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A PARTICULAR SOLUTION FOR POLARIZATION CALCULATIONS IN RADIATIVE TRANSFER

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Abstract—A pair of biorthogonality relations, relevant to some elements of a generalized spherical-harmonics solution of the homogeneous equation of transfer, is derived and used, along with linear-algebra techniques, to develop a particular solution appropriate to the scattering of polarized light. The considered radiative-transfer model is based on a general treatment of polarization effects in a plane-parallel medium that contains a source that varies with position and both direction variables.

1. INTRODUCTION

A particular solution is useful in developing solutions to boundary-value problems since it enables one to solve the inhomogeneous radiative-transfer equation using methods developed for solving the homogeneous equation, provided of course that the boundary conditions of the new problem are appropriately adjusted. This flexibility becomes important for problems involving inhomogeneous transport equations which have complicated sources.

We consider in this work situations in radiative transfer for which a complete azimuthally-dependent solution is required to analyze the radiation field[1] in a homogeneous plane-parallel medium, and we extend our earlier work[2] on particular solutions (for a theory that did not include polarization effects) so as to be able to solve the polarization case for a general inhomogeneous source term in the equation of transfer.

For the case of a radiative-transfer model that includes polarization effects and a source of radiation located within the medium, we consider the equation of transfer[1]

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}(\tau,\mu,\phi)+\mathbf{I}(\tau,\mu,\phi)=\frac{\varpi}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\mathbf{P}(\mu,\mu',\phi-\phi')\mathbf{I}(\tau,\mu',\phi')\,\mathrm{d}\phi'\,\mathrm{d}\mu'+\mathbf{S}(\tau,\mu,\phi),\tag{1}$$

for $\tau\in(0,\tau_0)$, $\mu\in[-1,1]$ and $\phi\in[0,2\pi]$, and the boundary conditions

$$\mathbf{I}(0,\mu,\phi)=\mathbf{F}_1(\mu,\phi)\tag{2a}$$

and

$$\mathbf{I}(\tau_0,-\mu,\phi)=\mathbf{F}_2(\mu,\phi),\tag{2b}$$

for $\mu\in[0,1]$ and $\phi\in[0,2\pi]$. Here ϖ is the albedo for single scattering ($\varpi<1$) and τ_0 is the optical thickness of the layer. In addition, μ is the direction cosine as measured from the positive τ axis, ϕ is the azimuthal angle and $\mathbf{P}(\mu,\mu',\phi-\phi')$ is the phase matrix. We allow phase matrices that have an expansion of the form that was derived by Siewert[3,4] from the fundamental paper of Kuščer and Ribarič.[5] We therefore write

$$\mathbf{P}(\mu,\mu',\phi-\phi')=\frac{1}{2}\sum_{m=0}^{L}(2-\delta_{0,m})[\mathbf{C}^m(\mu,\mu')\cos m(\phi-\phi')+\mathbf{S}^m(\mu,\mu')\sin m(\phi-\phi')].\tag{3}$$

Here

$$\mathbf{C}^m(\mu,\mu')=\mathbf{A}^m(\mu,\mu')+\mathbf{D}\mathbf{A}^m(\mu,\mu')\mathbf{D}\tag{4a}$$

and

$$\mathbf{S}^m(\mu,\mu') = \mathbf{A}^m(\mu,\mu')\mathbf{D} - \mathbf{D}\mathbf{A}^m(\mu,\mu') \tag{4b}$$

where

$$\mathbf{A}^m(\mu,\mu') = \sum_{l=m}^{L} \mathbf{\Pi}_l^m(\mu)\mathbf{B}_l\mathbf{\Pi}_l^m(\mu') \tag{5}$$

and

$$\mathbf{D} = \mathrm{diag}\{1, 1, -1, -1\}. \tag{6}$$

In addition

$$\mathbf{\Pi}_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} \begin{bmatrix} P_l^m(\mu) & 0 & 0 & 0 \\ 0 & R_l^m(\mu) & -T_l^m(\mu) & 0 \\ 0 & -T_l^m(\mu) & R_l^m(\mu) & 0 \\ 0 & 0 & 0 & P_l^m(\mu) \end{bmatrix} \tag{7}$$

where

$$P_l^m(\mu) = (1-\mu^2)^{1/2}\frac{\mathrm{d}^m}{\mathrm{d}\mu^m}P_l(\mu) \tag{8}$$

is used to denote the associated Legendre functions[6] and where the functions $R_l^m(\mu)$ and $T_l^m(\mu)$ are as defined in Refs. 3 and 4. To complete the definition of the phase matrix we note that the so-called "Greek constants" $\{\alpha_l, \beta_l, \gamma_l, \delta_l, \epsilon_l, \zeta_l\}$ defined in Refs. 3 and 4 are to be used in

$$\mathbf{B}_l = \begin{bmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\epsilon_l \\ 0 & 0 & \epsilon_l & \delta_l \end{bmatrix}, \tag{9}$$

with $\beta_0 = 1$ and $\gamma_l = \alpha_l = \zeta_l = \epsilon_l = 0$, for $l = 0$ and 1.

We assume here that the boundary data $\mathbf{F}_1(\mu,\phi)$ and $\mathbf{F}_2(\mu,\phi)$ and the source term $\mathbf{S}(\tau,\mu,\phi)$ are given, and so we will be looking for a particular solution that will account for the presence in Eq. (1) of a prescribed inhomogeneous driving force. We note that Chandrasekhar[1] separated the collided and the uncollided components of the radiation intensity and that, in so doing, he was led to consider Eq. (1) for a case where the source term depended on τ, μ and ϕ in a special way. Here we allow the source term to depend on the three independent variables in a more general manner, and we seek the required particular solution appropriate to the use of the generalized spherical-harmonics method (see Ref. 7 and the references quoted therein).

