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# An Exact Solution of Equations of Radiative Transfer for Local Thermodynamic Equilibrium in the Non-Gray Case. Picket Fence Approximation*

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Case's method of singular normal modes is used to obtain an exact solution to the equations of radiative transfer in the "uniform" or "random" picket fence model assuming Local Thermodynamic Equilibrium. Completeness and orthogonality theorems are proved, and explicit results are obtained for the extrapolated endpoint, the temperature distribution, and law of darkening for the Milne problem. In addition, the method for solving other half-space problems is sketched. The case when the determinant of the "transfer matrix" does not vanish (corresponding to radiative transfer *without* Complete Local Thermodynamic Equilibrium or the neutron transport problem) is also discussed. Full range completeness and orthogonality theorems are presented for this case; however, no convenient form for the half-range case has been found.

## I. INTRODUCTION

In this paper we present a solution of the equation of radiative transfer in the picket fence model (*1*). In this model, the absorption coefficient for the radiation is assumed to be representable as a set of two different constant values over the frequency spectrum.[1] This represents to some approximation absorption by resonance lines. In Section II we show how in this approximation, under the assumption of Local Thermodynamic Equilibrium (*2*), the equation of radiative transfer reduces to a set of two coupled transport equations similar to the two-group neutron transport equations considered by Żelazny and Kuszell (*3*).[2] The radia-

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[1] The generalization to $N$ different values of the absorption coefficient is discussed in a forthcoming paper.

[2] These authors used a highly inconvenient form of the (degenerate) eigensolutions. A

tive case, however, is simpler because the "transfer matrix" takes a particularly simple (degenerate) form and has a vanishing determinant. Our method of solution is given in some detail, because it will serve as a guide to the more general case mentioned in footnote 1.

The method we use is similar to that of ref. *3*, i.e., Case's method of singular eigensolutions (*4–7*).

A number of solutions to the picket fence model exist. Chandrasekhar, in ref. *1*, solves the equations using the Eddington approximation. Actually, his model is somewhat more general than ours in that he also considers scattering. Ours, in fact, corresponds to his case $\epsilon = 1$. The case $\epsilon \neq 1$ involves a transfer matrix whose determinant does not vanish; this case is discussed in Section VII. Various other approximate and numerical solutions have been developed and have been summarized by Gingerich (*9*) who considers numerical solutions. We will not attempt to review any of this work, except to note that numerical solutions are extremely difficult. Stewart (*10*) has obtained an exact solution; he used a Wick-Chandrasekhar discrete ordinates procedure (*11*) and took the limit as the number of ordinates approached infinity. However, we believe our method to be simpler and, in addition, posesses the merit that it can be readily generalized in the manner already mentioned.

There have also been a number of recent papers which consider the solution of the radiative transfer equations for various models in the nongray case, but these are not, strictly speaking, picket fence models. See, for example, refs. *12–16*

As we have stated, Section II is devoted to a brief derivation of the picket fence equations. Thus, in Section III we begin the solution by introducing normal modes. In Section IV a completeness proof is sketched and orthogonality relations are deduced in Section V. In Section VI explicit solutions for the extrapolated endpoint, the temperature distribution, and the law of darkening are developed for the Milne problem (*17*). Actually, we show how to solve any of the standard half-range problems, e.g., the Green's function and the albedo problem (the slab albedo problem can be solved by extending the method of McCormick and Mendelson (*18*)), but the Milne problem is of primary interest here.[3] In Section VII we make a few remarks about the problem in which the transfer matrix is completely arbitrary (this case is of considerable interest in neutron physics and also corresponds to Chandrasekhar's $\epsilon \neq 1$ (*1*)), but we have been able to obtain explicit solutions only for full-space problems. More work is needed on the half-space solutions.

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similar inconvenient form has been used by Ferziger and Leonard (*8*) who consider a related problem.

[3] The solution of the half-space Green's function requires a special technique. This is discussed in Appendix B.

Our notation is that of neutron physics because Case's paper (*4*), which we draw so heavily upon, uses this notation.[4]

Although the physical model considered here is idealized, the exact solutions obtained can be used to test numerical methods which may then be applied to more realistic physical situations.

