4. F_N Method for Solving Transport Problems, *R. D. M. Garcia (CTA–Brazil), C. E. Siewert (NCSU), J. R. Thomas, Jr. (VPI&SU), invited*

INTRODUCTION

The F_N method ($F \Rightarrow$ facile) is a semi-analytical method suitable for obtaining highly accurate solutions for one-dimensional transport problems. It was introduced in 1979 by Siewert and Benoist[1] in the context of neutron transport theory. At that time, a few one-speed problems for plane-parallel media with isotropic scattering[1,2] were used to test the method; 5 yr later the method had already been extended to treat multilayer problems with general anisotropic scattering, homogeneous spheres and cylinders with isotropic scattering, radiative transfer problems with reflective boundary conditions and polarization, multigroup slowing down in plane geometry, azimuthally dependent problems, radiative transfer in inhomogeneous media, neutral particle transport in plasmas, and basic problems in rarefied gas dynamics and kinetic theory (see the review by Garcia[3]).

In more recent years, the method has been used to solve a class of radiative transfer problems with internal heat generation and emitting and reflecting surfaces in spherical and cylindrical geometries,[4] a radiative transfer problem coupled with convection,[5] the problem of multiple light scattering with polarization,[6] an inverse problem in radiative transfer,[7] and a radiative transfer problem relevant to the study of plant canopies.[8,9] It has also been used to solve the fully coupled kinetic equations for half-space[10] and finite plane-parallel media,[11] two basic rarefied gas dynamics problems in cylindrical geometry,[12] the searchlight problem,[13,14] and a scalar transport equation basic to neutral particle transport in ducts.[15] Moreover, a reported F_N solution to fully coupled two-group problems in plane and spherical geometries[16] has been extended and applied to a more refined model of neutral particle transport in ducts.[17]

In addition to the comprehensive review[3] of the F_N method already mentioned, specific reviews of the use of the method in neutron transport theory,[18] atmospheric radiative transfer,[19] and radiative transfer in inhomogeneous media[20] are also available.

THE METHOD

In this section, we summarize the fundamental aspects of the F_N method; for details, the reader is referred to Ref. 3.

Singular Integral Equations and Constraints

The starting point for the F_N method is the integro-differential form of the transport equation. The basic idea behind the method consists of reducing the original equation to a set of singular integral equations and constraints that can be solved for the unknown exit distributions without having to deal with distributions at interior points of the medium. There are three reported ways of accomplishing this.[3] The first two[1,2] rely on a knowledge of the singular eigenfunction expansion technique[21] (Case's method), while the third[22] is based on an integral-transform technique and does not require such knowledge. However, it does require some familiarity with complex variable theory.

Approximations for the Exit Distributions

To solve the set of singular integral equations and constraints described earlier in an efficient manner, the unknown exit distributions are approximated in terms of a finite set of basis functions. For example, in the case of a single layer $[0,a]$ subject to an incident distribution $f(\mu)$ from the left, one would use the approximations, for $\mu > 0$,

$$\Psi(0,-\mu) = \sum_{\alpha=0}^{N} a_\alpha \Phi_\alpha(\mu) \tag{1}$$

and

$$\Psi(a,\mu) = f(\mu)\exp(-a/\mu) + \sum_{\alpha=0}^{N} b_\alpha \Phi_\alpha(\mu) , \tag{2}$$

where $\Phi_\alpha(\mu)$ is the $(\alpha+1)$'th element of the set of basis functions and a_α and b_α, $\alpha = 0,1,\ldots,N$, are coefficients to be determined.

In regard to the choice of the basis functions, the F_N method is very flexible. If the problem being solved is relatively simple, the power basis $\{\mu^\alpha\}$ and, in the case of a half-space with isotropic scattering, the rational basis $\{1/(\mu+\nu_\alpha)\}$ should be considered first since they are very easy to implement. However, experience has shown that it is not possible to go very high in the order of the approximation (N) with these two classes of basis functions before numerical instabilities develop—typically, one cannot expect to go much higher than $N = 15$ when implementing the method in a short-word machine with double precision (16 decimal digits). If one needs higher orders of the approximation to solve a problem, an alternative is to use an orthogonal basis as, e.g., the shifted Legendre basis $\{P_\alpha(2\mu-1)\}$, which is not so easy to implement but is very stable in high order (it has been used for N as high as 700). An interesting new option, especially for parallel-processing environments, is the use of splines as basis functions.[23]

Collocation and Linear Algebraic Equations

Since the set of singular integral equations derived for the method has infinite dimension (it depends on a parameter ξ that can take any value on $[0,1]$), the result of the previous step is an infinite set of linear algebraic equations. This set can be made finite, to match the number of equations to the number of unknowns, by using a collocation technique. For a typical one-speed problem, considering that there are $\aleph$ constraints, where $\aleph$ is the number of positive zeros of the dispersion function,[3] one picks $(N+1-\aleph)$ points in $[0,1]$ (see details about collocation schemes in Ref. 3), to obtain a system of linear algebraic equations of order $2(N+1)$ for the coefficients a_α and b_α, $\alpha = 0,1,\ldots,N$. Normally, the solution of this system can be accomplished by Gauss elimination, although there is an example in the literature where a more sophisticated linear equation solver (singular-value decomposition) had to be employed.[24]

Postprocessing

Once the coefficients $\{a_\alpha\}$ and $\{b_\alpha\}$ have been determined, Eqs. (1) and (2) could be used to compute the exit distributions. However, some time ago it was found that *postprocessing* Eqs. (1) and (2) with the singular integral equations results in improved approximations that converge faster and more uniformly in μ as N increases.[25]

Interior Distributions

If interior distributions for the problem being solved are needed, the F_N method can also be used to compute them. To this end, one can derive a set of singular integral equations and constraints (very similar to the set for the exit distributions)

that relates the particle distribution at any interior point of the medium to the boundary distributions.[3] For the case considered here, the approximations, for $x \in (0,a)$ and $\mu > 0$,

$$\Psi(x,-\mu) = \sum_{\alpha=0}^{N} c_\alpha(x)\Phi_\alpha(\mu) \tag{3}$$

and

$$\Psi(x,\mu) = f(\mu)\exp(-x/\mu) + \sum_{\alpha=0}^{N} d_\alpha(x)\Phi_\alpha(\mu) , \tag{4}$$

and a procedure similar to the one described for the exit distributions can be used to find a system of linear algebraic equations for the coefficients $\{c_\alpha(x)\}$ and $\{d_\alpha(x)\}$, for any desired value of x. It is worthwhile to mention that the correspondent matrix of coefficients does not depend on x (only the right-hand side vector does). Therefore, the determination of the distribution at, say, J interior points requires the solution of a linear system for which the augmented matrix has $(2N + J + 2)$ columns.

CONCLUDING REMARKS

In conclusion, the F_N method is a flexible and economical tool for solving transport problems in several fields. Frequently, the performance of the method has exceeded that of concurrent methods (e.g., the spherical harmonics method) in the sense that, for a given level of precision, the F_N method needs fewer terms in the approximation and thus involves less computational work. In a few instances (e.g., multigroup problems with upscattering, azimuthally dependent problems with highly anisotropic scattering), however, the complexity of implementing the F_N method has made it look less attractive than some alternative methods. This may change as the method is further developed.

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