THE FOURIER DECOMPOSITION FOR A RADIATIVE-TRANSFER PROBLEM WITH AN ASYMMETRICALLY REFLECTING GROUND

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Abstract — Elementary considerations are used to carry out a Fourier decomposition of the radiation intensity for a model atmosphere that has a ground defined by a general reflection function. In contrast to simpler problems, the intensity here has both *sine* and *cosine* components that are coupled by way of the boundary conditions at the ground. Copyright ©1996 Elsevier Science Ltd

1. INTRODUCTION

In a recent paper,[1] Godsalve carried out a Fourier analysis for an anisotropically scattering atmosphere illuminated by an incident solar beam. Interestingly, in order to represent better a remote-sensing problem, Godsalve allowed the ground to reflect in an asymmetrical manner, and this ground condition is such that both *sine* and *cosine* components are required in a Fourier decomposition of the radiation intensity. However, in reading the paper of Godsalve, we have discovered that serious errors were made in that work in regard to the coupling, by way of the boundary condition at the ground, between the *sine* and *cosine* components of the solution. In this work we report what we believe to be a correct Fourier decomposition that can be used to solve this new class of radiative-transfer problems. We note also that Godsalve introduced the discrete-ordinates approximation at the beginning of his analysis, and so a set of well-posed component problems that can be used with other solution techniques is not readily available from his paper. Thus here for a general model of ground reflectance, we develop a complete formulation, for all of the component problems related to a Fourier decomposition of the radiation intensity, that is independent of the solution technique to be used.

We let $I(\tau, \mu, \varphi)$ denote the intensity (radiance) of the radiation field and utilize the equation of transfer[2] for a plane-parallel medium to model our atmosphere. We write

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu, \varphi) + I(\tau, \mu, \varphi) = \frac{\varpi}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos \Theta) I(\tau, \mu', \varphi') d\varphi' d\mu' \qquad (1)$$

where $\tau \in (0, \tau_0)$ is the optical variable and ϖ is the albedo for single scattering. In addition, $\mu \in [-1, 1]$ and $\varphi \in [0, 2\pi]$ are, respectively, the cosine of the polar angle (as measured from the *positive* τ axis) and the azimuthal angle which describe the direction of propagation of the radiation. We note also that the phase function $p(\cos \Theta)$ is represented by a finite Legendre expansion in terms of the scattering angle Θ, viz.

$$p(\cos \Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos \Theta) \qquad (2)$$

where the coefficients are such that $\beta_0 = 1$ and $|\beta_l| < 2l + 1$.

We assume that the atmosphere is illuminated uniformly by a solar beam with a direction specified by (μ_0, φ_0), and so we seek a solution to Eq. (1) that satisfies the boundary conditions

$$I(0, \mu, \varphi) = \pi\delta(\mu - \mu_0)\delta(\varphi - \varphi_0) \tag{3a}$$

and

$$I(\tau_0, -\mu, \varphi) = \int_0^{2\pi} \int_0^1 R(\mu', \mu, \varphi', \varphi)I(\tau_0, \mu', \varphi')\mu'd\mu'd\varphi' \tag{3b}$$

for $\mu \in (0, 1]$ and $\varphi \in [0, 2\pi]$.

As we wish to include the possibility that there could be some phenomenon (e.g., rows of plants, or ocean waves) related to the ground that introduces a special direction into the problem, we make no assumptions here regarding the symmetry of the reflection function $R(\mu', \mu, \varphi', \varphi)$. We note that, in writing the boundary condition at the ground as we have in Eq. (3b), we are considering that the reflection function describes the reflection of radiation from some direction defined by the variables (μ', φ'), with $\mu' \in (0, 1]$ and $\varphi' \in [0, 2\pi]$, to another direction defined by the variables $(-\mu, \varphi)$, with $\mu \in (0, 1]$ and $\varphi \in [0, 2\pi]$.