## II. THE EQUATIONS IN THE PICKET FENCE MODEL[5]

The equation of radiative transfer under the assumption of local thermodynamic equilibrium can be written in the form

$$\mu \frac{\partial \psi_\nu}{\partial z} (z, \mu) + \rho(z) K_\nu \psi_\nu(z, \mu) = \rho(z) K_\nu B_\nu(T(z)). \tag{1}$$

This equation can be deduced from ref. *11* with the aid of footnote 4. Briefly, $2\pi\psi_\nu(z, \mu)\, dz\, d\nu\, d\mu$ represents the radiant energy contained in position $dz$ at $z$, in solid angle $d\mathbf{\Omega} = 2\pi\, d\mu$ at $\mathbf{\Omega}$ with frequency between $\nu$ and $\nu + d\nu$. It is the analogue of the angular density in neutron transport. $K_\nu$ is the absorption coefficient for radiation of frequency $\nu$, $\rho(z)$ is density of the medium, and $T(z)$ is the local temperature. In this equation $B_\nu(T(z))$ is the Planck black body function:

$$B_\nu(T(z)) = \frac{2h\nu^3}{c^2} \left( \exp \frac{h\nu}{kT(z)} - 1 \right)^{-1}. \tag{2}$$

To solve Eq. (1), a subsidiary condition is needed. This is the Schwarzschild condition, which states local energy conservation, thus

$$\int_0^\infty d\nu K_\nu B_\nu(T(z)) = \frac{1}{2} \int_0^\infty d\nu K_\nu \int_{-1}^1 \psi_\nu(z, \mu)\, d\mu. \tag{3}$$

Equations (1) and (3) are two equations for the two unknowns $\psi_\nu(z, \mu)$ and $T(z)$. Obviously they are nonlinear. However, if the absorption coefficient takes a certain form the problem becomes much simpler. The so-called gray case refers to $K_\nu$ = constant. For this situation Eq. (1) reduces to the form

$$\mu \frac{\partial \psi}{\partial x} (x, \mu) + \psi(x, \mu) = \frac{1}{2} \int_{-1}^1 \psi(x, \mu')\, d\mu', \tag{4}$$

where $x$ is the optical thickness and $\psi(x, \mu)$ is the integrated energy density, i.e.,

$$\psi(x, \mu) \triangleq \int_0^\infty \psi_\nu(x, \mu)\, d\nu. \tag{5}$$

[4] In order to compare our equations with those used in Astrophysics (e.g. ref *2*) one must change $\mu$ to $-\mu$, $\psi_\nu$ to $I_\nu$.

[5] The derivation of the picket fence equations are given in a number of places, e.g. ref. *1*. A brief sketch is presented here for the sake of completeness.

Equation (4) is precisely the equation considered by Case (*4*), and refers to the gray atmosphere in L.T.E. or to the case of pure isotropic scattering. We say no more about it.

In the picket fence model we have two constant values $K_1$ and $K_2$. Let $\Delta\nu_i$ represent the frequency region over which $K_\nu$ has the value $K_i$, and integrate Eqs. (1) and (3) obtaining

$$\mu \frac{\partial \psi_i}{\partial z}(z, \mu) + \rho(z) K_i \psi_i(z, \mu) = \frac{\rho(z) K_i w_i}{2 \sum_{j=1}^{2} K_j w_j} \sum_{j=1}^{2} K_j \int_{-1}^{1} \psi_j(z, \mu') \, d\mu', \qquad i = 1, 2. \tag{6}$$

Here $\psi_i(z, \mu)$ is defined by

$$\psi_i(z, \mu) \triangleq \int_{\Delta\nu_i} \psi_\nu(z, \mu) \, d\nu, \tag{7}$$

and $w_i$ is given by

$$w_i \triangleq \frac{\pi}{\sigma T^4(z)} \int_{\Delta\nu_i} d\nu B_\nu(T(z)), \tag{8}$$

where $\sigma$ is the Stefan-Boltzman constant. To obtain Eq. (6) the Schwarzschild condition has been used. In the present model it takes the form

$$\sum_{j=1}^{2} K_j w_j \frac{\sigma T^4(z)}{\pi} = \frac{1}{2} \sum_{j=1}^{2} K_j \int_{-1}^{1} \psi_j(z, \mu') \, d\mu'. \tag{9}$$

We note that in general the $w_i$ are functions of $z$. In order to solve the transport equation, Eq. (6), we shall have to assume the $w_i$ are constant. This implies either the *uniform* picket fence model (cf. ref. *9*) or a "random" model (*10*). In the uniform model, it is assumed that the frequency spectrum can be divided into ranges, $\Delta\nu_i$, so small that $B_\nu(T(z))$ may be considered constant in each range. In each range it is assumed that the fractional frequency width covered by each $k_i$ is constant. From these two assumptions, one readily verifies from Eq. (8) that the $w_i$ are independent of $z$. In the random model, one assumes that there are so many lines of random width randomly distributed throughout the spectrum that the distribution may be considered uniform on the average. Chandrasekhar, in ref. *1*, implicitly assumed one of these two models.