To preview what follows, we note that Secs. 2 and 3 are preliminary to this current work in that we first carry out a useful Fourier decomposition of our problem, and then we review the generalized spherical-harmonics method relevant to the developed homogeneous equations. In Sec. 4 we develop some new and useful biorthogonality relations, and in Sec. 5 we report the particular solution. Finally, in Sec. 6 we develop an existence proof to support the use of biorthogonality relations to find the particular solution.

2. THE FOURIER DECOMPOSITION

In Ref. 8 it was shown, for a radiative-transfer model that did not take into account polarization effects, how to carry out a Fourier decomposition of the solution to Eq. (1) for the case of an isotropically emitting inhomogeneous source term and for general boundary conditions that depend on μ and ϕ. In Ref. 3 the Fourier decomposition for the polarization case was reported in the context of an incident beam of polarized light and no inhomogeneous source term. Such Fourier decompositions have proved useful since the complete solution can be constructed from the solution of $L + 1$ transport problems that are independent of the angle ϕ. Here we develop a similar Fourier decomposition for the polarization case that has an inhomogeneous source term that depends on all three independent variables.

To start, we write

$$\Xi(\tau, \mu, \phi) = \frac{1}{\mu} \int_0^{\tau} e^{-(\tau - x)/\mu} \mathbf{S}(x, \mu, \phi)\, dx \tag{10a}$$

and

$$\Xi(\tau, -\mu, \phi) = \frac{1}{\mu} \int_{\tau}^{\tau_0} e^{-(x - \tau)/\mu} \mathbf{S}(x, -\mu, \phi)\, dx, \tag{10b}$$

for $\mu \in [0, 1]$, and note that $\Xi(\tau, \mu, \phi)$ satisfies

$$\mu \frac{\partial}{\partial \tau} \Xi(\tau, \mu, \phi) + \Xi(\tau, \mu, \phi) = \mathbf{S}(\tau, \mu, \phi). \tag{11}$$

We next express $\Xi(\tau, \mu, \phi)$ in the Fourier series

$$\Xi(\tau, \mu, \phi) = \mathbf{a}_0(\tau, \mu) + \sum_{n=1}^{\infty} [\mathbf{a}_n(\tau, \mu) \cos n(\phi - \phi_r) + \mathbf{b}_n(\tau, \mu) \sin n(\phi - \phi_r)] \tag{12}$$

where

$$\mathbf{a}_0(\tau, \mu) = \frac{1}{2\pi} \int_0^{2\pi} \Xi(\tau, \mu, \phi)\, d\phi \tag{13a}$$

and, for $n \geqslant 1$,

$$\mathbf{a}_n(\tau, \mu) = \frac{1}{\pi} \int_0^{2\pi} \Xi(\tau, \mu, \phi) \cos n(\phi - \phi_r)\, d\phi \tag{13b}$$

and

$$\mathbf{b}_n(\tau, \mu) = \frac{1}{\pi} \int_0^{2\pi} \Xi(\tau, \mu, \phi) \sin n(\phi - \phi_r)\, d\phi. \tag{13c}$$

Here ϕ_r denotes an arbitrary reference angle. We also expand the boundary data in the Fourier series

$$\mathbf{F}_\alpha(\mu, \phi) = \mathbf{a}_{\alpha,0}(\mu) + \sum_{n=1}^{\infty} [\mathbf{a}_{\alpha,n}(\mu) \cos n(\phi - \phi_r) + \mathbf{b}_{\alpha,n}(\mu) \sin n(\phi - \phi_r)] \tag{14}$$

where, for $\alpha = 1$ or 2,

$$\mathbf{a}_{\alpha,0}(\mu) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{F}_\alpha(\mu, \phi)\, d\phi \tag{15a}$$

and, for $n \geqslant 1$,

$$\mathbf{a}_{\alpha,n}(\mu) = \frac{1}{\pi} \int_0^{2\pi} \mathbf{F}_\alpha(\mu, \phi) \cos n(\phi - \phi_r)\, d\phi \tag{15b}$$

and

$$\mathbf{b}_{\alpha,n}(\mu) = \frac{1}{\pi} \int_0^{2\pi} \mathbf{F}_\alpha(\mu, \phi) \sin n(\phi - \phi_r)\, d\phi. \tag{15c}$$

We now let

$$\mathbf{f}_\alpha(\mu, \phi) = \mathbf{a}_{\alpha,0}(\mu) + \sum_{n=1}^{L} [\mathbf{a}_{\alpha,n}(\mu) \cos n(\phi - \phi_r) + \mathbf{b}_{\alpha,n}(\mu) \sin n(\phi - \phi_r)] \tag{16}$$

and substitute

$$\mathbf{I}(\tau, \mu, \phi) = [\mathbf{F}_1(\mu, \phi) - \mathbf{f}_1(\mu, \phi)]\, e^{-\tau/\mu} + \Upsilon(\tau, \mu, \phi) + \Xi(\tau, \mu, \phi) \tag{17a}$$

and

$$\mathbf{I}(\tau, -\mu, \phi) = [\mathbf{F}_2(\mu, \phi) - \mathbf{f}_2(\mu, \phi)]\, e^{-(\tau_0 - \tau)/\mu} + \Upsilon(\tau, -\mu, \phi) + \Xi(\tau, -\mu, \phi), \tag{17b}$$

for $\mu \in [0, 1]$, into Eqs. (1) and (2) to find

$$\mu \frac{\partial}{\partial \tau} \Upsilon(\tau, \mu, \phi) + \Upsilon(\tau, \mu, \phi) = \frac{\varpi}{4\pi} \int_{-1}^{1} \int_0^{2\pi} \mathbf{P}(\mu, \mu', \phi - \phi') \Upsilon(\tau, \mu', \phi')\, d\phi'\, d\mu' + \mathbf{Q}(\tau, \mu, \phi), \tag{18}$$

for $\tau \in (0, \tau_0)$, $\mu \in [-1, 1]$ and $\phi \in [0, 2\pi]$, and

$$\Upsilon(0, \mu, \phi) = \mathbf{f}_1(\mu, \phi) \quad \text{and} \quad \Upsilon(\tau_0, -\mu, \phi) = \mathbf{f}_2(\mu, \phi), \tag{19a,b}$$

for $\mu \in [0, 1]$ and $\phi \in [0, 2\pi]$. Here

$$\mathbf{Q}(\tau, \mu, \phi) = \frac{\varpi}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} \mathbf{P}(\mu, \mu', \phi - \phi')\mathbf{\Xi}(\tau, \mu', \phi')\, d\phi'\, d\mu'. \tag{20}$$

