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AN INVERSE SOURCE PROBLEM IN RADIATIVE TRANSFER

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Abstract—The spherical-harmonics method is used to develop a solution to an inverse source problem in radiative transfer. It is assumed that, with the exception of the inhomogeneous source term, all aspects of the radiation-transport problem are known, and we seek to determine the inhomogeneous source term from specified angular distributions of radiation exiting the two surfaces of a homogeneous plane-parallel medium. Anisotropic scattering is included in the monochromatic radiative-transfer model and general reflecting boundary conditions are considered.

INTRODUCTION

To analyze a radiative-transfer problem in a homogeneous plane-parallel medium for the case when there is a source of radiation, we consider the equation of transfer[1,2]

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu) + I(\tau, \mu) = \frac{\varpi}{2} \sum_{l=0}^{L} \beta_l P_l(\mu) \int_{-1}^{1} P_l(u) I(\tau, u)\, \mathrm{d}u + S(\tau), \tag{1}$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$, and the boundary conditions

$$I(0, \mu) = F_1(\mu) + \rho_1^{s} I(0, -\mu) + 2\rho_1^{d} \int_0^1 I(0, -u) u\, \mathrm{d}u \tag{2a}$$

and

$$I(\tau_0, -\mu) = F_2(\mu) + \rho_2^{s} I(\tau_0, \mu) + 2\rho_2^{d} \int_0^1 I(\tau_0, u) u\, \mathrm{d}u \tag{2b}$$

for $\mu \in (0, 1]$. Here ϖ is the albedo for single scattering ($\varpi < 1$), the elements β_l are the coefficients in a Legendre expansion of the scattering law and τ_0 is the optical thickness of the layer. In addition, ρ_α^{s} and ρ_α^{d}, $\alpha = 1$ and 2, are the coefficients for specular and diffuse reflection. We consider that the functions $F_1(\mu)$ and $F_2(\mu)$ are given, and we seek to determine the inhomogeneous source term $S(\tau)$ given that we know the boundary results $I(0, -\mu)$ and $I(\tau_0, \mu)$ for $\mu \in (0, 1]$.

Inverse problems in radiative transfer have defined a subject of interest for the past 20 or so years and there exists a considerable body of knowledge surrounding the subject that has been extensively reviewed in a series of papers by McCormick.[3–6] The specific inverse problem of interest examined here is one in which we seek to deduce an inhomogeneous source term within a medium of known properties. Thus here, in contrast to papers devoted to the more often studied inverse problems where one seeks, for example, to find the $\{\beta_l\}$ that define the scattering law and/or the albedo for single scattering ϖ, we assume that everything except the source is known and we attempt to determine the inhomogeneous source term $S(\tau)$ from the distributions, $I(0, -\mu)$ and $I(\tau_0, \mu)$, for $\mu \in (0, 1]$, of radiation that exit the host medium. In Refs. 7 and 8 McCormick and co-workers have reported on methods for determining an unknown inhomogeneous source that require that measurements be made in the interior of the medium. There clearly are applications where a method, such as the one developed here, that determines the internal source term from measurements of radiation exiting the boundaries could be more useful.

It seems clear that an inverse problem where only the source term is unknown is inherently much simpler than an inverse problem where either the albedo for single scattering or the scattering law is unknown.

We note that, as with essentially all inverse problems in radiative transfer, basic questions that concern the existence and the uniqueness of the solutions and the sensitivity of the established algorithm to measurement errors should be addressed. In reviewing the literature on inverse problems in radiative transfer, we have found only one paper[9] that has addressed the important issues of the existence and the uniqueness of the solution of the inverse source problem. Larsen's paper[9] is devoted to the case of a semi-infinite plane-parallel layer, and so is not immediately applicable here; however there are reasons to believe that Larsen's analysis could be generalized to the case of a finite layer.