Equation (6) can be put in more convenient notation. Let $K_2$ be the smaller of the $K_i$. Then define an optical variable $x$ in terms of $K_2$, i.e.

$$x \triangleq K_2 \int^{z} \rho(z') \, dz'. \tag{10}$$

Further denote $K_i/K_2$ by $\sigma_i(\sigma_2 \equiv 1)$. Equation (6) can thus be written in form

$$\mu \frac{\partial}{\partial x} \mathbf{\Psi}(x, \mu) + \mathbf{\Sigma}\mathbf{\Psi}(x, \mu) = \mathbf{C} \int_{-1}^{1} \mathbf{\Psi}(x, \mu') \, d\mu'. \tag{11}$$

Here $\mathbf{\Psi}(x, \mu)$ is a 2-component vector with elements $\psi_i(x, \mu)$ while $\mathbf{\Sigma}$ and $\mathbf{C}$ are the following matrices:

$$(\mathbf{\Sigma})_{ij} = \sigma_i \, \delta_{ij} \tag{12a}$$

$$(\mathbf{C})_{ij} = \frac{\sigma_i \, w_i \, \sigma_j}{2 \sum_{j=1}^{2} \sigma_j \, w_j} . \tag{12b}$$

$\mathbf{C}$ is the transfer matrix; we note that det $\mathbf{C} = 0$.

We turn now to the solution of Eq. (11).

## III. EIGENVALUES AND EIGENSOLUTIONS

We proceed as Case did (*4*) by noting the translational invariance of Eq. (11). This suggests that the eigensolutions should transform according to the irreducible representations of the translation group. We first try the one-dimensional representation $e^{-x/\eta}$, i.e., we assume a solution to Eq. (11) of the form

$$\mathbf{\Psi}(x, \mu) = e^{-x/\eta}\mathbf{F}(\eta, \mu). \tag{13}$$

When this *ansatz* is substituted into Eq. (11), we obtain an eigenequation for $\mathbf{F}(\eta, \mu)$, where $\eta$ is the eigenvalue:

$$\frac{1}{\eta} \begin{pmatrix} \sigma_1 \eta - \mu & 0 \\ 0 & \eta - \mu \end{pmatrix} \mathbf{F}(\eta, \mu) = \mathbf{C} \int_{-1}^{1} \mathbf{F}(\eta, \mu') \, d\mu'. \tag{14}$$

The eigenvalue spectrum of Eq. (14) must be considered as three separate regions. There are two continuum regions and a discrete spectrum. Consider then the first of the continuous spectra.

*Region 1:* $\eta \in [-1/\sigma_1 , 1/\sigma_1]$.

Here we write

$$\mathbf{F}^{(1)}(\eta, \mu) = \begin{pmatrix} \dfrac{\eta P}{\sigma_1 \eta - \mu} + \lambda_{11}(\eta)\delta(\sigma_1 \eta - \mu) & 0 \\ 0 & \dfrac{\eta P}{\eta - \mu} + \lambda_{12}(\eta)\delta(\eta - \mu) \end{pmatrix} \cdot \mathbf{C} \int_{-1}^{1} \mathbf{F}^{(1)}(\eta, \mu') \, d\mu'. \tag{15}$$

The symbol $P$ indicates that the Cauchy principal value is to be taken when integrals over these function are performed. The functions $\lambda_{1i}(\eta)$ are chosen such that Eq. (15) is satisfied. Since Eq. (15) is homogeneous the normalization is arbitrary. We select

$$\int_{-1}^{1} \mathbf{F}^{(1)}(\eta, \mu')\, d\mu' = \begin{pmatrix} a_1(\eta) \\ a_2(\eta) \end{pmatrix} \tag{16}$$

where, as yet, the $a_i(\eta)$ are unspecified functions of $\eta$. Integrating Eq. (15) we find

$$\begin{pmatrix} a_1(\eta) \\ a_2(\eta) \end{pmatrix} = \begin{pmatrix} 2\eta T(\sigma_1 \eta) + \lambda_{11}(\eta) & 0 \\ 0 & 2\eta T(\eta) + \lambda_{12}(\eta) \end{pmatrix} \mathbf{C} \begin{pmatrix} a_1(\eta) \\ a_2(\eta) \end{pmatrix}, \tag{17}$$

where we have used the abbreviation $T(x)$ for $\tanh^{-1} x$.

Equation (17) is really two equations for the four unknowns, $\lambda_{1i}(\eta)$ and $a_i(\eta)$. Thus we note that the eigensolutions in this region are twofold degenerate. (At this point we deviate from the treatment of ref. *3* in which the eigensolution was left arbitrary. We explicitly take the degeneracy into account.) Clearly the two linearly independent choices for $\mathbf{F}^{(1)}(\eta, \mu)$ correspond to $a_1(\eta) = 1$, $a_2(\eta) = 0$, and $a_1(\eta) = 0$, $a_2(\eta) = 1$. Therefore for region 1 we can write (solving for $\lambda_{1i}(\eta)$ and multiplying out Eq. (15)),

