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ON THE ALBEDO PROBLEM FOR A FINITE PLANE-PARALLEL RAYLEIGH-SCATTERING ATMOSPHERE

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ABSTRACT

The complete radiation field at any optical depth within a plane-parallel finite Rayleigh-scattering atmosphere is constructed, subject to generalized free-surface boundary conditions. The method of solution used here is based on Case's normal mode expansion technique and Chandrasekhar's results for the reflected and transmitted distributions. The various unknown expansion coefficients are shown to take quite simple forms when expressed in terms of the appropriate X- and Y-functions introduced by Chandrasekhar.

I. INTRODUCTION

In one of his classic papers on radiative transfer, Chandrasekhar (1946) formulated the equations of transfer for two components of a polarized radiation field in a semi-infinite, plane-parallel, free-electron atmosphere. In a following work (Chandrasekhar 1947) the exact solutions for the laws of darkening in the Milne problem were obtained.

In general, four Stokes parameters are needed to describe completely the radiation field in an atmosphere illuminated by an arbitrarily polarized incident beam. The formulation of the equations involved for this case was presented explicitly by Chandrasekhar (1950) in his treatise on radiative transfer. In addition, Chandrasekhar (1950) gave the results for the reflected and transmitted components of the radiation field in a finite slab due to a monodirectional beam incident on one of the free surfaces. These solutions are expressed ultimately in terms of various X- and Y-functions and will be used in subsequent sections of this paper to construct the desired solutions *within* the medium.

With attention focused more directly toward obtaining solutions interior to the surfaces, Mullikin (1966) made a very comprehensive study of the equations involved; in addition to proving the necessary existence and uniqueness theorems for this problem, he suggested a procedure for evaluating numerically all quantities of interest. Mullikin's paper (1966) is complete and extremely useful in that asymptotic formulae, applicable to wide slabs, are given. There has been, in addition, considerable interest in numerical solutions to this problem (Sekera 1956; Caulson, Dave, and Sekera 1960); however, the principal effort has been directed toward evaluating the surface quantities involved.

The technique of singular eigenfunction expansion, introduced by Case (1960) in relation to problems in neutron-transport theory and plasma physics, has been applied recently to several problems in radiative transfer (Siewert and Zweifel 1966; Siewert and Fraley 1967). The eigenfunction method is particularly useful when solutions at any optical depth are sought. However, if emphasis is placed on the various surface quantities of interest, Chandrasekhar's approach appears to be more advantageous; the reflected and transmitted distributions are obtained in a somewhat more direct manner. Several recent papers by Pahor (1966, 1967, 1968) have illustrated a procedure by which the respective merits of the two aforementioned methods of solutions could be utilized simultaneously. It follows that a given problem might be solved systematically in two steps: first, to determine the reflected and transmitted distributions using Chandrasekhar's method and, second, to use the full-range completeness and orthogonality theorems for the Case-type eigenfunctions to generate the desired solution interior to the surfaces. One of the more obvious benefits of this technique is that all unknown expansion coefficients are expressed in terms of Chandrasekhar's X- and Y-functions, which satisfy quite simple integral equations.

In the present work we solve the five equations of transfer for the considered problem by utilizing the results of Chandrasekhar (1950) for the surface quantities and the eigenfunctions developed by Siewert and Fraley (1967) for the vector equation and by Mika (1965) for the scalar equations. We believe the procedure used here has merit because (*a*) it leads to the complete solution at any optical depth, (*b*) all unknown expansion coefficients are expressed in terms of Chandrasekhar's X- and Y-functions, rather than as solutions to the very complex integral equations which result in typical half-range problems for finite slabs, and (*c*) it appears to be simpler than previously derived results.

II. THE BASIC EQUATIONS

The equation of transfer, as formulated by Chandrasekhar (1950), for the four parameters that characterize an arbitrarily polarized radiation field was written initially as a four-component vector equation. However, this equation may be decomposed into a set consisting of a two-component vector equation and four scalar equations (Chandrasekhar 1950; Mullikin 1966). We consider these equations in forms analogous to those used by Mullikin (1966):