It is not unusual to find in the general field of inverse radiative-transfer problems methods that rely on trial and error techniques and/or iterative procedures (see, for example, Ref. 10.). Our goal here is to provide a more deterministic algorithm for the considered inverse-source problem.

THE SPHERICAL HARMONICS SOLUTION

As discussed, for example, in Refs. 11–14, the spherical-harmonics solution to Eq. (1) can be expressed as

$$I(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} [A_j \mathrm{e}^{-\tau/\xi_j} + (-1)^l B_j \mathrm{e}^{-(\tau_0 - \tau)/\xi_j}] g_l(\xi_j) + I_p(\tau, \mu) \tag{3}$$

where the arbitrary constants A_j and B_j are to be fixed by the boundary conditions. Here we write the particular solution appropriate to the spherical-harmonics method as[12]

$$I_p(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \frac{C_j}{\xi_j} [U(\tau, \xi_j) + (-1)^l V(\tau, \xi_j)] g_l(\xi_j) \tag{4}$$

where, in general,

$$U(\tau, \xi) = \int_0^{\tau} S(x)\, \mathrm{e}^{-(\tau - x)/\xi}\, \mathrm{d}x \tag{5a}$$

and

$$V(\tau, \xi) = \int_{\tau}^{\tau_0} S(x)\, \mathrm{e}^{-(x - \tau)/\xi}\, \mathrm{d}x. \tag{5b}$$

To define Eqs. (3) and (4) we note[11-14] that the Chandrasekhar polynomials are denoted by $\{g_l(\xi)\}$, that the eigenvalues $\{\xi_j\}$ are, with N odd, the $J = (N+1)/2$ positive zeros of $g_{N+1}(\xi)$ and that the constants $\{C_j\}$ are given by

$$C_j = \left(\sum_{k=1}^{J} h_{2k-2} g_{2k-2}^2(\xi_j) \right)^{-1}, \quad j = 1, 2, \ldots, J, \tag{6}$$

with $h_l = 2l + 1 - \varpi\beta_l$, $0 \leqslant l \leqslant L$, and $h_l = 2l + 1$ for $l > L$.

If the source term $S(\tau)$ is known we can determine the arbitrary constants required in Eq. (3) by substituting Eq. (3) into Eqs. (2) and using, for example, the Mark or Marshak projections to generate a system of linear algebraic equations that could be solved to find $\{A_j\}$ and $\{B_j\}$. Here, since $S(\tau)$ is not known, we require a variation on this procedure.

First of all, we choose to express the unknown source term $S(\tau)$ in terms of a set of linearly independent basis functions $\{\phi_k(\xi)\}$ defined on the interval [0, 1] so that we can write

$$S(\tau) = \sum_{k=0}^{\mathcal{N}} a_k \phi_k(\tau/\tau_0). \tag{7}$$

It follows that all we need to do is to determine the unknown coefficients $\{a_k\}$ so that the exit distributions we compute match in some sense the exit distributions that are presumed known.

Using Eq. (7), we can rewrite Eq. (4) as

$$I_p(\tau, \mu) = \sum_{k=0}^{\mathcal{N}} a_k Y_k(\tau, \mu) \tag{8}$$

where the known functions $\Upsilon_k(\tau, \mu)$ are given by

$$\Upsilon_k(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \frac{C_j}{\xi_j} [U_k(\tau, \xi_j) + (-1)^l V_k(\tau, \xi_j)] g_l(\xi_j) \tag{9}$$

where

$$U_k(\tau, \xi) = \int_0^{\tau} \phi_k(x/\tau_0) \mathrm{e}^{-(\tau - x)/\xi} \, \mathrm{d}x \tag{10a}$$

and

$$V_k(\tau, \xi) = \int_{\tau}^{\tau_0} \phi_k(x/\tau_0) \mathrm{e}^{-(x - \tau)/\xi} \, \mathrm{d}x. \tag{10b}$$