$$\mathbf{F}_1^{(1)}(\eta, \mu) = \begin{pmatrix} \dfrac{c_{11}\, \eta P}{\sigma_1 \eta - \mu} + \delta(\sigma_1 \eta - \mu)(1 - 2\eta c_{11}\, T(\sigma_1 \eta)) \\ \dfrac{c_{21}\, \eta P}{\eta - \mu} + \delta(\eta - \mu)(-2\eta c_{21}\, T(\eta)) \end{pmatrix} \tag{18a}$$

and

$$\mathbf{F}_2^{(1)}(\eta, \mu) = \begin{pmatrix} \dfrac{c_{12}\, \eta P}{\sigma_1 \eta - \mu} + \delta(\sigma_1 \eta - \mu)(-2\eta c_{12}\, T(\sigma_1 \eta)) \\ \dfrac{c_{22}\, \eta P}{\eta - \mu} + \delta(\eta - \mu)(1 - 2\eta c_{22}\, T(\eta)) \end{pmatrix}. \tag{18b}$$

We now consider the second continuum.

*Region 2.* $\eta \in [-1, -1/\sigma_1]$ and $[1/\sigma_1, 1]$

Solving Eq. (14) for region 2, the factor $\sigma_1\eta - \mu$, we note, is not singular. Thus only one coefficient, $\lambda_{22}(\eta)$, enters; and there is no degeneracy.

We easily find

$$\mathbf{F}^{(2)}(\eta, \mu) = \begin{pmatrix} \dfrac{c_{12}\, \eta}{\sigma_1 \eta - \mu} \\ \dfrac{c_{22}\, \eta P}{\eta - \mu} + \delta(\eta - \mu)\left(1 - 2\eta c_{11}\, T\left(\dfrac{1}{\sigma_1 \eta}\right) - 2\eta c_{22}\, T(\eta)\right) \end{pmatrix}. \tag{19}$$

Finally we have the discrete spectrum.

*Region 3.* $\eta \notin [-1, 1]$

Solving Eq. (14) for $\mathbf{F}(\eta, \mu)$ and integrating over $\mu$ we find the dispersion relation,

$$\Omega(z) = 1 - 2zc_{11}T(1/\sigma_1 z) - 2zc_{22}T(1/z). \tag{20}$$

That is, the discrete eigenvalues are the zeros of $\Omega(z)$. It can be shown readily that $\Omega(z)$ has only two zeros, $\eta_0 = \pm\infty$. Thus, we find that the discrete eigensolutions are also twofold degenerate. In this case, however, unlike the degeneracy of region 1, we can no longer use the one-dimensional representation of the translation group, Eq. (13). Since we are dealing with twofold degeneracy, the two dimensional representation is appropriate. It can be generated by the basis set (7)

$$Y_1(x, \eta) = e^{-x/\eta} \rightarrow 1 \tag{21}$$

$$Y_2(x, \eta) = xe^{-x/\eta} \rightarrow x \tag{22}$$

where the arrows indicate the limit for $\eta_0 \rightarrow \infty$. The appropriate linear combinations which satisfy Eq. (11) we find to be

$$\mathbf{\Psi}_+(x, \mu) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \triangleq \mathbf{\Phi}_+ \tag{23}$$

and

$$\mathbf{\Psi}_-(x, \mu) = \begin{pmatrix} w_1(x - \mu/\sigma_1) \\ w_2(x - \mu) \end{pmatrix}. \tag{24}$$

Instead of the continuum eigenfunctions, $\mathbf{F}^{(i)}(\eta, \mu)$ which we have just calculated, Eqs. (18) and (19), it is more convenient to use certain linear combinations. The use of these linear combinations simplifies both the completeness proof, (Sec. IV) and the calculation of the normalization integrals (Sec. V). We define therefore

$$\mathbf{\Phi}_1(\eta, \mu) = \frac{1}{c_{11}} \mathbf{F}_1^{(1)}(\eta, \mu) - \frac{1}{c_{12}} \mathbf{F}_2^{(1)}(\eta, \mu) \tag{25a}$$

$$= \begin{pmatrix} \dfrac{1}{c_{11}} \delta(\sigma_1 \eta - \mu) \\ -\dfrac{1}{c_{12}} \delta(\eta - \mu) \end{pmatrix} \tag{25b}$$

$$\mathbf{\Phi}_2^{(1)}(\eta, \mu) = \mathbf{F}_2^{(1)}(\eta, \mu) \tag{25c}$$

and

$$\mathbf{\Phi}_2^{(2)}(\eta, \mu) = \mathbf{F}^{(2)}(\eta, \mu) \tag{25d}$$

We note that, if we wished, we could consider that we have only *two* continuum eigensolutions, namely, $\mathbf{\Phi}_1(\eta, \mu)$ and

$$\mathbf{\Phi}_2(\eta, \mu) \triangleq \mathbf{\Phi}_2^{(1)}(\eta, \mu)\, \Theta_1(\eta) + \mathbf{\Phi}_2^{(2)}(\eta, \mu)\, \Theta_2(\eta). \tag{26}$$

Here

$$\begin{aligned} \Theta_i(\eta) &= 1,\ \eta \in \text{region } i \\ &= 0, \text{ otherwise.} \end{aligned} \tag{27}$$

Having determined the eigenvalue spectrum and the corresponding eigensolutions, we consider next the completeness theorem.