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}(\tau,\mu) + \boldsymbol{I}(\tau,\mu) = \tfrac{1}{2}\int_{-1}^{1} \boldsymbol{K}(\mu,\mu')\boldsymbol{I}(\tau,\mu')d\mu' , \tag{1}$$

where $\boldsymbol{I}(\tau,\mu)$ is a vector whose components, $I_l(\tau,\mu)$ and $I_r(\tau,\mu)$, are two of the desired intensities, and

$$\mu \frac{\partial}{\partial \tau} \Phi^{(i)}(\tau,\mu) + \Phi^{(i)}(\tau,\mu) = \int_{-1}^{1} \Psi^{(i)}(\mu')\Phi^{(i)}(\tau,\mu')d\mu' , \quad i = 1,\ 2,\ 3, \quad \text{and} \quad 4 , \tag{2}$$

where the $\Phi^{(i)}(\tau,\mu)$ are to be used in conjunction with $\boldsymbol{I}(\tau,\mu)$ to reconstruct the Stokes parameters in the manner described by Chandrasekhar (1950).

Here, τ is the optical variable, and μ is the direction cosine, as measured from the *positive* τ-axis. In addition, the scattering matrix is given by

$$\boldsymbol{K}(\mu,\mu') = \tfrac{3}{4}\begin{vmatrix} 2(1-\mu^2)(1-\mu'^2) + \mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{vmatrix} , \tag{3}$$

and $\Psi^{(i)}(\mu)$ denotes the various characteristic functions, viz.,

$$\Psi^{(1)}(\mu) = \tfrac{3}{8}(1-\mu^2)(1+2\mu^2) , \quad \Psi^{(2)}(\mu) = \tfrac{3}{16}(1+\mu^2)^2 ,$$
$$\Psi^{(3)}(\mu) = \tfrac{3}{4}\mu^2 , \quad \text{and} \quad \Psi^{(4)}(\mu) = \tfrac{3}{8}(1-\mu^2) . \tag{4}$$

In contrast to Chandrasekhar (1950) and Mullikin (1966), our equations (1) and (2) are homogeneous; the appropriate boundary conditions to be used here are thus

$$\boldsymbol{I}(0,\mu) = \tfrac{1}{2}\delta(\mu-\mu_0)\boldsymbol{F} , \quad 0 < \mu_0 < 1 , \quad \mu \in (0,1) , \tag{5a}$$

$$\boldsymbol{I}(\tau_1,-\mu) = 0 , \quad \mu \in (0,1) , \tag{5b}$$

$$\Phi^{(i)}(0,\mu) = [\Psi^{(i)}(\mu)]^{-1}\delta(\mu-\mu_0) , \quad 0 < \mu_0 < 1 , \quad \mu \in (0,1) , \tag{5c}$$

and

$$\Phi^{(i)}(\tau_1,-\mu) = 0 , \quad \mu \in (0,1) . \tag{5d}$$

Also, τ_1 is the optical thickness of the considered atmosphere, and $\boldsymbol{F}$ is a vector with two constant components, F_l and F_r.

The problem is now completely defined mathematically; however, basic to our analysis are Chandrasekhar's solutions for the emergent distributions. Although these results can be obtained by the Case method, to do so is laborious and, in fact, would be contrary to the proposed procedure. We write, therefore, the emergent quantities (Chandrasekhar 1950) as

$$I(0,-\mu) = \frac{3}{16\mu} S(\mu,\mu_0;\tau_1)\boldsymbol{F} , \quad \mu \in (0,1) , \tag{6a}$$

$$I(\tau_1,\mu) = \frac{3}{16\mu} T(\mu,\mu_0;\tau_1)\boldsymbol{F} + \tfrac{1}{2}e^{-\tau_1/\mu_0}\delta(\mu - \mu_0)\boldsymbol{F} , \quad \mu \in (0,1) , \tag{6b}$$

$$\Phi^{(i)}(0,-\mu) = \frac{1}{2\mu} S^{(i)}(\mu,\mu_0;\tau_1) , \quad \mu \in (0,1) , \tag{6c}$$

and

$$\Phi^{(i)}(\tau_1,\mu) = \frac{1}{2\mu} T^{(i)}(\mu,\mu_0;\tau_1) + [\Psi(\mu)]^{-1}e^{-\tau_1/\mu_0}\delta(\mu - \mu_0) , \quad \mu \in (0,1) . \tag{6d}$$

The various S- and T-functions, we note, have been determined explicitly in terms of X- and Y-functions (see Chandrasekhar 1950, chap. x).