We thus can express our solution to Eq. (1) as

$$I(\tau, \mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} [A_j \mathrm{e}^{-\tau/\xi_j} + (-1)^l B_j \mathrm{e}^{-(\tau_0 - \tau)/\xi_j}] g_l(\xi_j)$$
$$+ \sum_{k=0}^{\mathcal{N}} a_k \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \frac{C_j}{\xi_j} [U_k(\tau, \xi_j) + (-1)^l V_k(\tau, \xi_j)] g_l(\xi_j). \tag{11}$$

In order to find the constants required in Eq. (11) we substitute Eq. (11) into Eqs. (2) and use the Marshak projection scheme in the manner of Ref. 13 to obtain, for $\alpha = 0, 1, \ldots, (N-1)/2$, a system of linear algebraic equations given by

$$\sum_{j=1}^{J} \sum_{l=0}^{N} \frac{2l+1}{2} \{[1 - (-1)^l \rho_1^s] S_{\alpha,l} - 2(-1)^l \rho_1^d S_{0,l} S_{\alpha,0}\} [A_j + (-1)^l B_j \mathrm{e}^{-\tau_0/\xi_j}] g_l(\xi_j) = R_{1,\alpha} \tag{12a}$$

and

$$\sum_{j=1}^{J} \sum_{l=0}^{N} \frac{2l+1}{2} \{[1 - (-1)^l \rho_2^s] S_{\alpha,l} - 2(-1)^l \rho_2^d S_{0,l} S_{\alpha,0}\} [B_j + (-1)^l A_j \mathrm{e}^{-\tau_0/\xi_j}] g_l(\xi_j) = R_{2,\alpha} \tag{12b}$$

where the constants $S_{\alpha,l}$ are given by

$$S_{\alpha,l} = \int_0^1 P_{2\alpha+1}(\mu) P_l(\mu) \, \mathrm{d}\mu. \tag{13}$$

Here the right-hand sides of Eqs. (12) are given by

$$R_{1,\alpha} = \int_0^1 P_{2\alpha+1}(\mu) F_1(\mu) \, \mathrm{d}\mu + \sum_{k=0}^{\mathcal{N}} a_k \sum_{j=1}^{J} \sum_{l=0}^{N} \frac{2l+1}{2}$$
$$\times \{2\rho_1^d S_{0,l} S_{\alpha,0} + [\rho_1^s - (-1)^l] S_{\alpha,l}\} \frac{C_j}{\xi_j} V_k(0, \xi_j) g_l(\xi_j) \tag{14a}$$

and

$$R_{2,\alpha} = \int_0^1 P_{2\alpha+1}(\mu) F_2(\mu) \, \mathrm{d}\mu + \sum_{k=0}^{\mathcal{N}} a_k \sum_{j=1}^{J} \sum_{l=0}^{N} \frac{2l+1}{2}$$
$$\times \{2\rho_2^d S_{0,l} S_{\alpha,0} + [\rho_2^s - (-1)^l] S_{\alpha,l}\} \frac{C_j}{\xi_j} U_k(\tau_0, \xi_j) g_l(\xi_j). \tag{14b}$$

If we now write

$$A_j = A_j^* + \sum_{k=0}^{\mathcal{N}} a_k A_j^k \tag{15a}$$

and

$$B_j = B_j^* + \sum_{k=0}^{\mathcal{N}} a_k B_j^k \tag{15b}$$

then it is clear that $\{A_j^*\}$ and $\{B_j^*\}$ are the solutions of Eqs. (12) with the right-hand sides given by

$$R_{1,\alpha}^* = \int_0^1 P_{2\alpha+1}(\mu)F_1(\mu)\,d\mu \tag{16a}$$

and

$$R_{2,\alpha}^* = \int_0^1 P_{2\alpha+1}(\mu)F_2(\mu)\,d\mu. \tag{16b}$$

In a similar manner the constants $\{A_j^k\}$ and $\{B_j^k\}$ are the solutions of Eqs. (12) with the right-hand sides given by