## IV. COMPLETENESS

THEOREM I. *The functions* $\mathbf{\Phi}_1(\eta, \mu)$, $\mathbf{\Phi}_2^{(i)}(\eta, \mu)$, $\eta \in [0, 1]$, *and* $\mathbf{\Phi}_+$ *form a complete set for functions defined on the "half-range"* $0 \leqq \mu \leqq 1$, *in the sense that an arbitrary function,* $\mathbf{\Psi}(\mu)$, *defined for* $0 \leqq \mu \leqq 1$ *can be expanded in the form*

$$\begin{aligned} \mathbf{\Psi}(\mu) = A_+ \mathbf{\Phi}_+ + \int_{①'} \alpha_1(\eta)\mathbf{\Phi}_1(\eta, \mu)\, d\eta + \int_{①'} \alpha_2(\eta)\mathbf{\Phi}_2^{(1)}(\eta, \mu)\, d\eta \\ + \int_{②'} \alpha_2(\eta)\mathbf{\Phi}_2^{(2)}(\eta, \mu)\, d\eta, \qquad 0 \leqq \mu \leqq 1. \end{aligned} \tag{28}$$

*Here the ranges of integration 1′ and 2′ refer to those portions of regions 1 and 2 for which* $\eta \geqq 0$.

This is the half-range completeness theorem.[6] Before proving this theorem, we should make two comments. First, the expansion coefficients $\alpha_1(\eta)$ and $\alpha_2(\eta)$ are taken to be scalars (i.e., multiples of the unit matrix). This is important because we wish to construct solutions to the transport equation (Eq. (11)) from the eigensolutions $\mathbf{\Psi}(\eta, x, \mu)$. In order that these combinations be solutions, it is necessary, in general, that the coefficients commute with the Boltzmann operator $\mathbf{B}$,

$$\mathbf{B} \triangleq \mu \frac{\partial}{\partial x}\mathbf{E} + \mathbf{\Sigma} - \mathbf{C}\int_{-1}^{1} \cdot\, d\mu'$$

(where $\mathbf{E}$ is the unit matrix). This will be true only if the expansion coefficients are scalars.

The second comment has to do with the arbitrariness of the expansion function $\mathbf{\Psi}(\mu)$. Actually, the solution of Eq. (28) incorporates results which are valid only if $\mathbf{\Psi}(\mu)$ obeys a Hölder condition on the interval [0, 1] (19). How-

[6] A full-range completeness theorem can also be proved but the restriction det $\mathbf{C} = 0$ is not necessary, cf. Section VII. Also, there is an analogous theorem for the other half-range ($-1 \leqq \mu \leqq 0$).

ever, we shall wish to apply the method when $\boldsymbol{\Psi}(\mu)$ is a distribution. A distribution is a weak limit of a sequence of Hölder functions; this implies that any results obtained when $\boldsymbol{\Psi}(\mu)$ is a distribution are to be interpreted in the weak topological sense, which is appropriate for transport theory. This point is discussed in more detail in ref. *7*.

We turn now to the proof of Theorem I. Equation (28) can be considered a singular integral equation for the expansion coefficients. To prove the theorem, it is sufficient to prove that a solution exists. This, in turn is done by solving the equation using the methods of ref. *19*. This yields expressions for the expansion coefficients; however, they are more conveniently obtained from the orthogonality relations developed in Section V.

We begin by attempting an expansion in terms of the continuum modes alone, i.e.,

$$\boldsymbol{\Psi}(\mu) = \int_{①'} \alpha_1(\eta)\boldsymbol{\Phi}_1(\eta,\mu)\, d\eta + \int_{①'} \alpha_2(\eta)\boldsymbol{\Phi}_2^{(1)}(\eta,\mu)\, d\eta + \int_{②'} \alpha_2(\eta)\boldsymbol{\Phi}_2^{(2)}(\eta,\mu)\, d\eta. \tag{29}$$

