Pergamon

J. Quant. Spectrosc. Radiat. Transfer Vol. 52, No. 2, pp. 157–160, 1994

0022-4073(94)E0025-2
0022-4073/94 $7.00 + 0.00

# A RADIATIVE-TRANSFER INVERSE-SOURCE PROBLEM FOR A SPHERE

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(Received 2 December 1993)

**Abstract**—A sphere-to-plane transformation technique and the spherical-harmonics method are used to develop a solution to an inverse-source problem for radiative transfer in a spherical body. It is assumed that, with the exception of the inhomogeneous source term, all aspects of the radiation-transport problem are known, and we seek to determine the source term from a specified angular distribution of radiation that exits the surface of the sphere.

## 1. INTRODUCTION

To analyze a radiation transport problem in a homogeneous sphere for the case when there is a source of radiation we consider the equation of transfer[1,2]

$$\mu \frac{\partial}{\partial r} I(r, \mu) + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} I(r, \mu) + I(r, \mu) = \frac{\varpi}{2} \int_{-1}^{1} I(r, u)\,\mathrm{d}u + S(r) \tag{1}$$

for $r \in (0, R)$ and $\mu \in [-1, 1]$ and the boundary condition

$$I(R, -\mu) = 0 \tag{2}$$

for $\mu \in (0, 1]$. Here $\varpi$ is the albedo for single scattering ($\varpi < 1$) and $R$ is the optical radius of the sphere. We seek to determine the inhomogeneous source term $S(r)$ given that we know the boundary result $I(R, \mu)$ for $\mu \in (0, 1]$.

In a recent paper,[3] the inverse-source problem for a plane-parallel medium was discussed, and a deterministic algorithm for computing a solution was reported. Also in Ref. 3 numerous references to basic works and to review articles were made, and so here, since this work is very similar to that reported in Ref. 3, we will be brief and note only how we have extended the previous paper to cover the case of a spherical body.

## 2. THE EXIT DISTRIBUTION AND THE PSEUDO PROBLEM

Pomraning and Siewert[4] derived the integral form of the transport equation applicable to a homogeneous sphere with an internal source and illuminated by an external source. As we intend to be brief here, we can specialize the results of Pomraning and Siewert to the case of no external illumination and use from Ref. 4 the following expression for the distribution of radiation exiting the sphere:

$$I(R, \mu) = \int_{R\sqrt{(1-\mu^2)}}^{R} \left[\frac{\varpi}{2} I(x) + S(x)\right] \Pi^{-1}(x, R, \mu) \left[\mathrm{e}^{-R\mu + \Pi(x, R, \mu)} + \mathrm{e}^{-R\mu - \Pi(x, R, \mu)}\right] x\,\mathrm{d}x \tag{3}$$

for $\mu \in (0, 1]$. Here

$$I(r) = \int_{-1}^{1} I(r, \mu)\,\mathrm{d}\mu \tag{4}$$

and

$$\Pi(x, r, \mu) = [x^2 - r^2(1 - \mu^2)]^{1/2}. \tag{5}$$

As we intend to use Eq. (3) to compute the exit distribution it is clear that we must first compute the flux $I(r)$ for $r \in [0, R]$.

Mitsis[5] developed, in the context of critical problems in neutron transport theory, a pseudo-slab problem the solution of which yielded the flux distribution $I(r)$ in a related sphere. The idea of Mitsis has subsequently been used, see for example Refs. 6 and 7, to solve some basic problems in radiative transfer. Here for the purpose of solving the considered inverse-source problem we can compute the required flux distribution $I(r)$ from[6,7]

$$I(r) = \frac{1}{r}\int_{-1}^{1} \Phi(r, \mu)\,\mathrm{d}\mu \tag{6}$$

where $\Phi(r, \mu)$ is defined by the pseudo-slab problem based on the equation of transfer

$$\mu\frac{\partial}{\partial r}\Phi(r, \mu) + \Phi(r, \mu) = \frac{\varpi}{2}\int_{-1}^{1} \Phi(r, u)\,\mathrm{d}u + rS(|r|), \tag{7}$$

for $r \in (-R, R)$ and $\mu \in [-1, 1]$, the symmetry condition $\Phi(-r, -\mu) = -\Phi(r, \mu)$ and the boundary condition

$$\Phi(R, -\mu) = 0 \tag{8}$$

for $\mu \in (0, 1]$.