Following Refs. 3, 7, and 9, we now introduce

$$\mathbf{\Phi}_1^m(\chi) = (2 - \delta_{0,m})\text{diag}\{\cos m\chi, \cos m\chi, \sin m\chi, \sin m\chi\} \tag{21a}$$

and

$$\mathbf{\Phi}_2^m(\chi) = (2 - \delta_{0,m})\text{diag}\{-\sin m\chi, -\sin m\chi, \cos m\chi, \cos m\chi\} \tag{21b}$$

and express the phase matrix as

$$\mathbf{P}(\mu, \mu', \phi - \phi') = \sum_{m=0}^{L} [\mathbf{\Phi}_1^m(\phi - \phi')\mathbf{A}^m(\mu, \mu')\mathbf{D}_1 + \mathbf{\Phi}_2^m(\phi - \phi')\mathbf{A}^m(\mu, \mu')\mathbf{D}_2] \tag{22}$$

where

$$\mathbf{D}_1 = \text{diag}\{1, 1, 0, 0\} \quad \text{and} \quad \mathbf{D}_2 = \text{diag}\{0, 0, 1, 1\}. \tag{23a,b}$$

We can now substitute Eqs. (12) and (22) in Eq. (20) to find

$$\mathbf{Q}(\tau, \mu, \phi) = \sum_{m=0}^{L} [\mathbf{\Phi}_1^m(\phi - \phi_r)\mathbf{Q}_1^m(\tau, \mu) + \mathbf{\Phi}_2^m(\phi - \phi_r)\mathbf{Q}_2^m(\tau, \mu)] \tag{24}$$

where

$$\mathbf{Q}_1^m(\tau, \mu) = \frac{\varpi}{4} \int_{-1}^{1} \mathbf{A}^m(\mu, \mu')[(1 + \delta_{0,m})\mathbf{D}_1 \mathbf{a}_m(\tau, \mu') + (1 - \delta_{0,m})\mathbf{D}_2 \mathbf{b}_m(\tau, \mu')]\, d\mu' \tag{25a}$$

and

$$\mathbf{Q}_2^m(\tau, \mu) = \frac{\varpi}{4} \int_{-1}^{1} \mathbf{A}^m(\mu, \mu')[(1 + \delta_{0,m})\mathbf{D}_2 \mathbf{a}_m(\tau, \mu') - (1 - \delta_{0,m})\mathbf{D}_1 \mathbf{b}_m(\tau, \mu')]\, d\mu'. \tag{25b}$$

After substituting

$$\Upsilon(\tau, \mu, \phi) = \sum_{m=0}^{L} [\mathbf{\Phi}_1^m(\phi - \phi_r)\Upsilon_1^m(\tau, \mu) + \mathbf{\Phi}_2^m(\phi - \phi_r)\Upsilon_2^m(\tau, \mu)] \tag{26}$$

into Eq. (18) and making use of Eq. (24), we find that the components of $\Upsilon(\tau, \mu, \phi)$ must satisfy

$$\mu \frac{\partial}{\partial \tau} \Upsilon_\beta^m(\tau, \mu) + \Upsilon_\beta^m(\tau, \mu) = \frac{\varpi}{2} \int_{-1}^{1} \mathbf{A}^m(\mu, \mu')\Upsilon_\beta^m(\tau, \mu')\, d\mu' + \mathbf{Q}_\beta^m(\tau, \mu) \tag{27}$$

for $\beta = 1$ and 2 and for $m = 0, 1, 2, \ldots, L$. It is clear that the Fourier decomposition is now complete.

In order to abbreviate the notation, we henceforth suppress the sub- and superscripts on Υ and the source term $\mathbf{Q}$ in Eq. (27) and consider, after noting Eq. (5),

$$\mu \frac{\partial}{\partial \tau} \Upsilon(\tau, \mu) + \Upsilon(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \mathbf{\Pi}_l^m(\mu)\mathbf{B}_l \int_{-1}^{1} \mathbf{\Pi}_l^m(\mu')\Upsilon(\tau, \mu')\, d\mu' + \mathbf{Q}(\tau, \mu). \tag{28}$$

3. THE SOLUTION OF THE HOMOGENEOUS EQUATION

We are now in a position to be able to extend our previous work[2] in order to find the particular solutions of Eq. (28) that are required in Eq. (26) to establish the complete solution given by Eqs. (17). As we intend to use the variation of parameters method to establish our particular solution, we first require the generalized spherical-harmonics solution of the homogeneous version of Eq. (28). We note that in Ref. 7 Garcia and Siewert made use of some basic properties of the matrix $\mathbf{\Pi}_l^m(\mu)$, viz. the three-term recursion formula

$$[(2l + 1)\mu\mathbf{I} + \mathbf{V}_l^m]\mathbf{\Pi}_l^m(\mu) = \mathbf{U}_{l+1}^m\mathbf{\Pi}_{l+1}^m(\mu) + \mathbf{U}_l^m\mathbf{\Pi}_{l-1}^m(\mu) \tag{29}$$

for $l = m, m+1, \ldots,$ where

$$\mathbf{U}_l^m = (l^2 - m^2)^{1/2}\,\mathrm{diag}\{1, (1-\delta_{0,l})(1-\delta_{1,l})(l^2-4)^{1/2}/l, (1-\delta_{0,l})(1-\delta_{1,l})(l^2-4)^{1/2}/l, 1\} \quad (30)$$

and

$$\mathbf{V}_l^m = \frac{2m(2l+1)}{l(l+1)}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (31)$$

and the orthogonality condition[10]

$$\int_{-1}^{1} \mathbf{\Pi}_l^m(\mu)\mathbf{\Pi}_{l'}^m(\mu)\,\mathrm{d}\mu = \frac{2}{2l+1}\delta_{l,l'}\,\mathbf{\Delta}_l, \quad (32)$$

where

$$\mathbf{\Delta}_l = \mathrm{diag}\{1, (1-\delta_{0,l})(1-\delta_{1,l}), (1-\delta_{0,l})(1-\delta_{1,l}), 1\}, \quad (33)$$

to find the solution we use.