2. THE UNSCATTERED COMPONENT OF THE INTENSITY

Since the incident beam for the considered problem is represented by delta *functions*, the resulting intensity will also have a component containing generalized functions, and so, in order to avoid the impossible task of trying to compute generalized functions, we use first a decomposition of the form

$$I(\tau, \mu, \varphi) = I_0(\tau, \mu, \varphi) + I_*(\tau, \mu, \varphi) \tag{4}$$

where $I_0(\tau, \mu, \varphi)$ denotes the solution of Eqs. (1) and (3) for the case $\varpi = 0$ and where $I_*(\tau, \mu, \varphi)$ is the complementary component of the solution. We note that even if the reflection function at the ground has a component for specular reflection, the decomposition given by Eq. (4) is such that all of the generalized functions in the solution $I(\tau, \mu, \varphi)$ will be contained in $I_0(\tau, \mu, \varphi)$.

Solving Eqs. (1) and (3) for the case $\varpi = 0$, we find

$$I_0(\tau, \mu, \varphi) = \pi\delta(\mu - \mu_0)\delta(\varphi - \varphi_0)e^{-\tau/\mu} \tag{5a}$$

and

$$I_0(\tau, -\mu, \varphi) = \pi\mu_0 R(\mu_0, \mu, \varphi_0, \varphi)e^{-\tau_0/\mu_0}e^{-(\tau_0-\tau)/\mu} \tag{5b}$$

for $\mu \in (0, 1]$ and $\varphi \in [0, 2\pi]$.

3. THE COMPLEMENTARY COMPONENT OF THE INTENSITY

We can now substitute Eq. (4) into Eqs. (1) and (3) and deduce, after noting Eqs. (5), that the complementary component $I_*(\tau, \mu, \varphi)$ is defined by

$$\mu\frac{\partial}{\partial\tau}I_*(\tau, \mu, \varphi) + I_*(\tau, \mu, \varphi) = \frac{\varpi}{4\pi}\int_{-1}^1 \int_0^{2\pi} p(\cos\Theta)I_*(\tau, \mu', \varphi')d\varphi'd\mu' + F(\tau, \mu, \varphi), \tag{6}$$

for $\tau \in (0, \tau_0)$, $\mu \in [-1, 1]$ and $\varphi \in [0, 2\pi]$, and the boundary conditions

$$I_*(0, \mu, \varphi) = 0 \tag{7a}$$

and

$$I_*(\tau_0, -\mu, \varphi) = \int_0^{2\pi} \int_0^1 R(\mu', \mu, \varphi', \varphi)I_*(\tau_0, \mu', \varphi')\mu'd\mu'd\varphi' \tag{7b}$$

for $\mu \in (0, 1]$ and $\varphi \in [0, 2\pi]$. Here, the known inhomogeneous term is

$$F(\tau, \mu, \varphi) = \frac{\varpi}{4\pi}\int_{-1}^1 \int_0^{2\pi} p(\cos\Theta)I_0(\tau, \mu', \varphi')d\varphi'd\mu'. \tag{8}$$

Before attempting a Fourier decomposition of the complementary component of the intensity, we make use of the addition theorem[3] for the Legendre polynomials and express the scattering law as

$$p(\cos \Theta) = \sum_{m=0}^{L} (2 - \delta_{0,m}) \sum_{l=m}^{L} \beta_l^m P_l^m(\mu') P_l^m(\mu) \cos[m(\varphi' - \varphi)]$$ (9)

where

$$P_l^m(\mu) = (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu)$$ (10)

denotes an associated Legendre function, and where

$$\beta_l^m = \frac{(l - m)!}{(l + m)!} \beta_l.$$ (11)

Using Eqs. (5) and (9), we rewrite Eq. (6) as

$$\mu \frac{\partial}{\partial \tau} I_*(\tau, \mu, \varphi) + I_*(\tau, \mu, \varphi) = \frac{\varpi}{4\pi} \sum_{m=0}^{L} (2 - \delta_{0,m}) \sum_{l=m}^{L} \beta_l^m P_l^m(\mu)$$

$$\times \int_{-1}^{1} \int_{0}^{2\pi} P_l^m(\mu') I_*(\tau, \mu', \varphi') \cos[m(\varphi' - \varphi)] d\varphi' d\mu' + F(\tau, \mu, \varphi)$$ (12)

where now we write the inhomogeneous term as

$$F(\tau, \mu, \varphi) = F_1(\tau, \mu, \varphi) + F_2(\tau, \mu, \varphi)$$ (13)

