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NOTE

ON INTENSITY CALCULATIONS IN RADIATIVE TRANSFER

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Abstract—A post-processing technique is used with the spherical-harmonics method to develop an accurate result for the radiation intensity in a homogeneous plane-parallel medium that contains a source that varies with position. Anisotropic scattering is included in the monochromatic radiative-transfer model, and general reflecting boundary conditions are considered.

INTRODUCTION

To analyze the radiation intensity in a homogeneous plane-parallel medium for the case when there is a source of radiation, we consider the equation of transfer[1,2]

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(u) I(\tau, u) \, du + S(\tau), \tag{1}$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$, and the boundary conditions

$$I(0, \mu) = F_1(\mu) + \rho_1^s I(0, -\mu) + 2\rho_1^d \int_0^1 I(0, -u) u \, du \tag{2a}$$

and

$$I(\tau_0, -\mu) = F_2(\mu) + \rho_2^s I(\tau_0, \mu) + 2\rho_2^d \int_0^1 I(\tau_0, u) u \, du \tag{2b}$$

for $\mu \in (0, 1]$. Here ϖ is the albedo for single scattering ($\varpi < 1$), the elements β_l are the coefficients in a Legendre expansion of the scattering law and τ_0 is the optical thickness of the layer. In addition, ρ_α^s and ρ_α^d, $\alpha = 1$ and 2, are the coefficients for specular and diffuse reflection. We consider that the functions $F_1(\mu)$ and $F_2(\mu)$ and the inhomogeneous source term $S(\tau)$ are given, and we seek, in general, the intensity $I(\tau, \mu)$ for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$ as well as the boundary results $I(0, -\mu)$ and $I(\tau_0, \mu)$ for $\mu \in (0, 1]$.

Generally in the field of radiative heat transfer, we are interested in heat-flux calculations, and to this end the spherical-harmonics, or P_N, method has proved to be accurate and easy to use (see Refs. 3–7). However, as we intend in the future to use the spherical-harmonics method to solve some inverse problems in radiative transfer, we wish to report the details of a post-processing procedure that can improve the most elementary spherical-harmonics result for the intensity. The post-processing procedure we use has been discussed by Kourganoff[8] and is known in the radiative-transfer literature as the method of source-function integration. We note that Dave and Armstrong[9] and Karp[10] have demonstrated that source-function integration can greatly improve the basic spherical-harmonics method for intensity calculations. Some of the details of our use of the method of source-function integration have already been reported in Refs. 4 and 7; however, the work in Ref. 4 does not make use of the general particular solution we now have available for the P_N method, and although the analysis in Ref. 7 is based on a general non-gray model of the equation of transfer, the boundary conditions used in that work are not as general as we consider here.

THE SPHERICAL HARMONICS SOLUTION AND POST PROCESSING

As discussed, for example, in Ref. 6, the spherical-harmonics solution to Eq. (1) can be expressed as

$$I(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} [A_j e^{-\tau/\xi_j} + (-1)^l B_j e^{-(\tau_0-\tau)/\xi_j}] g_l(\xi_j) + I_p(\tau, \mu) \quad (3)$$

where the arbitrary constants A_j and B_j are to be fixed by the boundary conditions. Here we write the particular solution appropriate to the spherical-harmonics method as[5]

$$I_p(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \frac{C_j}{\xi_j} [U(\tau, \xi_j) + (-1)^l V(\tau, \xi_j)] g_l(\xi_j) \quad (4)$$

where, in general,

$$U(\tau, \xi) = \int_0^{\tau} S(x) e^{-(\tau-x)/\xi} \, dx \quad (5a)$$

and

$$V(\tau, \xi) = \int_{\tau}^{\tau_0} S(x) e^{-(x-\tau)/\xi} \, dx. \quad (5b)$$

To define Eqs. (3) and (4), we note[3-5] that the Chandrasekhar polynomials are denoted by $\{g_l(\xi)\}$, that the eigenvalues $\{\xi_j\}$ are, with N odd, the $J = (N+1)/2$ positive zeros of $g_{N+1}(\xi)$ and that the constants $\{C_j\}$ are given by

$$C_j = \left(\sum_{k=1}^{J} h_{2k-2} g_{2k-2}^2(\xi_j) \right)^{-1}, \quad j = 1, 2, \ldots, J, \quad (6)$$

with $h_l = 2l + 1 - \varpi\beta_l$, $0 \leqslant l \leqslant L$, and $h_l = 2l + 1$ for $l > L$.