Putting in the explicit forms of the eigensolutions, we obtain the two equations

$$\psi_1(\mu) = \frac{1}{c_{11}\,\sigma_1}\,\alpha_1(\mu/\sigma_1) + c_{12}\int_{①'} \alpha_2(\eta)\left\{\frac{\eta P}{\sigma_1\,\eta - \mu} + \delta(\sigma_1\,\eta - \mu)(-2\eta T(\sigma_1\,\eta))\right\} d\eta + c_{12}\int_{②'} \alpha_2(\eta)\left\{\frac{\eta}{\sigma_1\,\eta - \mu}\right\} d\eta, \tag{30a}$$

and

$$\psi_2(\mu) = -\frac{\alpha_1(\mu)}{c_{12}}\,\Theta_1(\mu) + \int_{①'} \alpha_2(\eta)\left\{\frac{c_{22}\,\eta P}{\eta - \mu} + \delta(\eta - \mu)(1 - 2\eta c_{22}\,T(\eta))\right\} d\eta + \int_{②'} \alpha_2(\eta)\left\{\frac{c_{22}\,\eta P}{\eta - \mu} + \delta(\eta - \mu)\left(1 - 2\eta c_{11}\,T\left(\frac{1}{\sigma_1\eta}\right) - 2\eta c_{22}\,T(\eta)\right)\right\} d\eta. \tag{30b}$$

Here $\psi_1(\mu)$ and $\psi_2(\mu)$ are the two components of $\boldsymbol{\Psi}(\mu)$. Making the change of variable $\mu \to \sigma_1\,\mu$, solving for $\alpha_1(\mu)$ from Eq. (30a) and substituting into Eq. (30b) we obtain

$$\psi_2(\mu) + \frac{\sigma_1\,c_{11}}{c_{12}}\,\psi_1(\sigma_1\,\mu)\Theta_1(\mu) = \omega(\mu)\alpha_2(\mu) + [c_{22} + c_{11}\,\Theta_1(\mu)]P\int_0^1 \frac{\alpha_2(\eta)\eta}{\eta - \mu}\, d\eta, \tag{31}$$

where we have defined

$$\omega(\mu) \overset{\Delta}{=} 1 - 2\mu c_{22}\, T(\mu) - 2\mu c_{11}\, T(\sigma_1\,\mu)\Theta_1(\mu) - 2\mu c_{11}\, T\left(\frac{1}{\sigma_1\,\mu}\right)\Theta_2(\mu). \quad (32)$$

Taking the boundary values on the branch cut of the dispersion function, Eq. (20), we see that

$$\Omega_1^{\pm}(\mu) = 1 - 2\mu c_{11}T(\sigma_1\mu) - 2\mu c_{22}T(\mu) \pm \pi i\mu(c_{11} + c_{22}) \quad (33a)$$

and

$$\Omega_2^{\pm}(\mu) = 1 - 2\mu c_{11}T(1/\sigma_1\mu) - 2\mu c_{22}T(\mu) \pm \pi i\mu c_{22}\,. \quad (33b)$$

Here the subscripts on $\Omega^{\pm}(\mu)$ refer to the region in which the boundary value is evaluated. Thus, we find

$$\omega(\mu) = \tfrac{1}{2}\{\Omega^{+}(\mu) + \Omega^{-}(\mu)\} \quad (34a)$$

and

$$c_{22} + c_{11}\,\Theta_1\,(\mu) = \frac{1}{2\pi i\mu}\,\{\Omega^{+}(\mu) - \Omega^{-}(\mu)\}. \quad (34b)$$

Proceeding in standard fashion (*19*, *4*), we introduce the auxiliary function $N(z)$ defined by

$$N(z) \overset{\Delta}{=} \frac{1}{2\pi i}\int_0^1 \frac{\alpha_2(\eta)\eta\, d\eta}{\eta - z}, \quad (35)$$

with boundary values

$$N^{\pm}(\mu) = \frac{P}{2\pi i}\int_0^1 \frac{\alpha_2(\eta)\eta\, d\eta}{\eta - \mu} \pm \frac{1}{2}\,\mu\alpha_2(\mu). \quad (36)$$

From Eq. (35), $N(z)$ should be analytic in the cut plane and vanish as $1/z$ for large $z$. Substituting Eqs. (34) and (36) into Eq. (31) we find

$$\psi_2(\mu) + \sigma_1\frac{c_{11}}{c_{12}}\,\psi_1(\sigma_1\,\mu)\Theta_1(\mu) = \Omega^{+}(\mu)N^{+}(\mu) - \Omega^{-}(\mu)N^{-}(\mu). \quad (37)$$

The solution to Eq. (37) is well known. It is expressed in terms of a function $X(z)$ which has the properties

$$\frac{X^{+}(\mu)}{X^{-}(\mu)} = \frac{\Omega^{+}(\mu)}{\Omega^{-}(\mu)}, \qquad \mu \geqq 0 \quad (38)$$

but with $X^{+}(\mu) = X^{-}(\mu)$ for $\mu < 0$. That is, $X(z)$ is analytic in the complex plane cut, on the real line, from 0 to 1. $X(z)$ must also obey other restrictions as discussed in ref. *4*. The appropriate $X$ function for this case is

$$X(z) = \frac{1}{1 - z}\exp\left(\frac{1}{\pi}\int_0^1 \arg\,\Omega^{+}(\mu)\,\frac{d\mu}{\mu - z}\right). \quad (39)$$