$$R_{1,\alpha}^k = \sum_{j=1}^J \sum_{l=0}^N \frac{2l+1}{2}\{2\rho_1^d S_{0,l}S_{\alpha,0} + [\rho_1^s - (-1)^l]S_{\alpha,l}\}\frac{C_j}{\xi_j}V_k(0,\xi_j)g_l(\xi_j) \tag{17a}$$

and

$$R_{2,\alpha}^k = \sum_{j=1}^J \sum_{l=0}^N \frac{2l+1}{2}\{2\rho_2^d S_{0,l}S_{\alpha,0} + [\rho_2^s - (-1)^l]S_{\alpha,l}\}\frac{C_j}{\xi_j}U_k(\tau_0,\xi_j)g_l(\xi_j). \tag{17b}$$

For a fixed order N of the spherical-harmonics method and a fixed value of $\mathcal{N}$ we can solve our systems of linear algebraic equations to find all $\{A_j^*\}$ and $\{B_j^*\}$ as well as $\{A_j^k\}$ and $\{B_j^k\}$, for $k = 0, 1, \ldots, \mathcal{N}$, independent of the source term. We consider that we now have done just that.

Having solved the various systems of linear algebraic equations for the constants $\{A_j^*\}$, $\{B_j^*\}$, $\{A_j^k\}$ and $\{B_j^k\}$, for $k = 0, 1, \ldots, \mathcal{N}$, we can rewrite Eq. (11) as

$$I(\tau,\mu) = \sum_{l=0}^N \frac{2l+1}{2}P_l(\mu)\sum_{j=1}^J [A_j^* e^{-\tau/\xi_j} + (-1)^l B_j^* e^{-(\tau_0-\tau)/\xi_j}]g_l(\xi_j) + \sum_{k=0}^{\mathcal{N}} a_k I_k(\tau,\mu) \tag{18}$$

where

$$I_k(\tau,\mu) = \sum_{l=0}^N \frac{2l+1}{2}P_l(\mu)\sum_{j=1}^J \left\{A_j^k e^{-\tau/\xi_j} + (-1)^l B_j^k e^{-(\tau_0-\tau)/\xi_j} + \frac{C_j}{\xi_j}[U_k(\tau,\xi_j) + (-1)^l V_k(\tau,\xi_j)]\right\}g_l(\xi_j). \tag{19}$$

Note that at this point everything in Eq. (18) is known except the source constants $\{a_k\}$.

THE INVERSE SOURCE PROBLEM

Having developed our spherical-harmonics solution in terms of the source constants $\{a_k\}$, we are now ready to express these constants in terms of the presumed given distributions of radiation leaving the surfaces. We let $\Xi_1(\mu)$ and $\Xi_2(\mu)$ for $\mu \in (0, 1]$ denote the intensities that have been observed exiting respectively the surfaces $\tau = 0$ and $\tau = \tau_0$. We thus can use Eq. (18) and write

$$\sum_{k=0}^{\mathcal{N}} a_k I_k(0,-\mu) = \Xi_1(\mu) - I_*(0,-\mu) \tag{20a}$$

and

$$\sum_{k=0}^{\mathcal{N}} a_k I_k(\tau_0,\mu) = \Xi_2(\mu) - I_*(\tau_0,\mu) \tag{20b}$$

for $\mu \in (0, 1]$. Here we use the notation

$$I_*(\tau,\mu) = \sum_{l=0}^N \frac{2l+1}{2}P_l(\mu)\sum_{j=1}^J [A_j^* e^{-\tau/\xi_j} + (-1)^l B_j^* e^{-(\tau_0-\tau)/\xi_j}]g_l(\xi_j). \tag{21}$$