Letting $\mathbf{\Upsilon}_h(\tau, \mu)$ denote the solution of the homogeneous version of Eq. (28), we write[7]

$$\mathbf{\Upsilon}_h(\tau, \mu) = \sum_{l=m}^{M} \frac{2l+1}{2}\mathbf{\Pi}_l^m(\mu) \sum_{j=1}^{J} \{A_j \mathrm{e}^{-\tau/\xi_j} + (-1)^{l-m}\mathbf{D}B_j \mathrm{e}^{-(\tau_0-\tau)/\xi_j}\}\mathbf{T}_l^m(\xi_j) \quad (34)$$

where $M = m + N$ and where the constants $\{A_j\}$ and $\{B_j\}$ are arbitrary. In writing the generalized spherical-harmonics solution as Eq. (34), we have assumed that the order of the approximation N is odd. We also note[7] that ξ_j, with $\mathscr{R}(\xi_j) > 0$, for $j = 1, 2, \ldots, J$, are the eigenvalues relevant to the generalized spherical-harmonics method and that the vectors $\mathbf{T}_l^m(\xi_j)$ satisfy the three-term recursion formula

$$[\xi_j \mathbf{h}_l + \mathbf{V}_l^m]\mathbf{T}_l^m(\xi_j) = \mathbf{U}_{l+1}^m \mathbf{T}_{l+1}^m(\xi_j) + \mathbf{U}_l^m \mathbf{T}_{l-1}^m(\xi_j) \quad (35)$$

for $l = m, m+1, \ldots$. Here

$$\mathbf{h}_l = (2l+1)\mathbf{I} - \varpi \mathbf{B}_l. \quad (36)$$

In addition, we note that the vectors $\mathbf{T}_l^m(\xi_j)$ are elements of a certain eigenvector. More specifically, if the vectors $\mathbf{T}_l^m(\xi_j)$ for $l = m, l = m+1, \ldots, M$ are used as the elements of a vector $\mathbf{T}(\xi_j)$, i.e.

$$\mathbf{T}(\xi_j) = \begin{bmatrix} \mathbf{T}_m^m(\xi_j) \\ \mathbf{T}_{m+1}^m(\xi_j) \\ \vdots \\ \mathbf{T}_M^m(\xi_j) \end{bmatrix}, \quad (37)$$

then, as reported in Ref. 7, the eigenvalues ξ_j and the eigenvectors $\mathbf{T}(\xi_j)$ are subsets of the eigenvalues and eigenvectors defined by

$$\mathbf{W}\mathbf{T}(\xi_j) = \xi_j \mathbf{T}(\xi_j) \quad (38)$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{B}_m^m & \mathbf{C}_m^m & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{A}_{m+1}^m & \mathbf{B}_{m+1}^m & \mathbf{C}_{m+1}^m & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{m+2}^m & \mathbf{B}_{m+2}^m & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_{M-2}^m & \mathbf{C}_{M-2}^m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}_{M-1}^m & \mathbf{B}_{M-1}^m & \mathbf{C}_{M-1}^m \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A}_M^m & \mathbf{B}_M^m \end{bmatrix}. \quad (39)$$

Here the elements of the block tridiagonal $\mathbf{W}$ matrix are given by[7]

$$\mathbf{A}_l^m = \mathbf{h}_l^{-1}\mathbf{U}_l^m, \quad \mathbf{B}_l^m = -\mathbf{h}_l^{-1}\mathbf{V}_l^m \quad \text{and} \quad \mathbf{C}_l^m = \mathbf{h}_l^{-1}\mathbf{U}_{l+1}^m. \quad (40a,b,c)$$

As we intend to use a formulation of the generalized spherical-harmonics solution that is slightly different from that given in Ref. 7, some additional comments are appropriate. First of all, if we let $\mathbf{\Delta}$ be the block diagonal matrix

$$\mathbf{\Delta} = \text{diag}\{\mathbf{D}, -\mathbf{D}, \mathbf{D}, -\mathbf{D}, \ldots, -\mathbf{D}\} \tag{41}$$

then we see that

$$\mathbf{\Delta W \Delta} = -\mathbf{W}, \tag{42}$$

and so from Eq. (38) we conclude that

$$\mathbf{W}\,\mathbf{\Delta T}(\xi_j) = -\xi_j\,\mathbf{\Delta T}(\xi_j) \tag{43}$$

and thus that the eigenvalues of $\mathbf{W}$ occur in $\pm$ pairs. In writing our solution of the homogeneous equation as Eq. (34) we have already taken into account the fact that the eigenvalues $\{\xi_j\}$ occur in $\pm$ pairs, and so, in general, $J = 2N + 2$. However, as noted in Ref. 7, the cases $m = 0$ and $m = 1$ are special. For $m = 0$ the matrix $\mathbf{W}$ has an eigenvalue $\xi = 0$ of multiplicity 4 and 4 corresponding linearly independent eigenvectors. It turns out that because of the special form of the matrices $\mathbf{\Pi}_l^0(\mu)$, for $l = 0$ and $l = 1$, the eigenvectors corresponding to $\xi = 0$ make no contribution to the solution. And so for the case $m = 0$ we can take $J = 2N$ and use only the eigenvalues of $\mathbf{W}$ that lie in the right half-plane. In a similar manner, for the case $m = 1$ the matrix $\mathbf{W}$ has eigenvalues ± 1 whose corresponding eigenvectors are such that again they make no contribution to the homogeneous solution. For this reason we can use, for the $m = 1$ case, $J = 2N + 1$ and we can ignore the eigenvalue $\xi = 1$.