The solution to Eq. (37) is then

$$N(z) = \frac{1}{2\pi i X(z)} \int_0^1 \gamma(\mu) \left\{ \psi_2(\mu) + \sigma_1 \frac{c_{11}}{c_{12}} \psi_1(\sigma_1 \mu)\Theta_1(\mu) \right\} \frac{d\mu}{\mu - z} \tag{40}$$

where

$$\gamma(\mu) \stackrel{\Delta}{=} \mu \frac{X^+(\mu)}{\Omega^+(\mu)}. \tag{41}$$

We see from Eq. (40) that $N(z)$ does not vanish at infinity as $1/z$ unless

$$\int_0^1 \gamma(\mu) \left\{ \psi_2(\mu) + \sigma_1 \frac{c_{11}}{c_{12}} \psi_1(\sigma_1 \mu)\Theta_1(\mu) \right\} d\mu = 0. \tag{42}$$

In general, this condition will not be satisfied. However, we recall that we are actually expanding not $\mathbf{\Psi}(\mu)$ but $\mathbf{\Psi}(\mu) - A_+\mathbf{\Phi}_+$ (comparing Eqs. (28) and (29)). Replacing the $\psi_i(\mu)$ in Eq. (42) by $\psi_i(\mu) - A_+w_i$, we see that Eq. (42) will be satisfied if $A_+$ is defined by

$$A_+ = \int_0^1 \tilde{\mathbf{\Phi}}_+^\dagger \mathbf{W}(\mu)\mathbf{\Psi}(\mu)\, d\mu \Big/ \int_0^1 \tilde{\mathbf{\Phi}}_+^\dagger \mathbf{W}(\mu)\mathbf{\Phi}_+\, d\mu \tag{43}$$

where

$$\mathbf{W}(\mu) = \begin{pmatrix} \sigma_1\gamma(\mu/\sigma_1) & 0 \\ 0 & \gamma(\mu) \end{pmatrix} \tag{44}$$

and

$$\mathbf{\Phi}_+^\dagger = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{45}$$

The coefficients are now determined. $A_+$ is given by Eq. (43) while the continuum coefficients may be determined from $N(z)$, Eq. (40), using Eq. (36) to find $\alpha_2(\eta)$ and then obtaining $\alpha_1(\eta)$ from Eq. (30a). In principle the expansion coefficients could be found in this manner. However, the orthogonality relations derived in the next section provide a more convenient method. The existence of such relations is already strongly suggested by the form of Eq. (43), particularly because one verifies immediately that $\mathbf{\Phi}_+^\dagger$ is a solution of the "adjoint" equation, i.e., Eq. (14) with $\mathbf{C}$ replaced by $\tilde{\mathbf{C}}$ corresponding to eigenvalue $\eta_0 = \pm\infty$. In this case the scalar product must be defined including a "weight function," $\mathbf{W}(\mu)$.

## V. ORTHOGONALITY AND NORMALIZATION

THEOREM II. *The functions* $\mathbf{\Phi}_1(\eta, \mu)$, $\mathbf{\Phi}_2^{(i)}(\eta, \mu)$, *and* $\mathbf{\Phi}_+$ *are orthogonal to the corresponding solutions of the adjoint equation on the range* [0, 1] *with weight function* $\mathbf{W}(\mu)$. *That is,*

$$\int_0^1 \tilde{\mathbf{\Phi}}^\dagger(\eta', \mu)\mathbf{W}(\mu)\mathbf{\Phi}(\eta, \mu)\, d\mu = 0, \qquad \eta \neq \eta'. \tag{46}$$

We shall prove this theorem using the original continuum eigensolutions, $\mathbf{F}(\eta, \mu)$, rather than the $\mathbf{\Phi}(\eta, \mu)$. This is clearly immaterial to the proof but the choice is made for convenience.

We rewrite Eq. (14) and the adjoint equation below in symbolic form:

$$\left(\frac{\mathbf{\Sigma}}{\mu} - \frac{\mathbf{C}}{\mu}\int_{-1}^{1} \cdot\, d\mu'\right)\mathbf{F}(\eta, \mu) = \frac{1}{\eta}\,\mathbf{F}(\eta, \mu) \tag{47a}$$

$$\left(\frac{\mathbf{\Sigma}}{\mu} - \frac{\tilde{\mathbf{C}}}{\mu}\int_{-1}^{1} \cdot\, d\mu'\right)\mathbf{F}^\dagger(\eta', \mu) = \frac{1}{\eta'}\,\mathbf{F}^\dagger(\eta', \mu). \tag{47b}$$