with

$$F_1(\tau, \mu, \varphi) = \frac{\varpi}{4} \sum_{m=0}^{L} (2 - \delta_{0,m}) \sum_{l=m}^{L} \beta_l^m P_l^m(\mu_0) P_l^m(\mu) \cos[m(\varphi - \varphi_0)] e^{-\tau/\mu_0}$$ (14a)

and

$$F_2(\tau, \mu, \varphi) = \frac{\varpi}{4} \mu_0 \sum_{m=0}^{L} (2 - \delta_{0,m}) \sum_{l=m}^{L} \beta_l^m P_l^m(\mu)(-1)^{l-m} e^{-\tau_0/\mu_0}$$

$$\times \int_{0}^{1} \int_{0}^{2\pi} P_l^m(\mu') R(\mu_0, \mu', \varphi_0, \varphi') \cos[m(\varphi' - \varphi)] e^{-(\tau_0-\tau)/\mu'} d\varphi' d\mu'.$$ (14b)

In order to simplify Eq. (14b), we first let φ_r denote a reference direction, and then we expand the reflection function in a Fourier series of the form

$$R(\mu', \mu, \varphi', \varphi) = \frac{1}{2} \sum_{n=0}^{\infty} (2 - \delta_{0,n}) R_n^c(\mu', \mu, \varphi') \cos[n(\varphi - \varphi_r)]$$

$$+ \sum_{n=1}^{\infty} R_n^s(\mu', \mu, \varphi') \sin[n(\varphi - \varphi_r)],$$ (15)

where, of course,

$$R_n^c(\mu', \mu, \varphi') = \frac{1}{\pi} \int_{0}^{2\pi} R(\mu', \mu, \varphi', \varphi) \cos[n(\varphi - \varphi_r)] d\varphi, \quad n \geq 0,$$ (16a)

and

$$R_n^s(\mu', \mu, \varphi') = \frac{1}{\pi} \int_{0}^{2\pi} R(\mu', \mu, \varphi', \varphi) \sin[n(\varphi - \varphi_r)] d\varphi, \quad n \geq 1.$$ (16b)

Since later in this work we will use a double Fourier series representation of the reflection function, we also write here

$$R_n^c(\mu', \mu, \varphi') = \frac{1}{2} \sum_{m=0}^{\infty} (2 - \delta_{0,m}) R_{n,m}^{cc}(\mu', \mu) \cos[m(\varphi' - \varphi_r)]$$

$$+ \sum_{m=1}^{\infty} R_{n,m}^{cs}(\mu', \mu) \sin[m(\varphi' - \varphi_r)]$$ (17a)

and

$$R_n^s(\mu', \mu, \varphi') = \frac{1}{2} \sum_{m=0}^{\infty} (2 - \delta_{0,m}) R_{n,m}^{sc}(\mu', \mu) \cos[m(\varphi' - \varphi_r)]$$

$$+ \sum_{m=1}^{\infty} R_{n,m}^{ss}(\mu', \mu) \sin[m(\varphi' - \varphi_r)], \quad (17b)$$

with

$$R_{n,m}^{cc}(\mu', \mu) = \frac{1}{\pi} \int_0^{2\pi} R_n^c(\mu', \mu, \varphi') \cos[m(\varphi' - \varphi_r)] d\varphi', \quad n \geq 0, \ m \geq 0, \quad (18a)$$

$$R_{n,m}^{cs}(\mu', \mu) = \frac{1}{\pi} \int_0^{2\pi} R_n^c(\mu', \mu, \varphi') \sin[m(\varphi' - \varphi_r)] d\varphi', \quad n \geq 0, \ m \geq 1, \quad (18b)$$

$$R_{n,m}^{sc}(\mu', \mu) = \frac{1}{\pi} \int_0^{2\pi} R_n^s(\mu', \mu, \varphi') \cos[m(\varphi' - \varphi_r)] d\varphi', \quad n \geq 1, \ m \geq 0, \quad (18c)$$

and

$$R_{n,m}^{ss}(\mu', \mu) = \frac{1}{\pi} \int_0^{2\pi} R_n^s(\mu', \mu, \varphi') \sin[m(\varphi' - \varphi_r)] d\varphi', \quad n \geq 1, \ m \geq 1. \quad (18d)$$