As we intend to solve our inverse-source problem in terms of moments of the radiation intensity exiting the sphere, we multiply Eq. (3) by $W_\alpha(\mu)$ and integrate over $\mu$ to obtain

$$\int_0^1 I(R, \mu)W_\alpha(\mu)\,\mathrm{d}\mu = \int_0^R x\left[\frac{\varpi}{2}I(x) + S(x)\right]F_\alpha(x)\,\mathrm{d}x \tag{9}$$

where

$$F_\alpha(x) = \frac{1}{R}\int_{R-x}^{R+x} W_\alpha\left(\frac{\tau^2 + R^2 - x^2}{2R\tau}\right)\mathrm{e}^{-\tau}\frac{\mathrm{d}\tau}{\tau}. \tag{10}$$

Though other good choices clearly are possible, we take

$$W_\alpha(\mu) = P_{2\alpha-1}(\mu),\ \alpha = 1, 2, \ldots, \tag{11}$$

where $P_l(\mu)$ is used to denote the Legendre polynomial of order $l$.

In order to solve our inverse-source problem, we now expand the unknown source in terms of a set of linearly independent basis functions $\{\phi_k(x)\}$ so that we can write

$$S(r) = \sum_{k=0}^{K} a_k\phi_k(r/R),\ r \in [0, R]. \tag{12}$$

We consider now the pseudo problem relevant to each basis function $\phi_k(r/R)$, viz.

$$\mu\frac{\partial}{\partial r}\Phi_k(r, \mu) + \Phi_k(r, \mu) = \frac{\varpi}{2}\int_{-1}^{1} \Phi_k(r, u)\,\mathrm{d}u + r\phi_k(|r/R|), \tag{13}$$

for $r \in (-R, R)$ and $\mu \in [-1, 1]$, $\Phi_k(-r, -\mu) = -\Phi_k(r, \mu)$ and

$$\Phi_k(R, -\mu) = 0 \tag{14}$$

for $\mu \in (0, 1]$. If we let

$$I_k(r) = \frac{1}{r}\int_{-1}^{1} \Phi_k(r, \mu)\,\mathrm{d}\mu \tag{15}$$

and note that

$$\Phi(r, \mu) = \sum_{k=0}^{K} a_k\Phi_k(r, \mu) \tag{16}$$

then we can write Eq. (9) as

$$\int_0^1 I(R,\mu)W_\alpha(\mu)\,d\mu = \sum_{k=0}^{K} a_k \int_0^R x\left[\frac{\varpi}{2}I_k(x) + \phi_k(x/R)\right]F_\alpha(x)\,dx. \tag{17}$$

If we consider Eq. (17) for $\alpha = 1, 2, 3, \ldots, K$ we can solve that system of linear algebraic equations to find the coefficients $\{a_k\}$ required in Eq. (12) to define the desired source.

## 3. THE SPHERICAL HARMONICS SOLUTION

Following, for example, Refs. 8–10, we express our spherical-harmonics solution to Eq. (13) as

$$\Phi_k(r,\mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} A_{k,j}[e^{-(R+r)/\xi_j} - (-1)^l e^{-(R-r)/\xi_j}]g_l(\xi_j) + \Phi_{k,p}(r,\mu) \tag{18}$$

where the arbitrary constants $A_{k,j}$ are to be fixed by the boundary conditions. Here we write the particular solution appropriate to the spherical-harmonics method as[9]

$$\Phi_{k,p}(r,\mu) = \sum_{l=0}^{N} \frac{2l+1}{2} P_l(\mu) \sum_{j=1}^{J} \frac{C_j}{\xi_j}[U_k(r,\xi_j) + (-1)^l V_k(r,\xi_j)]g_l(\xi_j) \tag{19}$$

where, in general,

$$U_k(r,\xi) = \int_{-R}^{r} x\phi_k(|x/R|)e^{-(r-x)/\xi}\,dx \tag{20a}$$

and

$$V_k(r,\xi) = \int_{r}^{R} x\phi\,(|x/R|)e^{-(x-r)/\xi}\,dx. \tag{20b}$$

To define Eqs. (18) and (19), we note[8–10] that the Chandrasekhar polynomials are denoted by $\{g_l(\xi)\}$, that the eigenvalues $\{\xi_j\}$ are, with $N$ odd, the $J = (N+1)/2$ positive zeros of $g_{N+1}(\xi)$ and that the constants $\{C_j\}$ are given by

$$C_j = \left(\sum_{k=1}^{J} h_{2k-2} g^2_{2k-2}(\xi_j)\right)^{-1}, \quad j = 1, 2, \ldots, J, \tag{21}$$

with $h_0 = 1 - \varpi$ and $h_l = 2l + 1$ for $l > 0$.

