK[8] collection to compute the eigenvalues and eigenvectors of the $\mathbf{A}$ matrix given by Eq. (14). As the elements of these eigenvectors provide only the even-order $\mathbf{T}_l(\xi_j)$ we then use Eq. (10) to compute the odd-order $\mathbf{T}_l(\xi_j)$. We have also carried out a similar computation to find the vectors $\mathbf{T}_l^\dagger(\xi_j)$; however, at the end of the day, we found it faster to compute the inverses of the matrices $\mathbf{T}_e$ and $\mathbf{T}_o$ given by Eqs. (21) and to define the vectors $\mathbf{T}_l^\dagger(\xi_j)$ from these inverses. At this point the constants $\{C_j\}$ were computed from Eq. (40), and the integrals given by Eqs. (42) were evaluated by Gaussian quadrature. We then solved, using the subroutines DGEFA and DGESL from the LINPACK[12] collection, either Eqs. (46) or Eqs. (48) to find the constants $\{A_j\}$ and $\{B_j\}$, and finally the desired results were found by evaluating Eqs. (50) and (51).

Table 1. The total cross sections (in cm^{-1}).

1.50520	1.57051	3.51907	6.26226	1.40294(+1)	4.21492(+1)

In order to check out our solution, we first considered the 20-group problem defined and solved by Garcia and Siewert.[13] This problem has anisotropic scattering of order 10 but has no up-scattering. The problem is, however, an excellent test case because results accurate to five significant figures are given[13] for all group angular fluxes as well as group fluxes, group currents and albedo and transmission factors for each group. In regard to our results here, we found that with, say, $N = 99$ we could obtain results accurate to all five figures for the angular fluxes, the fluxes and the currents for all groups for all (of those considered in Ref. 13) of the points interior to the surface of the layer. On the other hand, as is typical of the spherical-harmonics method, our solution on the boundary was, again with, say, $N = 99$ good only to three significant figures. By using the alternative formulation discussed in the previous section, we were able to obtain some modest improvement in our boundary results, but, of course, this second solution required considerably more computational time.

In order to provide a test of our general multi-group solution we next consider a six-group problem that has anisotropic scattering and up-scattering. The cross sections for this problem were provided by R. D. M. Garcia[14] and are listed in Tables 1 and 2. The total cross sections and the transfer matrices given for this six-group problem were derived[14] for neutron scattering in water, and, of course, are subject to some approximations; however, here in our numerical calculations, we consider the input data to be exact.

Table 2(a). The transfer cross sections $\mathbf{T}_0$ (in cm^{-1}).

8.07750(−1)	0.0	0.0	0.0	0.0	0.0
6.26456(−1)	9.84040(−1)	9.79447(−2)	4.70592(−2)	1.26378(−1)	3.97098(−1)
6.16969(−2)	5.22489(−1)	3.09892	2.01540	3.24243	9.62747
6.16969(−3)	5.15850(−2)	2.97524(−1)	3.88257	8.22842	2.42148(+1)
6.16969(−4)	5.15850(−3)	3.48269(−3)	2.43839(−1)	2.09399	5.46759
6.83147(−5)	5.72919(−4)	9.81407(−5)	6.67680(−3)	1.27601(−1)	1.77484

Table 2(b). The transfer cross sections $\mathbf{T}_1$ (in cm^{-1}).

1.73809	0.0	0.0	0.0	0.0	0.0
1.00096	2.06953	3.65035(−2)	−7.17931(−3)	−6.57000(−3)	−6.21269(−3)
3.23741(−2)	7.97905(−1)	1.84248	−8.37651(−1)	−9.50478(−1)	−8.87500(−1)
1.16883(−3)	2.58418(−2)	−1.42702(−2)	1.51150	−1.42204(−1)	−5.72883(−1)
8.28375(−5)	9.49379(−4)	−1.60567(−3)	7.80252(−2)	1.18387	1.21816
2.28725(−5)	9.04254(−5)	−1.63418(−5)	3.64032(−4)	2.94178(−2)	5.93333(−1)

Table 2(c). The transfer cross sections $\mathbf{T}_2$ (in cm^{-1}).

1.89361	0.0	0.0	0.0	0.0	0.0
−1.16776(−2)	2.03360	−2.78866(−2)	−3.66853(−3)	−5.40201(−3)	−1.56529(−2)
−1.38579(−1)	−1.00130(−1)	4.28177(−1)	−4.78174(−1)	−1.90615(−1)	−3.91507(−1)
−1.52290(−2)	−1.16769(−1)	−6.28730(−2)	5.52373(−1)	−5.57599(−1)	−1.08378
−1.53296(−3)	−1.27412(−2)	−3.69913(−4)	4.15341(−2)	7.64138(−1)	1.96411(−1)
−1.61574(−4)	−1.41761(−3)	−4.41786(−6)	−2.23975(−4)	3.03330(−2)	2.98538(−1)

Table 2(d). The transfer cross sections $\mathbf{T}_3$ (in cm^{-1}).