III. INTERIOR SOLUTIONS

Having stated the problem of interest in the previous section and, in addition, having noted there Chandrasekhar's results for the exit distributions, we should like now to develop the interior solutions. We consider first the vector equation. The solution to equation (1) may be written as a linear superposition of the eigenvectors found by Siewert and Fraley (1967):

$$I(\tau,\mu) = A_+I_+ + A_-I_-(\tau,\mu) + \int_{-1}^{1} A_1(\eta)I_1(\eta,\mu)e^{-\tau/\eta}d\eta + \int_{-1}^{1} A_2(\eta)I_2(\eta,\mu)e^{-\tau/\eta}d\eta , \tag{7}$$

where A_+, A_-, $A_1(\eta)$, and $A_2(\eta)$ are arbitrary expansion coefficients to be determined from the boundary conditions. Also,

$$I_+ = \begin{vmatrix} 1 \\ 1 \end{vmatrix} , \quad I_-(\tau,\mu) = (\tau - \mu)\begin{vmatrix} 1 \\ 1 \end{vmatrix} , \tag{8a}$$

$$I_1(\eta,\mu) = \begin{vmatrix} \frac{3\eta}{2}(1 - \mu^2)\frac{P}{\eta - \mu} + \lambda_1(\eta)\delta(\eta - \mu) \\ 0 \end{vmatrix} , \tag{8b}$$

and

$$I_2(\eta,\mu) = \begin{vmatrix} -\frac{3\eta}{2}(\eta + \mu) \\ \frac{3\eta}{2}(1 - \eta^2)\frac{P}{\eta - \mu} + \lambda_2(\eta)\delta(\eta - \mu) \end{vmatrix} . \tag{8c}$$

Here,

$$\lambda_\alpha(\eta) = (-1)^\alpha + 3(1 - \eta^2)(1 - \eta \tanh^{-1}\eta), \quad \alpha = 1 \text{ or } 2 , \tag{9}$$

the Dirac delta function is denoted by $\delta(x)$, and P is a mnemonic symbol used to indicate that all integrals over η or μ are to be evaluated in the Cauchy principal-value sense.

The unknown expansion coefficients in equation (7) must be chosen such that the boundary conditions (viz., eqs. [5]) at the two surfaces are satisfied. Since we have Chandrasekhar's results to work with, we need consider only the boundary $\tau = 0$.

Equating τ to zero in equation (7) and utilizing equations (5a) and (6a), we find

$$\begin{aligned} I(\mu) = A_+ I_+ + A_- I_-(0,\mu) + \int_{-1}^{1} A_1(\eta) I_1(\eta,\mu) d\eta \\ + \int_{-1}^{1} A_2(\eta) I_2(\eta,\mu) d\eta \,, \quad \mu \in (-1,1) \,, \end{aligned} \tag{10}$$

where

$$I(\mu) = \tfrac{1}{2}\delta(\mu - \mu_0)\Theta(\mu) F + \frac{3}{16\mu} S(\mu,\mu_0;\tau_1)[1 - \Theta(\mu)] F \,. \tag{11}$$

Also,

$$\Theta(\mu) \triangleq 1 \,, \quad \mu \in (0,1) \,; \qquad \Theta(\mu) \triangleq 0 \,, \quad \mu \in (-1,0) \,.$$

The full-range completeness theorem given by Smith and Siewert (1967) insures that equation (10) has a solution. Further, the set I_+, $I_-(0,\mu)$, $I_1(\eta,\mu)$, and $I_2(\eta,\mu)$, $\eta \in (-1,1)$, are orthogonal on the full range, with respect to weight function μ.

All expansion coefficients may thus be determined by taking scalar products of equation (10). We use the results of Smith and Siewert (1967) to find ultimately the following:

$$\begin{aligned} A_+ = \tfrac{3}{8} \Big\langle F_l \Big\{ \tfrac{3}{8} \int_0^1 \mu [S_{ll}(\mu,\mu_0;\tau_1) + S_{rl}(\mu,\mu_0;\tau_1)] d\mu + \mu_0^2 \Big\} \\ + F_r \Big\{ \tfrac{3}{8} \int_0^1 \mu [S_{lr}(\mu,\mu_0;\tau_1) + S_{rr}(\mu,\mu_0;\tau_1)] d\mu + \mu_0^2 \Big\} \Big\rangle \,, \end{aligned} \tag{12a}$$