Following Ref. 7, we now reduce our eigenvalue problem as defined by Eq. (38) to a half-size eigenvalue problem for ξ_j^2. We first partition the matrices $\mathbf{A}_l^m$, $\mathbf{B}_l^m$ and $\mathbf{C}_l^m$ into 2×2 blocks, i.e.

$$\mathbf{A}_l^m = \begin{bmatrix} {}^1\mathbf{A}_l^m & \mathbf{0} \\ \mathbf{0} & {}^2\mathbf{A}_l^m \end{bmatrix}, \quad \mathbf{B}_l^m = \begin{bmatrix} \mathbf{0} & {}^1\mathbf{B}_l^m \\ {}^2\mathbf{B}_l^m & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \mathbf{C}_l^m = \begin{bmatrix} {}^1\mathbf{C}_l^m & \mathbf{0} \\ \mathbf{0} & {}^2\mathbf{C}_l^m \end{bmatrix}. \tag{44a,b,c}$$

Next we carry out a 2×2 row and column shuffle by writing down the 2×2 rows and columns in the order $1, 4, 5, 8, 9, 12, 13, \ldots$ and then $2, 3, 6, 7, 10, 11, \ldots$ so that we can write Eq. (38) as

$$\begin{bmatrix} \mathbf{0} & \mathbf{U} \\ \mathbf{L} & \mathbf{0} \end{bmatrix} \mathbf{T}_*(\xi_j) = \xi_j \mathbf{T}_*(\xi_j) \tag{45}$$

where the block tridiagonal $\mathbf{U}$ and $\mathbf{L}$ matrices are given by

$$\mathbf{U} = \begin{bmatrix} {}^1\mathbf{B}_m^m & {}^1\mathbf{C}_m^m & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ {}^2\mathbf{A}_{m+1}^m & {}^2\mathbf{B}_{m+1}^m & {}^2\mathbf{C}_{m+1}^m & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}^1\mathbf{A}_{m+2}^m & {}^1\mathbf{B}_{m+2}^m & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & {}^2\mathbf{B}_{M-2}^m & {}^2\mathbf{C}_{M-2}^m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & {}^1\mathbf{A}_{M-1}^m & {}^1\mathbf{B}_{M-1}^m & {}^1\mathbf{C}_{M-1}^m \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & {}^2\mathbf{A}_M^m & {}^2\mathbf{B}_M^m \end{bmatrix} \tag{46}$$

and

$$\mathbf{L} = \begin{bmatrix} {}^2\mathbf{B}_m^m & {}^2\mathbf{C}_m^m & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ {}^1\mathbf{A}_{m+1}^m & {}^1\mathbf{B}_{m+1}^m & {}^1\mathbf{C}_{m+1}^m & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & {}^2\mathbf{A}_{m+2}^m & {}^2\mathbf{B}_{m+2}^m & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & {}^1\mathbf{B}_{M-2}^m & {}^1\mathbf{C}_{M-2}^m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & {}^2\mathbf{A}_{M-1}^m & {}^2\mathbf{B}_{M-1}^m & {}^2\mathbf{C}_{M-1}^m \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & {}^1\mathbf{A}_M^m & {}^1\mathbf{B}_M^m \end{bmatrix}. \tag{47}$$

In addition,

$$\mathbf{T}_*(\xi_j) = \begin{bmatrix} \mathbf{T}_a(\xi_j) \\ \mathbf{T}_b(\xi_j) \end{bmatrix}, \tag{48}$$

where

$$\mathbf{T}_a(\xi_j) = \begin{bmatrix} {}^1\mathbf{T}_m^m(\xi_j) \\ {}^2\mathbf{T}_{m+1}^m(\xi_j) \\ {}^1\mathbf{T}_{m+2}^m(\xi_j) \\ \vdots \\ {}^2\mathbf{T}_M^m(\xi_j) \end{bmatrix} \quad \text{and} \quad \mathbf{T}_b(\xi_j) = \begin{bmatrix} {}^2\mathbf{T}_m^m(\xi_j) \\ {}^1\mathbf{T}_{m+1}^m(\xi_j) \\ {}^2\mathbf{T}_{m+2}^m(\xi_j) \\ \vdots \\ {}^1\mathbf{T}_M^m(\xi_j) \end{bmatrix}. \tag{49a,b}$$

In writing Eqs. (49), we have used, in a manner similar to that used in Eqs. (44), ${}^1\mathbf{T}_l^m(\xi_j)$ and ${}^2\mathbf{T}_l^m(\xi_j)$ respectively to denote the first two and the second two components of $\mathbf{T}_l^m(\xi_j)$.

It follows, since we can conclude from Eq. (45) that

$$\mathbf{M}_a\mathbf{T}_a(\xi_j) = \xi_j^2\mathbf{T}_a(\xi_j) \quad \text{and} \quad \mathbf{M}_b\mathbf{T}_b(\xi_j) = \xi_j^2\mathbf{T}_b(\xi_j), \tag{50a,b}$$

where $\mathbf{M}_a = \mathbf{UL}$ and $\mathbf{M}_b = \mathbf{LU}$, that the squares of the eigenvalues we seek are the eigenvalues of either $\mathbf{M}_a$ or $\mathbf{M}_b$.

From a computational point-of-view, it is clear that finding the J eigenvalues and eigenvectors of $\mathbf{M}_a$ or $\mathbf{M}_b$ and using

$$\mathbf{T}_b(\xi_j) = \frac{1}{\xi_j}\mathbf{L}\mathbf{T}_a(\xi_j) \quad \text{or} \quad \mathbf{T}_a(\xi_j) = \frac{1}{\xi_j}\mathbf{U}\mathbf{T}_b(\xi_j) \tag{51a,b}$$

is preferable to finding the $2J$ eigenvalues and eigenvectors of $\mathbf{W}$.