1.28480	0.0	0.0	0.0	0.0	0.0
−1.17471	1.09070	−2.12993(−2)	−7.43655(−4)	−5.91497(−4)	−5.66126(−4)
−1.05643(−1)	−1.00250	4.61562(−1)	−1.88398(−1)	−8.54669(−2)	−7.69580(−2)
−4.05881(−3)	−8.45487(−2)	−2.85371(−2)	7.24719(−1)	−1.03599(−2)	−6.50889(−2)
−2.88894(−4)	−3.29773(−3)	−1.53968(−4)	5.10368(−2)	7.61976(−1)	2.89226(−1)
−7.47744(−5)	−3.13307(−4)	−1.46113(−6)	5.06034(−5)	3.23582(−2)	2.85246(−1)

Table 3. The group fluxes $\Psi_0(\tau)$.

Group	$\tau/\tau_0 = 0.0$	$\tau/\tau_0 = 0.25$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1.0$
1	1.09	1.6205(–4)	4.8524(–8)	1.4567(–11)	4(–15)
2	2.30(–1)	3.7447(–2)	1.9639(–3)	1.0277(–4)	1.79(–6)
3	2.92(–1)	1.8547(–1)	9.7989(–3)	5.1278(–4)	4.37(–6)
4	3.06(–2)	2.6281(–2)	1.3884(–3)	7.2654(–5)	4.39(–7)
5	6.00(–4)	6.1078(–4)	3.2260(–5)	1.6882(–6)	7.91(–9)
6	7.31(–6)	7.2593(–6)	3.8325(–7)	2.0056(–8)	7.94(–11)

Table 4. The group currents $\Psi_1(\tau)$.

Group	$\tau/\tau_0 = 0.0$	$\tau/\tau_0 = 0.25$	$\tau/\tau_0 = 0.5$	$\tau/\tau_0 = 0.75$	$\tau/\tau_0 = 1.0$
1	4.7192(–1)	1.0441(–4)	3.1293(–8)	9.3942(–12)	3(–15)
2	–9.9291(–2)	5.9818(–3)	3.1144(–4)	1.6359(–5)	1.0985(–6)
3	–1.6297(–1)	8.7772(–3)	4.6933(–4)	2.4656(–5)	2.5198(–6)
4	–1.7026(–2)	5.9229(–4)	3.1693(–5)	1.6650(–6)	2.4654(–7)
5	–3.2783(–4)	6.7184(–6)	3.5924(–7)	1.8873(–8)	4.3167(–9)
6	–3.8625(–6)	2.8826(–8)	1.5337(–9)	8.0572(–11)	4.1041(–11)

In addition to the data given in Tables 1 and 2, we note that here we use, as was suggested by Garcia[14] for a health-physics application, $z_0 = 30$ cm which is equivalent to $\tau_0 = 45.156$. For the boundary conditions, we use

$$\mathbf{F}_1(\mu) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{F}_2(\mu) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{62a, b}$$

for $\mu \in [0, 1]$. Finally, for this problem, there is no inhomogeneous source term in Eq. (3), i.e. $\mathbf{Q}(\tau, \mu) = \mathbf{0}$.

In Tables 3 and 4 we report our results for the group fluxes and currents, as computed from Eqs. (50). To obtain the results shown in Tables 3 and 4, we have used both the Mark and Marshak boundary conditions. We have used both the straightforward formulation and, what we call, the Chandrasekhar formulation, and we have increased the order of the approximation until we have established some confidence that the reported results are accurate to ± 1 unit in the last digits given. For the considered test problem, we found the results given in Tables 3 and 4 to be stable as the order of the approximation varied from, say, $N = 199$ to 499. In conclusion, we note that we found, for the considered test problem, no appreciable difference (especially as the order of the approximation was increased) in the results obtained from the two standard approximations, the Marshak and the Mark, to the true boundary conditions.

In conclusion it should be noted that the results given in Tables 3 and 4 have not been confirmed, as we would have liked, by comparison with results from independent calculations. Though some effort has been made to find an existing computer code that can solve the considered multi-group problem, no success can be claimed. It appears (at least to the author) that several of the existing multi-group codes are not able to yield correct results for problems that have significant components of up-scattering. Needless to say, it is possible that errors have been made in the programs (written by the non-expert author) that yielded the results in Tables 3 and 4, and so the author would be grateful for communications that either confirm or dispute the reported results.

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