In order to determine the arbitrary constants required in Eq. (3) we can substitute Eq. (3) into Eqs. (2) and use, for example, the Mark or Marshak projections to generate a system of linear algebraic equations that can be solved to find $\{A_j\}$ and $\{B_j\}$. Considering now that we have so determined the $\{A_j\}$ and $\{B_j\}$, we note that any required moments

$$I_l(\tau) = \int_{-1}^{1} P_l(\mu) I(\tau, \mu) \, d\mu, \quad l = 0, 1, \ldots, N, \quad (7)$$

of the intensity can be obtained immediately by integrating Eq. (3). We find

$$I_l(\tau) = \sum_{j=1}^{J} \{A_j e^{-\tau/\xi_j} + (-1)^l B_j e^{-(\tau_0-\tau)/\xi_j} + \frac{C_j}{\xi_j} [U(\tau, \xi_j) + (-1)^l V(\tau, \xi_j)]\} g_l(\xi_j). \quad (8)$$

As Eq. (8) generally yields good results for the moments, the idea of the post processing is simply to use Eq. (8) to define the right-hand side of Eq. (1) and then to find the intensity by solving the resulting differential equation. We thus can write

$$I(\tau, \mu) = I(0, \mu) e^{-\tau/\mu} + \frac{1}{\mu} \int_0^{\tau} \mathscr{F}(x, \mu) e^{-(\tau-x)/\mu} \, dx \quad (9a)$$

and

$$I(\tau, -\mu) = I(\tau_0, -\mu) e^{-(\tau_0-\tau)/\mu} + \frac{1}{\mu} \int_{\tau}^{\tau_0} \mathscr{F}(x, -\mu) e^{-(x-\tau)/\mu} \, dx \quad (9b)$$

for $\tau \in [0, \tau_0]$ and $\mu \in (0, 1]$. Here

$$\mathscr{F}(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{N} \beta_l P_l(\mu) I_l(\tau) + S(\tau) \quad (10)$$

or, more explicitly,

$$\mathscr{F}(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{N} \beta_l P_l(\mu) \sum_{j=1}^{J} \{A_j e^{-\tau/\xi_j} + (-1)^l B_j e^{-(\tau_0 - \tau)/\xi_j} + \frac{C_j}{\xi_j}[U(\tau, \xi_j) + (-1)^l V(\tau, \xi_j)]\} g_l(\xi_j) + S(\tau). \quad (11)$$

Substituting Eq. (11) into Eqs. (9), we find

$$I(\tau, \mu) = I(0, \mu)e^{-\tau/\mu} + \frac{1}{\mu} U(\tau, \mu) + \Xi(\tau, \mu) + \frac{\varpi}{2} \sum_{l=0}^{N} \beta_l P_l(\mu) \sum_{j=1}^{J} \xi_j [A_j C(\tau: \mu, \xi_j) + (-1)^l B_j e^{-(\tau_0 - \tau)/\xi_j} S(\tau: \mu, \xi_j)] g_l(\xi_j) \quad (12a)$$

and

$$I(\tau, -\mu) = I(\tau_0, -\mu)e^{-(\tau_0 - \tau)/\mu} + \frac{1}{\mu} V(\tau, \mu) + \Xi(\tau, -\mu) + \frac{\varpi}{2} \sum_{l=0}^{N} \beta_l P_l(\mu) \sum_{j=1}^{J} \xi_j [(-1)^l A_j e^{-\tau/\xi_j} \times S(\tau_0 - \tau: \mu, \xi_j) + B_j C(\tau_0 - \tau: \mu, \xi_j)] g_l(\xi_j), \quad (12b)$$

for $\tau \in [0, \tau_0]$ and $\mu \in (0, 1]$, where

$$C(a: x, y) = \frac{e^{-a/x} - e^{-a/y}}{x - y} \quad (13a)$$

and

$$S(a: x, y) = \frac{1 - e^{-a/x} e^{-a/y}}{x + y}. \quad (13b)$$

In addition

$$\Xi(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{N} \beta_l P_l(\mu) \sum_{j=1}^{J} C_j \left\{ \int_0^{\tau} S(x) C(\tau - x: \mu, \xi_j) \, dx + (-1)^l [V(\tau, \xi_j) S(\tau: \mu, \xi_j) + \int_0^{\tau} S(x) e^{-(\tau - x)/\mu} S(x: \mu, \xi_j) \, dx] \right\} g_l(\xi_j) \quad (14a)$$

and

$$\Xi(\tau, -\mu) = \frac{\varpi}{2} \sum_{l=0}^{N} \beta_l P_l(\mu) \sum_{j=1}^{J} C_j \left\{ \int_{\tau}^{\tau_0} S(x) C(x - \tau: \mu, \xi_j) \, dx + (-1)^l [U(\tau, \xi_j) S(\tau_0 - \tau: \mu, \xi_j) + \int_{\tau}^{\tau_0} S(x) e^{-(x - \tau)/\mu} S(\tau_0 - x: \mu, \xi_j) \, dx] \right\} g_l(\xi_j) \quad (14b)$$

for $\tau \in [0, \tau_0]$ and $\mu \in (0, 1]$.