Substituting Eq. (15) into Eq. (14b), we find

$$F_2(\tau, \mu, \varphi) = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) F_{2,m}^c(\tau, \mu) \cos[m(\varphi - \varphi_r)] + \sum_{m=1}^{L} F_{2,m}^s(\tau, \mu) \sin[m(\varphi - \varphi_r)] \quad (19)$$

where

$$F_{2,m}^c(\tau, \mu) = \frac{\varpi\pi}{2} \mu_0 \sum_{l=m}^{L} \beta_l^m P_l^m(\mu)(-1)^{l-m} e^{-\tau_0/\mu_0} \int_0^1 P_l^m(\mu') R_m^c(\mu_0, \mu', \varphi_0) e^{-(\tau_0-\tau)/\mu'} d\mu' \quad (20a)$$

and

$$F_{2,m}^s(\tau, \mu) = \frac{\varpi\pi}{2} \mu_0 \sum_{l=m}^{L} \beta_l^m P_l^m(\mu)(-1)^{l-m} e^{-\tau_0/\mu_0} \int_0^1 P_l^m(\mu') R_m^s(\mu_0, \mu', \varphi_0) e^{-(\tau_0-\tau)/\mu'} d\mu'. \quad (20b)$$

By defining

$$F_{1,m}^c(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l^m P_l^m(\mu_0) P_l^m(\mu) \cos[m(\varphi_0 - \varphi_r)] e^{-\tau/\mu_0} \quad (21a)$$

and

$$F_{1,m}^s(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l^m P_l^m(\mu_0) P_l^m(\mu) \sin[m(\varphi_0 - \varphi_r)] e^{-\tau/\mu_0}, \quad (21b)$$

we can rewrite Eq. (14a) as

$$F_1(\tau, \mu, \varphi) = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) F_{1,m}^c(\tau, \mu) \cos[m(\varphi - \varphi_r)] + \sum_{m=1}^{L} F_{1,m}^s(\tau, \mu) \sin[m(\varphi - \varphi_r)]. \quad (22)$$

Now, with

$$F_m^c(\tau, \mu) = F_{1,m}^c(\tau, \mu) + F_{2,m}^c(\tau, \mu) \quad (23a)$$

and

$$F_m^s(\tau, \mu) = F_{1,m}^s(\tau, \mu) + F_{2,m}^s(\tau, \mu), \quad (23b)$$

we can write the inhomogeneous term in Eq. (12) as

$$F(\tau, \mu, \varphi) = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) F_m^c(\tau, \mu) \cos[m(\varphi - \varphi_r)] + \sum_{m=1}^{L} F_m^s(\tau, \mu) \sin[m(\varphi - \varphi_r)]. \quad (24)$$

Considering the use of Fourier series again, we substitute

$$I_*(\tau, \mu, \varphi) = \frac{1}{2} \sum_{m=0}^{\infty} (2 - \delta_{0,m}) I_{*,m}^c(\tau, \mu) \cos[m(\varphi - \varphi_r)] + \sum_{m=1}^{\infty} I_{*,m}^s(\tau, \mu) \sin[m(\varphi - \varphi_r)] \quad (25)$$

into Eq. (12) and note Eq. (24) to find, for $m = 0, 1, \ldots L$,

$$\mu \frac{\partial}{\partial \tau} I_{*,m}^c(\tau, \mu) + I_{*,m}^c(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l^m P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu') I_{*,m}^c(\tau, \mu') d\mu' + F_m^c(\tau, \mu) \quad (26a)$$

and, for $m = 1, 2, \ldots L$,

$$\mu \frac{\partial}{\partial \tau} I_{*,m}^s(\tau, \mu) + I_{*,m}^s(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l^m P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu') I_{*,m}^s(\tau, \mu') d\mu' + F_m^s(\tau, \mu). \quad (26b)$$

In addition, we find, for $m = L + 1, L + 2, \ldots$,

$$\mu \frac{\partial}{\partial \tau} I_{*,m}^c(\tau, \mu) + I_{*,m}^c(\tau, \mu) = 0 \quad (27a)$$

and

$$\mu \frac{\partial}{\partial \tau} I_{*,m}^s(\tau, \mu) + I_{*,m}^s(\tau, \mu) = 0. \quad (27b)$$