If we substitute Eq. (18) into Eq. (14) and use the Marshak projection scheme[8] we obtain a system of linear algebraic equations that can be solved to yield the constants $\{A_{k,j}\}$ required to complete the solution given by Eq. (18).

At this point we are ready to try some numerical experiments to see how well we can determine the source $S(r)$ from computed moments of the exiting distribution $I(R,\mu)$, $\mu \in (0, 1]$.

## 4. SOME TEST CASES

In order to test our algorithm for solving the inverse-source problem in a sphere we choose to use the Legendre polynomials $\{P_k(2r/R - 1)\}$ as our basis functions in Eq. (12), and we use Eq. (11) to define our projection of the radiation exiting the surface. We note that the required integrals

$$F_\alpha(x) = \frac{1}{R}\int_{R-x}^{R+x} P_{2\alpha-1}\left(\frac{\tau^2 + R^2 - x^2}{2R\tau}\right)e^{-\tau}\frac{d\tau}{\tau} \tag{22}$$

were evaluated accurately by utilizing the symbolic-computation software package *Maple V* and standard numerical integration techniques.

For our computations we use $R = 2$ and $\varpi = 0.9$, and, in regard to the inhomogeneous source term, we have tried three cases:

$$S_a(r) = 16[1 + (r/R)^2]^{-4}, \tag{23a}$$

$$S_b(r) = 1 + \sin\left(\frac{\pi r}{R}\right) \tag{23b}$$

Table 1. The given and the computed source term for $R = 2$ and $\varpi = 0.9$.

| $r/R$ | $S_a(r)$ | $\hat{S}_a(r)$ | $S_b(r)$ | $\hat{S}_b(r)$ | $S_c(r)$ | $\hat{S}_c(r)$ |
|---|---|---|---|---|---|---|
| 0.0 | 16.00 | 16.05 | 1.000 | 0.994 | 2.000 | 1.991 |
| 0.1 | 15.38 | 15.37 | 1.309 | 1.309 | 2.588 | 2.587 |
| 0.2 | 13.68 | 13.68 | 1.588 | 1.588 | 2.951 | 2.951 |
| 0.3 | 11.33 | 11.33 | 1.809 | 1.809 | 2.951 | 2.951 |
| 0.4 | 8.837 | 8.837 | 1.951 | 1.951 | 2.588 | 2.588 |
| 0.5 | 6.554 | 6.554 | 2.000 | 2.000 | 2.000 | 2.000 |
| 0.6 | 4.677 | 4.677 | 1.951 | 1.951 | 1.412 | 1.412 |
| 0.7 | 3.246 | 3.246 | 1.809 | 1.809 | 1.049 | 1.049 |
| 0.8 | 2.212 | 2.212 | 1.588 | 1.588 | 1.049 | 1.049 |
| 0.9 | 1.491 | 1.491 | 1.309 | 1.309 | 1.412 | 1.412 |
| 1.0 | 1.000 | 1.000 | 1.000 | 1.000 | 2.000 | 2.000 |

and

$$S_c(r) = 2 + \sin\left(\frac{2\pi r}{R}\right). \tag{23c}$$

For each of the foregoing source terms we have used the spherical-harmonics method to solve the direct problem in order to evaluate the left-hand side of Eq. (9). We then used our inverse solution to recompute the assumed source. In the accompanying table we list the exact values of $S_a(r)$, $S_b(r)$ and $S_c(r)$ along with the estimated values of the sources $\hat{S}_a(r)$, $\hat{S}_b(r)$ and $\hat{S}_c(r)$ obtained from our solution of the inverse problem.

While the results in our table are clearly very good, we must admit that there can exist inverse source problems that, without further numerical work, we cannot solve as well as the three test cases we considered.

*Acknowledgements*—The author would like to thank R. D. M. Garcia and N. J. McCormick for their careful reading of the manuscript and for communicating some comments that were used to improve the presentation of this work.

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