The elements of Eqs. (12) are all now defined except for $I(0, \mu)$ and $I(\tau_0, -\mu)$, and were it not for the reflection terms in Eqs. (2) these terms would be given by those equations. Thus because of the reflection allowed in Eqs. (2) we must work a little harder to find $I(0, \mu)$ and $I(\tau_0, -\mu)$ for $\mu \in (0, 1]$. First of all we can substitute Eqs. (9) into Eqs. (2) to obtain

$$I(0, \mu) - \rho_1^s I(\tau_0, -\mu)e^{-\tau_0/\mu} = F_1(\mu) + \frac{1}{\mu} \rho_1^s \int_0^{\tau_0} \mathscr{F}(x, -\mu) e^{-x/\mu} \, dx + 2\rho_1^d J_1 \quad (15a)$$

and

$$I(\tau_0, -\mu) - \rho_2^s I(0, \mu)e^{-\tau_0/\mu} = F_2(\mu) + \frac{1}{\mu} \rho_2^s \int_0^{\tau_0} \mathscr{F}(x, \mu) e^{-(\tau_0 - x)/\mu} \, dx + 2\rho_2^d J_2, \quad (15b)$$

for $\mu \in (0, 1]$, where

$$J_1 = \int_0^1 I(\tau_0, -\mu) e^{-\tau_0/\mu} \mu \, d\mu + \int_0^1 \int_0^{\tau_0} \mathscr{F}(x, -\mu) e^{-x/\mu} \, dx \, d\mu \tag{16a}$$

and

$$J_2 = \int_0^1 I(0, \mu) e^{-\tau_0/\mu} \mu \, d\mu + \int_0^1 \int_0^{\tau_0} \mathscr{F}(x, \mu) e^{-(\tau_0 - x)/\mu} \, dx \, d\mu. \tag{16b}$$

We can now solve Eqs. (15) to find

$$I(0, \mu) = M(\mu)[K_1(\mu) + \rho_1^s \, e^{-\tau_0/\mu} K_2(\mu)] \tag{17a}$$

and

$$I(\tau_0, -\mu) = M(\mu)[K_2(\mu) + \rho_2^s \, e^{-\tau_0/\mu} K_1(\mu)], \tag{17b}$$

for $\mu \in (0, 1]$, where

$$K_1(\mu) = F_1(\mu) + \frac{1}{\mu} \rho_1^s \int_0^{\tau_0} \mathscr{F}(x, -\mu) e^{-x/\mu} \, dx + 2\rho_1^d J_1 \tag{18a}$$

and

$$K_2(\mu) = F_2(\mu) + \frac{1}{\mu} \rho_2^s \int_0^{\tau_0} \mathscr{F}(x, \mu) e^{-(\tau_0 - x)/\mu} \, dx + 2\rho_2^d J_2 \tag{18b}$$

with

$$M(\mu) = (1 - \rho_1^s \rho_2^s \, e^{-2\tau_0/\mu})^{-1}. \tag{19}$$

We can now multiply Eqs. (9b) and (9a), evaluated respectively at $\tau = 0$ and $\tau = \tau_0$, by μ and integrate to obtain

$$J_1 = \int_0^1 I(0, -\mu) \mu \, d\mu \tag{20a}$$

and

$$J_2 = \int_0^1 I(\tau_0, \mu) \mu \, d\mu. \tag{20b}$$

In addition we can multiply Eqs. (2) by μ and integrate to find

$$\int_0^1 I(0, \mu) \mu \, d\mu = \int_0^1 F_1(\mu) \mu \, d\mu + (\rho_1^s + \rho_1^d) J_1 \tag{21a}$$

and

$$\int_0^1 I(\tau_0, -\mu) \mu \, d\mu = \int_0^1 F_2(\mu) \mu \, d\mu + (\rho_2^s + \rho_2^d) J_2. \tag{21b}$$