Finally we substitute Eqs. (15), (17) and (25) into Eqs. (7) and deduce the required boundary conditions on the Fourier components of $I_*(\tau, \mu, \varphi)$. We find from Eq. (7a)

$$I_{*,m}^c(0, \mu) = 0, \quad m = 0, 1, 2, \ldots, \quad (28a)$$

and

$$I_{*,m}^s(0, \mu) = 0, \quad m = 1, 2, 3, \ldots, \quad (28b)$$

for $\mu \in (0, 1]$. From Eq. (7b) we find, for $m = 0, 1, 2, \ldots$ and $\mu \in (0, 1]$,

$$I_{*,m}^c(\tau_0, -\mu) = \frac{\pi}{2} \sum_{n=0}^{\infty} (2 - \delta_{0,n}) \int_0^1 R_{m,n}^{cc}(\mu', \mu) I_{*,n}^c(\tau_0, \mu') \mu' d\mu'$$

$$+ \pi \sum_{n=1}^{\infty} \int_0^1 R_{m,n}^{cs}(\mu', \mu) I_{*,n}^s(\tau_0, \mu') \mu' d\mu' \quad (29a)$$

and, for $m = 1, 2, 3, \ldots$ and $\mu \in (0, 1]$,

$$I_{*,m}^s(\tau_0, -\mu) = \frac{\pi}{2} \sum_{n=0}^{\infty} (2 - \delta_{0,n}) \int_0^1 R_{m,n}^{sc}(\mu', \mu) I_{*,n}^c(\tau_0, \mu') \mu' d\mu'$$

$$+ \pi \sum_{n=1}^{\infty} \int_0^1 R_{m,n}^{ss}(\mu', \mu) I_{*,n}^s(\tau_0, \mu') \mu' d\mu'. \quad (29b)$$

Considering now the component problems for $m > L$, we can solve Eqs. (27) subject to the conditions of Eqs. (28) to find

$$I_{*,m}^c(\tau, \mu) = 0 \quad (30a)$$

and

$$I_{*,m}^s(\tau, \mu) = 0 \quad (30b)$$

for $\mu \in (0, 1]$. We also find

$$I_{*,m}^c(\tau, -\mu) = I_{*,m}^c(\tau_0, -\mu) e^{-(\tau_0 - \tau)/\mu} \quad (31a)$$

and

$$I_{*,m}^s(\tau, -\mu) = I_{*,m}^s(\tau_0, -\mu) e^{-(\tau_0 - \tau)/\mu} \quad (31b)$$

for $\mu \in (0, 1]$. Here $I_{*,m}^c(\tau_0, -\mu)$ and $I_{*,m}^s(\tau_0, -\mu)$ are to be determined.

Noting Eqs. (30), we see that the summations in Eqs. (29) truncate, so we write the boundary conditions at $\tau = \tau_0$ as

$$I_{*,m}^c(\tau_0, -\mu) = \frac{\pi}{2} \sum_{n=0}^{L} (2 - \delta_{0,n}) \int_0^1 R_{m,n}^{cc}(\mu', \mu) I_{*,n}^c(\tau_0, \mu') \mu' d\mu'$$

$$+ \pi \sum_{n=1}^{L} \int_0^1 R_{m,n}^{cs}(\mu', \mu) I_{*,n}^s(\tau_0, \mu') \mu' d\mu', \quad (32a)$$

for $m \geq 0$ and $\mu \in (0, 1]$, and

$$I_{*,m}^s(\tau_0, -\mu) = \frac{\pi}{2} \sum_{n=0}^{L} (2 - \delta_{0,n}) \int_0^1 R_{m,n}^{sc}(\mu', \mu) I_{*,n}^c(\tau_0, \mu') \mu' d\mu'$$

$$+ \pi \sum_{n=1}^{L} \int_0^1 R_{m,n}^{ss}(\mu', \mu) I_{*,n}^s(\tau_0, \mu') \mu' d\mu', \quad (32b)$$

for $m \geq 1$ and $\mu \in (0, 1]$. It follows that once we have solved our basic problems for $m = 0, 1, 2, \ldots L$, we can compute the functions required to complete the solutions given by Eqs. (30) and (31) from Eqs. (32) with $m > L$.