$$\begin{aligned} A_- = \tfrac{3}{8} \Big\langle F_l \Big\{ \tfrac{3}{8} \int_0^1 [S_{ll}(\mu,\mu_0;\tau_1) + S_{rl}(\mu,\mu_0;\tau_1)] d\mu - \mu_0 \Big\} \\ + F_r \Big\{ \tfrac{3}{8} \int_0^1 [S_{lr}(\mu,\mu_0;\tau_1) + S_{rr}(\mu,\mu_0;\tau_1)] d\mu - \mu_0 \Big\} \Big\rangle \,, \end{aligned} \tag{12b}$$

$$\begin{aligned} A_1(\pm\eta) = \pm \frac{1}{2N_1(\eta)} \Big\{ F_l [\mu_0 I_{11}(\pm\eta,\mu_0) - \tfrac{3}{8} \int_0^1 I_{11}(\mp\eta,\mu) S_{ll}(\mu,\mu_0;\tau_1) d\mu] \\ - \tfrac{3}{8} F_r \int_0^1 I_{11}(\mp\eta,\mu) S_{lr}(\mu,\mu_0;\tau_1) d\mu \Big\} \,, \quad \eta \in (0,1) \,, \end{aligned} \tag{12c}$$

and

$$\begin{aligned} A_2(\pm\eta) = \pm \frac{1}{2N_2(\eta)} \Big\langle F_l \Big\{ \mu_0 I_{21}(\pm\eta,\mu_0) - \tfrac{3}{8} \int_0^1 [I_{21}(\mp\eta,\mu) S_{ll}(\mu,\mu_0;\tau_1) \\ + I_{22}(\mp\eta,\mu_0) S_{rl}(\mu,\mu_0;\tau_1)] d\mu \Big\} + F_r \Big\{ \mu_0 I_{22}(\pm\eta,\mu_0) \\ - \tfrac{3}{8} \int_0^1 [I_{21}(\mp\eta,\mu) S_{lr}(\mu,\mu_0;\tau_1) + I_{22}(\mp\eta,\mu_0) S_{rr}(\mu,\mu_0;\tau_1)] d\mu \Big\} \Big\rangle \,, \\ \eta \in (0,1) \,. \end{aligned} \tag{12d}$$

The quantities $S_{\alpha\beta}(\mu,\mu_0;\tau_1)$ denote the elements of Chandrasekhar's S-matrix, and $I_{\alpha\beta}(\xi,\xi')$ is the βth component of $I_\alpha(\xi,\xi')$. In addition, the functions $N_1(\eta)$ and $N_2(\eta)$ are given by

$$N_\alpha(\eta) = \eta[\lambda_\alpha^2(\eta) + \tfrac{9}{4}\pi^2\eta^2(1 - \eta^2)^2] \,, \quad \alpha = 1 \text{ or } 2 \,. \tag{13}$$

Since the S-matrix is known in terms of X- and Y-functions, the coefficients may be considered known; the solution to the vector portion of the problem is thus completed.

We should now like to consider the solutions to the scalar equations. For economy in notation, we suppress the superscript which delineates the particular scalar equation of interest and write the general solution to equation (2) as

$$\Phi(\tau,\mu) = B_+\phi_+(\mu)e^{-\tau/\nu_0} + B_-\phi_-(\mu)e^{\tau/\nu_0} + \int_{-1}^{1} B(\nu)\phi_\nu(\mu)e^{-\tau/\nu}d\nu \,, \tag{14}$$

where we use the eigenfunctions in the forms given by McCormick and Kuščer (1966):

$$\phi_\pm(\mu) = \frac{\nu_0}{2}\frac{1}{\nu_0 \mp \mu} \tag{15a}$$

and

$$\phi_\nu(\mu) = \frac{\nu}{2}\frac{P}{\nu - \mu} + \frac{\lambda(\nu)}{2\Psi(\nu)}\delta(\nu - \mu) \,. \tag{15b}$$

In addition,

$$\lambda(\nu) = 1 + \nu P \int_{-1}^{1} \Psi(\mu)\frac{d\mu}{\mu - \nu} \tag{16a}$$

and ν_0 is the positive zero of the function

$$\Lambda(z) = 1 + z\int_{-1}^{1} \Psi(\mu)\frac{d\mu}{\mu - z} \,. \tag{16b}$$

We note that for the cases $i = 1$, 2, and 3, $\Lambda(z)$ has only one positive zero; $\Lambda(z)$ has no zeros for the case $i = 4$.