We can now multiply Eqs. (17) by μ and integrate over μ to obtain, after we use Eqs. (21),

$$[1 - \rho_1^d \rho_2^s \Gamma(2\tau_0)] J_1 - \rho_2^d \Gamma(\tau_0) J_2 = A(0) + f_2(\tau_0) + \rho_2^s [B(\tau_0) + f_1(2\tau_0)] \tag{22a}$$

and

$$[1 - \rho_2^d \rho_1^s \Gamma(2\tau_0)] J_2 - \rho_1^d \Gamma(\tau_0) J_1 = B(0) + f_1(\tau_0) + \rho_1^s [A(\tau_0) + f_2(2\tau_0)] \tag{22b}$$

where

$$f_\alpha(\tau) = \int_0^1 M(\mu)\mathrm{e}^{-\tau/\mu} F_\alpha(\mu)\mu\, \mathrm{d}\mu, \quad \alpha = 1, 2, \tag{23}$$

$$\Gamma(\tau) = 2\int_0^1 M(\mu)\mathrm{e}^{-\tau/\mu}\mu\, \mathrm{d}\mu, \tag{24}$$

$$A(\tau) = \int_0^1 M(\mu)\mathrm{e}^{-\tau/\mu} \int_0^{\tau_0} \mathscr{F}(x, -\mu)\mathrm{e}^{-x/\mu}\, \mathrm{d}x\, \mathrm{d}\mu \tag{25}$$

and

$$B(\tau) = \int_0^1 M(\mu)\mathrm{e}^{-\tau/\mu} \int_0^{\tau_0} \mathscr{F}(x, \mu)\mathrm{e}^{-(\tau_0 - x)/\mu}\, \mathrm{d}x\, \mathrm{d}\mu. \tag{26}$$

In order to complete the desired development, we can solve the two linear algebraic equations given as Eqs. (22) to find the values of J_1 and J_2 required in Eqs. (18) to complete the expressions given by Eqs. (17). With $I(0, \mu)$ and $I(\tau_0, -\mu)$ for $\mu \in (0, 1]$ so determined, our results given by Eqs. (12) for the intensity are now complete. However, to be more explicit, we can rewrite Eqs. (18) as

$$K_1(\mu) = F_1(\mu) + \rho_1^{\mathrm{s}} D_1(\mu) + 2\rho_1^{\mathrm{d}} J_1 \tag{27a}$$

and

$$K_2(\mu) = F_2(\mu) + \rho_2^{\mathrm{s}} D_2(\mu) + 2\rho_2^{\mathrm{d}} J_2, \tag{27b}$$

and we can now compare Eqs. (9) and Eqs. (12) to deduce that

$$D_1(\mu) = \frac{1}{\mu} V(0, \mu) + \Xi(0, -\mu) + \frac{\varpi}{2} \sum_{l=0}^{N} \beta_l P_l(\mu) \sum_{j=1}^{J} \xi_j[(-1)^l A_j S(\tau_0 : \mu, \xi_j) + B_j C(\tau_0 : \mu, \xi_j)]g_l(\xi_j) \tag{28a}$$

and

$$D_2(\mu) = \frac{1}{\mu} U(\tau_0, \mu) + \Xi(\tau_0, \mu) + \frac{\varpi}{2} \sum_{l=0}^{N} \beta_l P_l(\mu) \sum_{j=1}^{J} \xi_j[A_j C(\tau_0 : \mu, \xi_j) + (-1)^l B_j S(\tau_0 : \mu, \xi_j)]g_l(\xi_j) \tag{28b}$$

for $\mu \in (0, 1]$. At this point we can rewrite Eqs. (25) and (26) as

$$A(\tau) = \int_0^1 M(\mu)\mathrm{e}^{-\tau/\mu} D_1(\mu)\mu\, \mathrm{d}\mu \tag{29}$$

and

$$B(\tau) = \int_0^1 M(\mu)\mathrm{e}^{-\tau/\mu} D_2(\mu)\mu\, \mathrm{d}\mu. \tag{30}$$

Finally, we note that as a simplification to the procedure of solving Eqs. (22) to find J_1 and J_2, we can use Eq. (3) in Eqs. (20) to find the alternative expressions

$$J_1 = \sum_{l=0}^{N} \frac{2l+1}{2} S_{0,l} \sum_{j=1}^{J} [(-1)^l A_j + B_j \mathrm{e}^{-\tau_0/\xi_j} + \frac{C_j}{\xi_j} V(0, \xi_j)]g_l(\xi_j) \tag{31a}$$

and

$$J_2 = \sum_{l=0}^{N} \frac{2l+1}{2} S_{0,l} \sum_{j=1}^{J} [A_j \mathrm{e}^{-\tau_0/\xi_j} + (-1)^l B_j + \frac{C_j}{\xi_j} U(\tau_0, \xi_j)]g_l(\xi_j) \tag{31b}$$

where, in general, the constants $S_{\alpha,l}$ are given by[3]

$$S_{\alpha,l} = \int_0^1 P_{2\alpha+1}(\mu) P_l(\mu)\, d\mu. \tag{32}$$

Some calculations have indicated that Eqs. (31) can provide useful approximations for J_1 and J_2.

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