4. SPECIAL CASES

At this point we would like to show how the foregoing development reduces to simpler forms for three special cases. We consider first a mixture of specular and diffuse reflection. We thus write

$$R(\mu', \mu, \varphi', \varphi) = (\rho_s/\mu)\delta(\mu' - \mu)\delta(\varphi' - \varphi) + \rho_d/\pi \quad (33)$$

where ρ_s and ρ_d are, respectively, the coefficients for specular and diffuse reflection. In this (classic) example of reflection, there clearly is no preferred direction on the ground, and so we are free to choose $\varphi_r = \varphi_0$. Looking back to Eq. (5b), we find we can now write

$$I_0(\tau, -\mu, \varphi) = [\pi\rho_s\delta(\mu - \mu_0)\delta(\varphi - \varphi_0) + \mu_0\rho_d]e^{-\tau_0/\mu_0}e^{-(\tau_0-\tau)/\mu} \quad (34)$$

for $\mu \in (0, 1]$. We next observe that for this special case all *sine* components of $I_*(\tau, \mu, \varphi)$ are zero. In addition, Eq. (21a) reduces to

$$F_{1,m}^c(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l^m P_l^m(\mu_0) P_l^m(\mu) e^{-\tau/\mu_0} \quad (35a)$$

and Eq. (20a) becomes

$$F_{2,m}^c(\tau, \mu) = \frac{\varpi}{2}\rho_s \sum_{l=m}^{L} \beta_l^m P_l^m(\mu_0) P_l^m(\mu)(-1)^{l-m} e^{-(2\tau_0-\tau)/\mu_0}$$

$$+ \varpi\mu_0\rho_d\delta_{0,m} \sum_{l=0}^{L} \beta_l P_l(\mu)(-1)^l e^{-\tau_0/\mu_0} \int_0^1 P_l(\mu') e^{-(\tau_0-\tau)/\mu'} d\mu'. \quad (35b)$$

It follows that, with Eqs. (35) and

$$F_m^c(\tau, \mu) = F_{1,m}^c(\tau, \mu) + F_{2,m}^c(\tau, \mu), \quad (36)$$

the inhomogeneous source term in Eq. (26a) is explicitly available. We can now use Eq. (33) in Eqs. (16a) and (18a) to find

$$R_{m,n}^{cc}(\mu', \mu) = \frac{1}{\pi}(1 + \delta_{0,n})\delta_{m,n}[(\rho_s/\mu)\delta(\mu' - \mu) + 2\rho_d\delta_{0,n}], \quad m \geq 0, \ n \geq 0, \quad (37)$$

so that we can conclude, from Eqs. (28a) and (32a), that the boundary conditions subject to which we must solve Eq. (26a), for $m = 0, 1, 2, \ldots L$, are

$$I_{*,m}^c(0, \mu) = 0 \quad (38a)$$

and

$$I^c_{*,m}(\tau_0, -\mu) = \rho_s I^c_{*,m}(\tau_0, \mu) + 2\rho_d \delta_{0,m} \int_0^1 I^c_{*,0}(\tau_0, \mu')\mu' d\mu' \qquad (38b)$$

for $\mu \in (0, 1]$. For $m > L$, it is clear from Eqs. (30a), (31a), (32a) and (37) that $I^c_{*,m}(\tau, \mu) = 0$.

For a second special case, we consider a reflection function of the form

$$R(\mu', \mu, \varphi', \varphi) = \sum_{k=0}^K b_k P_k(\cos \Xi) \qquad (39)$$

where Ξ is the angle between the vectors that define the incident and emergent directions of the radiation. In addition, the $\{b_k\}$ are coefficients that are assumed known. As we did with the phase function, we can use the addition theorem[3] for the Legendre polynomials to rewrite Eq. (39) as

$$R(\mu', \mu, \varphi', \varphi) = \sum_{j=0}^K (2 - \delta_{0,j}) \sum_{k=j}^K (-1)^{k-j} b_k^j P_k^j(\mu') P_k^j(\mu) \cos[j(\varphi' - \varphi)] \qquad (40)$$