If we equate τ to zero in equation (14) and invoke equations (5c) and (6c), we find

$$\Phi(\mu) = B_+\phi_+(\mu) + B_-\phi_-(\mu) + \int_{-1}^{1} B(\nu)\phi_\nu(\mu)d\nu \,, \quad \mu \in (-1,1) \,, \tag{17}$$

where

$$\Phi(\mu) \triangleq [\Psi(\mu)]^{-1}\delta(\mu - \mu_0)\Theta(\mu) + \frac{1}{2\mu} S(\mu,\mu_0;\tau_1)[1 - \Theta(\mu)] \,. \tag{18}$$

The full-range completeness and orthogonality theorems for the currently used eigenfunctions were proved by Mika (1965) for a problem in neutron-thermalization theory. We are thus justified in taking scalar products of equation (17) to obtain the expansion coefficients. We find

$$B(\pm\nu) = \frac{\mu_0}{N(\pm\nu)}\left[\phi_{\pm\nu}(\mu_0) - \frac{1}{2\mu_0}\int_0^1 \Psi(\mu)\phi_{\mp\nu}(\mu)S(\mu,\mu_0;\tau_1)d\mu\right] , \quad \nu \in (0,1) \,, \tag{19a}$$

and

$$B_\pm = \frac{\mu_0}{N_\pm}\left[\phi_\pm(\mu_0) - \frac{1}{2\mu_0}\int_0^1 \Psi(\mu)\phi_\mp(\mu)S(\mu,\mu_0;\tau_1)d\mu\right] . \tag{19b}$$

We note again that for $i = 4$ there is no discrete eigenfunction; thus $B_\pm^{(4)} = 0$. The normalization factors appearing in equations (19) are given by

$$N(\xi)\delta(\xi - \xi') = \int_{-1}^{1} \mu\Psi(\mu)\phi_\xi(\mu)\phi_{\xi'}(\mu)d\mu \,, \quad \xi,\xi' \in (-1,1) \tag{20a}$$

and

$$N_{\pm} = \int_{-1}^{1} \mu \Psi(\mu) \phi_{\pm}^{2}(\mu) d\mu \; ; \tag{20b}$$

more explicitly, equations (20) yield

$$N(\xi) = \tfrac{1}{4}\xi[\Psi(\xi)]^{-1}[\lambda^{2}(\xi) + \pi^{2}\xi^{2}\Psi^{2}(\xi)] \tag{21a}$$

and

$$N_{\pm} = \pm \frac{\nu_0^{2}}{4} \frac{d}{dz} \Lambda(z) \Big|_{z=\nu_0} . \tag{21b}$$

Since the various S-functions appearing above have been written in terms of X- and Y-functions by Chandrasekhar (1950), the expansion coefficients, as given by equations (19), are determined explicitly. The four Stokes parameters describing the radiation field are easily constructed from these results.

We have concluded our basic analysis for this problem; however, several additional comments are in order.

First, it is clear that our results for the interior solutions were obtained quite easily. This, of course, follows from the fact that we have made use of Chandrasekhar's solutions for the exit distributions.

Second, if we use only the Case technique, two half-range expansions must be invoked: one at the surface $\tau = 0$ for $\mu \in (0,1)$, and the other at $\tau = \tau_1$ for $\mu \in (-1,0)$. If we consider the vector equation, for example, this procedure leads to a set of six coupled integral equations for the unknown expansion coefficients. Although these equations may be solved numerically to any desired degree of accuracy, we believe that the expansion coefficients take considerably more tractable forms when expressed in terms of the S-functions or, alternatively, in terms of X- and Y-functions. A numerical procedure is required to generate the necessary X- and Y-functions; however, these satisfy simple integral equations and, in fact, have already been determined for many cases (Chandrasekhar, 1950).

Third, we note from equations (12c), (12d), and (19a) that the continuum expansion coefficients are generalized functions. Thus, to determine them numerically would be, at least, ambiguous; however, since the integral terms in these equations are not generalized functions, the calculational ambiguity is not actually encountered.

Fourth, it should be noted that we have not used the T-functions or the boundary conditions at $\tau = \tau_1$. Since boundary conditions on the full range, $\mu \in (-1,1)$, are used, either surface yields sufficient information to determine the expansion coefficients. Identical results are obtained if, instead of $\tau = 0$ and the S-function, we use $\tau = \tau_1$ and the T-function.

Finally, it is our belief that asymptotic approximations are accomplished more easily in terms of X- and Y-functions than from the set of integral equations for the expansion coefficients that result when half-range boundary conditions are used.

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