4. BIORTHOGONALITY RELATIONS

We begin with the observation from Ref. 2 that orthogonality relations can be used to obtain the particular solution of the scalar equation of transfer. Since Eq. (28) contains an asymmetric matrix $\mathbf{B}_l$, we expect that some sort of biorthogonality relations may exist for the vectors $\mathbf{T}_l^m(\xi_i)$.

It can be shown that

$$\tilde{\mathbf{B}}_l = \mathbf{F}\mathbf{B}_l\mathbf{F} \quad \text{and} \quad \tilde{\mathbf{B}}_l = \mathbf{E}\mathbf{B}_l\mathbf{E} \tag{52a,b}$$

where the tilde denotes a transpose,

$$\mathbf{F} = \text{diag}\{1, 1, 1, -1\} \quad \text{and} \quad \mathbf{E} = \text{diag}\{1, 1, -1, 1\}. \tag{53a,b}$$

It is clear that $\mathbf{D} = \mathbf{EF}$, $\mathbf{F} = \mathbf{DE}$ and $\mathbf{E} = \mathbf{FD}$. As Eq. (35) contains the non-diagonal matrix $\mathbf{V}_l^m$ we note that

$$\mathbf{V}_l^m = \mathbf{F}\mathbf{V}_l^m\mathbf{F} \quad \text{and} \quad \mathbf{V}_l^m = -\mathbf{E}\mathbf{V}_l^m\mathbf{E}. \tag{54a,b}$$

If we now postmultiply the transpose of Eq. (35) by $\mathbf{F}$ and use Eqs. (52) and (54) we find that

$$\tilde{\mathbf{T}}_l^m(\xi_i)\mathbf{F}[\xi_i\mathbf{h}_l + \mathbf{V}_l^m] = \tilde{\mathbf{T}}_{l+1}^m(\xi_i)\mathbf{F}\mathbf{U}_{l+1}^m + \tilde{\mathbf{T}}_{l-1}^m(\xi_i)\mathbf{F}\mathbf{U}_l^m. \tag{55}$$

We next premultiply Eq. (35) by $\tilde{\mathbf{T}}_l^m(\xi_i)\mathbf{F}$, postmultiply Eq. (55) by $\mathbf{T}_l^m(\xi_j)$ and sum each equation over l. After subtraction of the two results and use of $\mathbf{T}_{M+1}^m(\xi_j) = \mathbf{0}$, we find that all the inner products on the right-hand side cancel as in a Christoffel–Darboux-type formula. Thus we find

$$(\xi_i - \xi_j)\sum_{l=m}^{M} \tilde{\mathbf{T}}_l^m(\xi_i)\mathbf{F}\mathbf{h}_l\mathbf{T}_l^m(\xi_j) = 0. \tag{56}$$

If we now assume that the eigenvalues $\{\xi_j\}$ are distinct, we can deduce from Eq. (56) the biorthogonality relation

$$\frac{1}{2}\sum_{l=m}^{M} \tilde{\mathbf{T}}_l^m(\xi_i)\mathbf{F}\mathbf{h}_l\mathbf{T}_l^m(\xi_j) = C_j^{-1}\delta_{i,j} \tag{57}$$

where the normalization constant C_j is given by

$$C_j = \left(\frac{1}{2}\sum_{l=m}^{M} \tilde{\mathbf{T}}_l^m(\xi_j)\mathbf{F}\mathbf{h}_l\mathbf{T}_l^m(\xi_j)\right)^{-1}. \tag{58}$$

We note that throughout this paper we follow the convention that the eigenvalues $\{\xi_j\}$ are labeled such that $\mathscr{R}(\xi_j) > 0$; however, we can consider Eq. (35) with ξ_j changed to $-\xi_j$ if we also consider, as was done in Ref. 7 to obtain the form of the solution given by Eq. (34), that

$$\mathbf{T}_l^m(-\xi_j) \propto (-1)^{l-m}\mathbf{D}\mathbf{T}_l^m(\xi_j). \tag{59}$$

We can thus rewrite Eq. (56) as

$$(\xi_i + \xi_j)\sum_{l=m}^{M} \tilde{\mathbf{T}}_l^m(-\xi_i)\mathbf{F}\mathbf{h}_l\mathbf{T}_l^m(\xi_j) = 0 \tag{60}$$

or

$$(\xi_i + \xi_j)\sum_{l=m}^{M} (-1)^{l-m}\tilde{\mathbf{T}}_l^m(\xi_i)\mathbf{D}\mathbf{F}\mathbf{h}_l\mathbf{T}_l^m(\xi_j) = 0. \tag{61}$$

Again, assuming that the eigenvalues are distinct, we find from Eq. (61) a second biorthogonality relation, viz.

$$\sum_{l=m}^{M} (-1)^{l-m}\tilde{\mathbf{T}}_l^m(\xi_i)\mathbf{E}\mathbf{h}_l\mathbf{T}_l^m(\xi_j) = 0. \tag{62}$$

Equation (62) can also be written as

$$\sum_{k=1}^{K} \tilde{\mathbf{T}}_{m+2k-2}^m(\xi_i)\mathbf{E}\mathbf{h}_{m+2k-2}\mathbf{T}_{m+2k-2}^m(\xi_j) = \sum_{k=1}^{K} \tilde{\mathbf{T}}_{m+2k-1}^m(\xi_i)\mathbf{E}\mathbf{h}_{m+2k-1}\mathbf{T}_{m+2k-1}^m(\xi_j) \tag{63}$$

where $K = (N+1)/2$. This is a generalization of an identity proved in Ref. 2 for the special case of $\xi_i = \xi_j$.

The biorthogonality relations given by Eqs. (57) and (62) are basic to our derivation of the required particular solution of Eq. (28).