Equations (20) now must be solved in some approximate manner. If $\Xi_1(\mu)$ and $\Xi_2(\mu)$ for $\mu \in (0, 1]$ are known at a sufficiently large number of points then we can simply evaluate Eqs. (20) at those points and solve the resulting system of linear algebraic equations for the $\mathcal{N} + 1$ unknown constants $\{a_k\}$. Of course other projection techniques are also possible. For a selected set of basis functions $\{H_\alpha(\mu)\}$, we can multiply Eqs. (20) by these basis functions and integrate to obtain

$$\sum_{k=0}^{\mathcal{N}} M_{\alpha,k} a_k = \int_0^1 H_\alpha(\mu)\Xi_1(\mu)\,d\mu - T_{\alpha,1}, \quad \alpha = 1, 2, \ldots, N_1, \tag{22a}$$

and

$$\sum_{k=0}^{\mathcal{N}} N_{\alpha,k} a_k = \int_0^1 H_\alpha(\mu)\Xi_2(\mu)\,d\mu - T_{\alpha,2}, \quad \alpha = 1, 2, \ldots, N_2, \tag{22b}$$

where

$$M_{\alpha,k} = \int_0^1 H_\alpha(\mu) I_k(0, -\mu)\,d\mu, \tag{23a}$$

$$N_{\alpha,k} = \int_0^1 H_\alpha(\mu) I_k(\tau_0, \mu)\,d\mu, \tag{23b}$$

$$T_{\alpha,1} = \int_0^1 H_\alpha(\mu) I_*(0, -\mu)\,d\mu \tag{24a}$$

and

$$T_{\alpha,2} = \int_0^1 H_\alpha(\mu) I_*(\tau_0, \mu)\,d\mu. \tag{24b}$$

Here in order to generate a "square" system of linear algebraic equations we take $N_1 + N_2 = \mathcal{N} + 1$.

We note that the elements defined by Eqs. (23) and (24) are independent of the inhomogeneous source term, and thus they would not have to be recomputed for a series of applications where only the source is changing.

SOME TEST CASES

In order to have a specific scattering law for testing our solution technique for the inverse source problem, and to avoid having to provide a table of scattering law coefficients $\{\beta_l\}$, we use here the binomial scattering law

$$p(\cos\Theta) = \frac{L+1}{2^L}(1 + \cos\Theta)^L \tag{25}$$

which can be represented with $L + 1$ Legendre coefficients that can be computed with $\beta_0 = 1$ and[15]

$$\beta_l = \left(\frac{2l+1}{2l-1}\right)\left(\frac{L+l-1}{L+l+1}\right)\beta_{l-1} \tag{26}$$

for $l = 1, 2, \ldots, L$. Here we use Eq. (25) with $L = 24$ for our scattering law, and we use $\varpi = 0.9$, $\tau_0 = 2.0$, $\rho_1^s = 0.1$, $\rho_1^d = 0.2$, $\rho_2^s = 0.3$ and $\rho_2^d = 0.1$.

To define the boundary conditions we use, for our test cases,

$$F_1(\mu) = \alpha + \beta\delta(\mu - \mu_0) \tag{27a}$$

and

$$F_2(\mu) = \gamma, \tag{27b}$$

for $\mu \in (0, 1]$, with the numerical values $\alpha = 1.0$, $\beta = 0.7$, $\mu_0 = 0.5$ and $\gamma = 0.6$.

In regard to the basis functions $\{\phi_k(x)\}$ used in Eq. (7), we have considered two cases: (i) the Hermite cubic splines[16] as used in Ref. (13) and (ii) the Legendre polynomials $\{P_k(2x - 1)\}$. For the projection technique that yields Eqs. (22)–(24), we have used

$$H_\alpha(\mu) = P_{2\alpha-1}(\mu) \tag{28}$$

and we have used Eqs. (22a) and (22b) respectively with $\alpha = 1, 2, \ldots, N_1$ and $\alpha = 1, 2, \ldots, N_2$ where $N_1 + N_2 = \mathcal{N} + 1$. Although the choice is somewhat arbitrary, we have generally taken $N_1 \approx N_2$ in our calculations.