where

$$b_k^j = \frac{(k-j)!}{(k+j)!} b_k. \qquad (41)$$

We note that if we choose $\varphi_r = \varphi_0$ then the complementary part of the intensity $I_*(\tau, \mu, \varphi)$ will not have any *sine* components. Here again Eq. (21a) reduces to

$$F^c_{1,m}(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^L \beta_l^m P_l^m(\mu_0) P_l^m(\mu) e^{-\tau/\mu_0} \qquad (42a)$$

and Eq. (20a) becomes

$$F^c_{2,m}(\tau, \mu) = \varpi \pi \mu_0 \sum_{l=m}^L \beta_l^m P_l^m(\mu) (-1)^{l-m} e^{-\tau_0/\mu_0}$$

$$\times \sum_{k=m}^K (-1)^{k-m} b_k^m P_k^m(\mu_0) \int_0^1 P_l^m(\mu') P_k^m(\mu') e^{-(\tau_0 - \tau)/\mu'} d\mu'. \qquad (42b)$$

We find that, with Eqs. (42) and

$$F^c_m(\tau, \mu) = F^c_{1,m}(\tau, \mu) + F^c_{2,m}(\tau, \mu), \qquad (43)$$

the inhomogeneous source term in Eq. (26a) is again explicitly available. Upon using Eq. (40) in Eqs. (16a) and (18a), we find

$$R^{cc}_{m,n}(\mu', \mu) = 2(1 + \delta_{0,n}) \delta_{m,n} \sum_{k=n}^K (-1)^{k-n} b_k^n P_k^n(\mu') P_k^n(\mu), \quad m \geq 0, \ n \geq 0. \qquad (44)$$

It now follows, from Eqs. (28a) and (32a), that the boundary conditions subject to which we must solve Eq. (26a), for $m = 0, 1, 2, \ldots L$, are

$$I^c_{*,m}(0, \mu) = 0 \qquad (45a)$$

and

$$I^c_{*,m}(\tau_0, -\mu) = \int_0^1 T_m(\mu', \mu) I^c_{*,m}(\tau_0, \mu') \mu' d\mu' \qquad (45b)$$

for $\mu \in (0, 1]$. Here

$$T_m(\mu', \mu) = 2\pi \sum_{k=m}^K (-1)^{k-m} b_k^m P_k^m(\mu') P_k^m(\mu). \qquad (46)$$

To complete the formulation for this special case, we note that $I^c_{*,m}(\tau, \mu) = 0$ for $m > L$.

Our third (and last) special case is defined by the reflection function

$$R(\mu', \mu, \varphi', \varphi) = \frac{1}{2} \sum_{k=0}^{K_c} (2 - \delta_{0,k}) a_k(\mu', \mu) \cos[k(\varphi' - \varphi)] + \sum_{k=1}^{K_s} b_k(\mu', \mu) \sin[k(\varphi' - \varphi)] \quad (47)$$

where the coefficients $\{a_k(\mu', \mu)\}$ and $\{b_k(\mu', \mu)\}$ are assumed known. We note that Eq. (47) covers the cases studied by Godsalve[1] and by Walthall et al.[4] as well as the scalar problem obtained by neglecting polarization effects in the formulation of Deuzé et al.[5]

Here again the reflection function does not involve a preferred direction so we can set $\varphi_r = \varphi_0$. With this choice of φ_r, Eqs. (21) become

$$F_{1,m}^c(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l^m P_l^m(\mu_0) P_l^m(\mu) e^{-\tau/\mu_0} \quad (48a)$$

and

$$F_{1,m}^s(\tau, \mu) = 0. \quad (48b)$$

In addition, after noting that Eqs. (16) yield, for this special case,

$$R_0^c(\mu', \mu, \varphi') = a_0(\mu', \mu), \quad (49a)$$

$$R_n^c(\mu', \mu, \varphi') = a_n(\mu', \mu) \cos[n(\varphi' - \varphi_0)] + b_n(\mu', \mu) \sin[n(\varphi' - \varphi_0)] \quad (49b)$$

and

$$R_n^s(\mu', \mu, \varphi') = a_n(\mu', \mu) \sin[n(\varphi' - \varphi_0)] - b_n(\mu', \mu) \cos[n(\varphi' - \varphi_0)], \quad (49c)$$