5. THE PARTICULAR SOLUTION

To develop our particular solution, we consider Eq. (34) and thus propose

$$\mathbf{\Upsilon}_p(\tau,\mu) = \sum_{l=m}^{M} \frac{2l+1}{2}\mathbf{\Pi}_l^m(\mu)\sum_{j=1}^{J}\{\mathscr{A}_j(\tau)\,e^{-\tau/\xi_j} + (-1)^{l-m}\mathbf{D}\mathscr{B}_j(\tau)\,e^{-(\tau_0-\tau)/\xi_j}\}\mathbf{T}_l^m(\xi_j) \tag{64}$$

where the functions $\{\mathscr{A}_j(\tau)\}$ and $\{\mathscr{B}_j(\tau)\}$ are to be determined. Substituting the solution proposed in Eq. (64) into Eq. (28), multiplying the resulting equation by $\mathbf{\Pi}_\beta^m(\mu)$, for $\beta = m, m+1, m+2, \ldots, M$, and integrating over μ from -1 to 1, we find, after using Eqs. (29), (32) and (35),

$$\sum_{j=1}^{J} \xi_j\{\mathscr{A}_j'(\tau)\,e^{-\tau/\xi_j} - (-1)^{\beta-m}\mathbf{D}\mathscr{B}_j'(\tau)\,e^{-(\tau_0-\tau)/\xi_j}\}\mathbf{T}_\beta^m(\xi_j) = 2\mathbf{h}_\beta^{-1}\mathbf{Q}_\beta^m(\tau) \tag{65}$$

where the superscript prime is used to denote differentiation with respect to τ and where

$$\mathbf{Q}_\beta^m(\tau) = \frac{2\beta+1}{2}\int_{-1}^{1}\mathbf{\Pi}_\beta^m(\mu)\mathbf{Q}(\tau,\mu)\,d\mu. \tag{66}$$

Considering that solutions to Eq. (65) exist (this point is addressed in Sec. 6), we can now premultiply Eq. (65) by $\tilde{\mathbf{T}}_\beta^m(\xi_i)\mathbf{F}$, sum over β and use Eqs. (57) and (62) to find

$$\mathscr{A}_j'(x)\,e^{-x/\xi_j} = \frac{C_j}{\xi_j}\sum_{l=m}^{M}\tilde{\mathbf{T}}_l^m(\xi_j)\mathbf{F}\mathbf{Q}_l^m(x). \tag{67a}$$

Similarly, we premultiply Eq. (65) by $(-1)^{\beta-m}\tilde{\mathbf{T}}_\beta^m(\xi_i)\mathbf{E}$ and sum over β to obtain

$$\mathscr{B}_j'(x)\,e^{-(\tau_0-x)/\xi_j} = -\frac{C_j}{\xi_j}\sum_{l=m}^{M}(-1)^{l-m}\tilde{\mathbf{T}}_l^m(\xi_j)\mathbf{E}\mathbf{Q}_l^m(x). \tag{67b}$$

We can now multiply Eq. (67a) by $\exp(x/\xi_j)$ and integrate over x from 0 to τ to find

$$\mathscr{A}_j(\tau)\,e^{-\tau/\xi_j} = \frac{C_j}{\xi_j}\int_0^{\tau}\sum_{l=m}^{M}\tilde{\mathbf{T}}_l^m(\xi_j)\mathbf{F}\mathbf{Q}_l^m(x)\,e^{-(\tau-x)/\xi_j}\,dx. \tag{68a}$$

In a similar way we find from Eq. (67b) that

$$\mathscr{B}_j(\tau)\,e^{-(\tau_0-\tau)/\xi_j} = \frac{C_j}{\xi_j}\int_{\tau}^{\tau_0}\sum_{l=m}^{M}(-1)^{l-m}\tilde{\mathbf{T}}_l^m(\xi_j)\mathbf{E}\mathbf{Q}_l^m(x)\,e^{-(x-\tau)/\xi_j}\,dx. \tag{68b}$$

In obtaining Eqs. (68) we have discarded the values of $\mathscr{A}_j(0)$ and $\mathscr{B}_j(\tau_0)$ since they give contributions already contained in Eq. (34).

To summarize our final results from Eqs. (64) and (68): A spherical harmonics particular solution for radiative transfer with polarization is given in terms of the eigenvalues $\{\xi_j\}$ and eigenvectors $\mathbf{T}_l^m(\xi_j)$ by

$$\Upsilon_p(\tau,\mu) = \sum_{l=m}^{M}\frac{2l+1}{2}\Pi_l^m(\mu)\sum_{j=1}^{J}\frac{C_j}{\xi_j}\sum_{\beta=m}^{M}\left\{\tilde{\mathbf{T}}_\beta^m(\xi_j)\int_0^{\tau}\mathbf{F}\mathbf{Q}_\beta^m(x)\,e^{-(\tau-x)/\xi_j}\,dx + (-1)^{l-\beta}\mathbf{D}\tilde{\mathbf{T}}_\beta^m(\xi_j)\int_{\tau}^{\tau_0}\mathbf{E}\mathbf{Q}_\beta^m(x)\,e^{-(x-\tau)/\xi_j}\,dx\right\}\mathbf{T}_l^m(\xi_j) \tag{69}$$

with the constants C_j given by Eq. (58). New biorthogonality relations for the vectors $\mathbf{T}_l^m(\xi_j)$ are given by Eqs. (57) and (62).

6. AN EXISTENCE PROOF

As the use of the biorthogonality relations in Sec. 5 to find expressions for the functions $\{\mathscr{A}_j'(\tau)\}$ and $\{\mathscr{B}_j'(\tau)\}$ presupposes, in fact, that there exist solutions to Eq. (65), we now wish to complete our development by providing an argument that solutions $\{\mathscr{A}_j'(\tau)\}$ and $\{\mathscr{B}_j'(\tau)\}$ to Eq. (65) do exist.