Table 1. The given and the computed inhomogeneous source term.

τ/τ_0	$S_a(\tau)$	$\hat{S}_a(\tau)$	$S_b(\tau)$	$\hat{S}_b(\tau)$	$S_c(\tau)$	$\hat{S}_c(\tau)$
0.0	1.000	0.993	1.000	0.997	2.000	2.005
0.1	1.041	1.043	1.309	1.310	2.588	2.587
0.2	1.170	1.168	1.588	1.589	2.951	2.958
0.3	1.412	1.410	1.809	1.809	2.951	2.949
0.4	1.811	1.813	1.951	1.950	2.588	2.580
0.5	2.441	2.445	2.000	1.999	2.000	2.000
0.6	3.421	3.421	1.951	1.951	1.412	1.420
0.7	4.929	4.924	1.809	1.810	1.049	1.052
0.8	7.234	7.233	1.588	1.588	1.049	1.043
0.9	10.73	10.74	1.309	1.308	1.412	1.413
1.0	16.00	15.98	1.000	1.001	2.000	1.990

In regard to the inhomogeneous source term, we have tried three cases:

$$S_a(\tau) = [1 + (\tau/\tau_0)^2]^4, \tag{29a}$$

$$S_b(\tau) = 1 + \sin(\pi\tau/\tau_0) \tag{29b}$$

and

$$S_c(\tau) = 2 + \sin(2\pi\tau/\tau_0). \tag{29c}$$

In the accompanying table we list the exact values of $S_a(\tau)$, $S_b(\tau)$ and $S_c(\tau)$ along with the estimated values of the sources $\hat{S}_a(\tau)$, $\hat{S}_b(\tau)$ and $\hat{S}_c(\tau)$ obtained from our solution of the inverse problem.

While the results in our table are clearly very good, we must admit that there can exist inverse source problems that, without further numerical work, we cannot solve as well as the three test cases we considered. We have obtained, while using both the Hermite cubic splines and the Legendre polynomials as basis functions to represent the unknown source, a system of linear algebraic equations that became poorly conditioned as we increased the number of terms $\mathcal{N} + 1$ in the expansion of the source term. In fact our best results were obtained by using $\mathcal{N} = 5$ with the Legendre basis. Of course it is often the case that we encounter a poorly conditioned system of equations in trying to solve inverse problems in radiative transfer; however, additional work is planned to try to improve the numerical aspects of this method of solving the inverse source problem.

In this, our first paper on the inverse source problem, we have not carried out in a definitive manner many of the studies that should be undertaken in order to establish the class of problems and experiments from which we could expect to extract with confidence the desired results. However, we can report several observations we have made: (i) for the test problems we considered we were able to reduce the accuracy of the exiting distributions to just two figures and still obtain meaningful results for the desired source term; (ii) we were able to obtain good results for the source term by using the exiting distributions at only 10 points on each of the two boundaries; (iii) although we generally have used $N_1 \approx N_2$ in our calculations, we also were able to obtain good results by taking either $N_1 = 0$ or $N_2 = 0$; in other words, we were able to deduce the source term from the intensity exiting from only one of the two surfaces. It is also worthwhile to note that we were able to deduce the unknown source term for the three considered test problems even though we had, in addition to the driving force of the inhomogeneous source, radiation incident on both sides of the layer and both specular and diffuse reflection on each of the surfaces.

To conclude this work we note that we could generalize the procedure developed here to allow both $F_1(\mu)$ and $F_2(\mu)$ in Eqs. (2) to be unknowns so that we could determine, in principle, $S(\tau)$, $F_1(\mu)$ and $F_2(\mu)$, for $\mu \in (0, 1]$, from given observations of the distributions of radiation exiting the layer. It is also clear that in the event that the algorithm developed here will not extract the source to the degree of accuracy desired, the method could be used to initiate an iterative procedure. The method developed here could also be extended to non-gray radiative-transfer models and to models that take into account polarization effects.

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