for $n \geq 1$, we write Eqs. (20) as

$$F_{2,m}^c(\tau, \mu) = \frac{\varpi\pi}{2} \mu_0 \sum_{l=m}^{L} \beta_l^m P_l^m(\mu)(-1)^{l-m} e^{-\tau_0/\mu_0} \int_0^1 P_l^m(\mu') a_m(\mu_0, \mu') e^{-(\tau_0-\tau)/\mu'} d\mu', \quad (50a)$$

for $m \geq 0$, and

$$F_{2,m}^s(\tau, \mu) = -\frac{\varpi\pi}{2} \mu_0 \sum_{l=m}^{L} \beta_l^m P_l^m(\mu)(-1)^{l-m} e^{-\tau_0/\mu_0} \int_0^1 P_l^m(\mu') b_m(\mu_0, \mu') e^{-(\tau_0-\tau)/\mu'} d\mu', \quad (50b)$$

for $m \geq 1$. From Eqs. (50), it is clear that, for this case, we must solve both the *sine* and *cosine* component problems defined by Eqs. (26), with the corresponding inhomogeneous source terms given explicitly by Eqs. (23), (48) and (50). In regard to the boundary conditions at the ground, we can use the expressions

$$R_{m,n}^{cc}(\mu', \mu) = (1 + \delta_{0,n}) \delta_{m,n} a_n(\mu', \mu), \quad m \geq 0, \ n \geq 0, \quad (51a)$$

$$R_{m,n}^{cs}(\mu', \mu) = \delta_{m,n} b_n(\mu', \mu), \quad m \geq 0, \ n \geq 1 \quad (51b)$$

$$R_{m,0}^{sc}(\mu', \mu) = 0, \quad m \geq 1, \quad (51c)$$

$$R_{m,n}^{sc}(\mu', \mu) = -\delta_{m,n} b_n(\mu', \mu), \quad m \geq 1, \ n \geq 1, \quad (51d)$$

and

$$R_{m,n}^{ss}(\mu', \mu) = \delta_{m,n} a_n(\mu', \mu), \quad m \geq 1, \ n \geq 1, \quad (51e)$$

obtained by substituting Eqs. (49) into Eqs. (18), to find that Eqs. (32) reduce to

$$I_{*,0}^c(\tau_0, -\mu) = \pi \int_0^1 a_0(\mu', \mu) I_{*,0}^c(\tau_0, \mu') \mu' d\mu', \quad (52a)$$

for $\mu \in (0, 1]$,

$$I_{*,m}^c(\tau_0, -\mu) = \pi \int_0^1 [a_m(\mu', \mu) I_{*,m}^c(\tau_0, \mu') + b_m(\mu', \mu) I_{*,m}^s(\tau_0, \mu')] \mu' d\mu' \quad (52b)$$

and

$$I^s_{*,m}(\tau_0, -\mu) = \pi \int_0^1 [a_m(\mu', \mu)I^s_{*,m}(\tau_0, \mu') - b_m(\mu', \mu)I^c_{*,m}(\tau_0, \mu')]\mu'd\mu', \qquad (52c)$$

for $m \geq 1$ and $\mu \in (0, 1]$. Finally, we complete the formulation for this case by noting that $I^c_{*,m}(\tau, \mu) = 0$ for $m > L$ and that $I^s_{*,m}(\tau, \mu) = 0$ for $m > \min\{L, K_s\}$.

5. CONCLUDING REMARKS

Having considered a fairly general reflection function, we find that the Fourier decomposition is considerably more complicated than what we are accustomed to seeing in radiative transfer calculations in plane geometry. In particular, we see that for the general case an infinite number of *sine* and *cosine* components of $I_*(\tau, \mu, \varphi)$ are required to establish the solution. We note also that the first L *sine* and the first $L + 1$ *cosine* components of $I_*(\tau, \mu, \varphi)$ are coupled by the boundary condition at the ground and that the solution for the remaining *sine* and *cosine* components of $I_*(\tau, \mu, \varphi)$ can be expressed in terms of the solutions for those $2L + 1$ components. Needless to say, some simplifying assumptions about the reflection function, as, for example, the ones we have made in Sec. 4, could make the formulation of the desired solution for the intensity a great deal more tractable.

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