In Ref. 11 a particular solution appropriate to a spherical harmonics solution of a multi-group radiation-transport problem was developed, and so here we can take advantage of some of the similarities between that work and this. We therefore let

$$X_j(\tau) = \xi_j\{\mathscr{A}_j'(\tau)\,e^{-\tau/\xi_j} - \mathscr{B}_j'(\tau)\,e^{-(\tau_0-\tau)/\xi_j}\} \tag{70a}$$

and

$$Y_j(\tau) = \xi_j\{\mathscr{A}_j'(\tau)\,e^{-\tau/\xi_j} + \mathscr{B}_j'(\tau)\,e^{-(\tau_0-\tau)/\xi_j}\} \tag{70b}$$

and rewrite Eq. (65), for $\beta = m, m+1, \ldots, M$, as

$$\sum_{j=1}^{J}X_j(\tau)\mathbf{T}_a(\xi_j) = \mathbf{V}_a(\tau) \quad\text{and}\quad \sum_{j=1}^{J}Y_j(\tau)\mathbf{T}_b(\xi_j) = \mathbf{V}_b(\tau). \tag{71a,b}$$

In order to obtain Eqs. (71) we first defined

$$\mathbf{V}_l^m(\tau) = 2\mathbf{h}_l^{-1}\mathbf{Q}_l^m(\tau) \tag{72}$$

for $l = m, m+1, m+2, \ldots, M$. We then let ${}^1\mathbf{V}_l^m(\tau)$ and ${}^2\mathbf{V}_l^m(\tau)$ denote, respectively, the first two and the second two components of $\mathbf{V}_l^m(\tau)$ and define

$$\mathbf{V}_a(\tau) = \begin{bmatrix} {}^1\mathbf{V}_m^m(\tau) \\ {}^2\mathbf{V}_{m+1}^m(\tau) \\ {}^1\mathbf{V}_{m+2}^m(\tau) \\ \vdots \\ {}^2\mathbf{V}_M^m(\tau) \end{bmatrix} \quad\text{and}\quad \mathbf{V}_b(\tau) = \begin{bmatrix} {}^2\mathbf{V}_m^m(\tau) \\ {}^1\mathbf{V}_{m+1}^m(\tau) \\ {}^2\mathbf{V}_{m+2}^m(\tau) \\ \vdots \\ {}^1\mathbf{V}_M^m(\tau) \end{bmatrix}. \tag{73a,b}$$

Here, in order to avoid having to deal explicitly with the special case $m = 0$ and $m = 1$, we consider $J = 2N + 2$ for all m. We now let $\mathbf{T}_a$ and $\mathbf{T}_b$ be $J \times J$ matrices that have, respectively, the vectors $\{\mathbf{T}_a(\xi_j)\}$ and $\{\mathbf{T}_b(\xi_j)\}$ as columns, so that, with

$$\mathbf{X}(\tau) = \begin{bmatrix} X_1(\tau) \\ X_2(\tau) \\ \vdots \\ X_J(\tau) \end{bmatrix} \quad \text{and} \quad \mathbf{Y}(\tau) = \begin{bmatrix} Y_1(\tau) \\ Y_2(\tau) \\ \vdots \\ Y_J(\tau) \end{bmatrix}, \tag{74a,b}$$

we can write Eqs. (71) as

$$\mathbf{T}_a\mathbf{X}(\tau) = \mathbf{V}_a(\tau) \quad \text{and} \quad \mathbf{T}_b\mathbf{Y}(\tau) = \mathbf{V}_b(\tau). \tag{75a,b}$$

We are now ready to solve Eqs. (75) to find $\mathbf{X}(\tau)$ and $\mathbf{Y}(\tau)$; however, lacking a required proof, we must now assume (a similar assumption was required in Ref. 11) that the matrices $\mathbf{M}_a$ and $\mathbf{M}_b$ are not defective so as to ensure that $\mathbf{T}_a$ and $\mathbf{T}_b$ are invertible. Following this assumption, we can write

$$\mathbf{X}(\tau) = \mathbf{T}_a^{-1}\mathbf{V}_a(\tau) \quad \text{and} \quad \mathbf{Y}(\tau) = \mathbf{T}_b^{-1}\mathbf{V}_b(\tau). \tag{76}$$

If we now let

$$\mathbf{A}(\tau) = \begin{bmatrix} \mathscr{A}_1(\tau) \\ \mathscr{A}_2(\tau) \\ \vdots \\ \mathscr{A}_J(\tau) \end{bmatrix} \quad \text{and} \quad \mathbf{B}(\tau) = \begin{bmatrix} \mathscr{B}_1(\tau) \\ \mathscr{B}_2(\tau) \\ \vdots \\ \mathscr{B}_J(\tau) \end{bmatrix} \tag{77a,b}$$

we can eliminate between Eqs. (70) to find

$$\mathbf{A}'(\tau) = \tfrac{1}{2}\mathbf{G}\,\mathrm{e}^{\tau\mathbf{G}}[\mathbf{T}_a^{-1}\mathbf{V}_a(\tau) + \mathbf{T}_b^{-1}\mathbf{V}_b(\tau)] \tag{78a}$$

and

$$\mathbf{B}'(\tau) = \tfrac{1}{2}\mathbf{G}\,\mathrm{e}^{(\tau_0-\tau)\mathbf{G}}[\mathbf{T}_b^{-1}\mathbf{V}_b(\tau) - \mathbf{T}_a^{-1}\mathbf{V}_a(\tau)] \tag{78b}$$

where

$$\mathbf{G} = \mathrm{diag}\left\{\frac{1}{\xi_1}, \frac{1}{\xi_2}, \ldots, \frac{1}{\xi_J}\right\}. \tag{79}$$

The development of Eqs. (78) provides, in the context of the assumption that the matrices $\mathbf{M}_a$ and $\mathbf{M}_b$ are not defective, proof of the existence of solutions to Eq. (65).

We note that the construction used here has been completed to find the same final results for the functions $\{\mathscr{A}_j(\tau)\}$ and $\{\mathscr{B}_j(\tau)\}$ as given in Sec. 5; however, since the use of the biorthogonality relations provides a more concise development, once the existence of solutions has been established, we need go no further.

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