(The superscript tilde means transpose.) It is easily verified that Eqs. (47) have identical eigenvalue spectra. Furthermore we see that

$$\mathbf{F}^\dagger(\eta, \mu, \mathbf{C}) = \mathbf{F}(\eta, \mu, \tilde{\mathbf{C}}), \tag{48}$$

i.e., the adjoint functions are obtained from $\mathbf{F}(\eta, \mu)$ by interchanging in the latter $c_{ij}$ and $c_{ji}$.[7] The discrete adjoint function is simply

$$\mathbf{\Phi}_+{}^\dagger = \begin{pmatrix}1\\1\end{pmatrix}. \tag{49}$$

To prove the theorem, multiply Eq. (47a) from the left by $\tilde{\mathbf{F}}^\dagger(\eta', \mu)\mathbf{W}(\mu)$. Then multiply the *transpose* of Eq. (47b) from the right with $\mathbf{W}(\mu)\mathbf{F}(\eta, \mu)$. Integrate both equations over $d\mu$ from 0 to 1 and subtract. The terms involving $\mathbf{\Sigma}$ are identical since $\mathbf{\Sigma}$ and $\mathbf{W}(\mu)$ commute, and so we obtain

$$\begin{aligned}\int_0^1 \frac{d\mu}{\mu}\,\tilde{\mathbf{F}}^\dagger(\eta', \mu)\mathbf{W}(\mu)\mathbf{C}\int_{-1}^1 \mathbf{F}(\eta,\mu')\, d\mu' \\ - \int_{-1}^1 \tilde{\mathbf{F}}^\dagger(\eta', \mu')\, d\mu'\mathbf{C}\int_0^1 \frac{d\mu}{\mu}\,\mathbf{W}(\mu)\mathbf{F}(\eta, \mu)\, d\mu \\ = \left(\frac{1}{\eta'} - \frac{1}{\eta}\right)\int_0^1 \tilde{\mathbf{F}}^\dagger(\eta', \mu)\mathbf{W}(\mu)\mathbf{F}(\eta, \mu)\, d\mu.\end{aligned} \tag{50}$$

Thus, to prove Theorem II it is necessary to show that the left hand side of Eq. (50) vanishes if $\eta \neq \eta'$. Denote

$$\int_{-1}^1 \mathbf{F}(\eta, \mu')\, d\mu' \triangleq \mathbf{A}(\eta) \tag{51a}$$

and

[7] In making this replacement $c_{ij} \Leftrightarrow c_{ji}$ one must be careful not to express the components of the $\mathbf{\Sigma}$ matrix in terms of the $c_{ij}$ until after the replacement has been made.

$$\int_{-1}^{1} \mathbf{F}^{\dagger}(\eta', \mu')\, d\mu' \triangleq \mathbf{A}^{\dagger}(\eta'). \tag{51b}$$

(With the proviso of footnote 7 the elements $a_i^{\dagger}(\eta)$ of $\mathbf{A}^{\dagger}(\eta)$ are obtained from the elements $a_i(\eta)$ of $\mathbf{A}(\eta)$ under the transformation $c_{ij} \rightarrow c_{ji}$). Furthermore, let

$$\int_{0}^{1} \frac{d\mu}{\mu} \mathbf{W}(\mu)\mathbf{F}(\eta, \mu) \triangleq \mathbf{B}(\eta) \tag{52a}$$

and

$$\int_{0}^{1} \frac{d\mu}{\mu} \mathbf{W}(\mu)\mathbf{F}^{\dagger}(\eta', \mu) \triangleq \mathbf{B}^{\dagger}(\eta'). \tag{52b}$$

In this notation the left hand side of Eq. (50) becomes

$$\text{L.H.S.} = \tilde{\mathbf{B}}^{\dagger}(\eta')\mathbf{C}\mathbf{A}(\eta) - \tilde{\mathbf{A}}^{\dagger}(\eta')\mathbf{C}\mathbf{B}(\eta), \tag{53}$$

or in summation form we have

$$\text{L.H.S.} = \sum_{i,j} b_i^{\dagger}(\eta')c_{ij}a_j(\eta) - \sum_{i,j} a_i^{\dagger}(\eta')c_{ij}b_j(\eta), \tag{54}$$

where $b_i(\eta)(b_i^{\dagger}(\eta'))$ are the elements of $\mathbf{B}(\eta)(\mathbf{B}^{\dagger}(\eta'))$. Now from the definition of $c_{ij}$, Eq. (12b), we note

$$c_{ij} = \frac{c_{2j}\, c_{i2}}{c_{22}}. \tag{55}$$

When this expression is substituted into Eq. (54), the sums separate:

$$c_{22}(\text{L.H.S.}) = \sum_i b_i^{\dagger}(\eta')c_{i2} \sum_j c_{2j}a_j(\eta) - \sum_i a_i^{\dagger}(\eta')c_{i2} \sum_j c_{2j}b_j(\eta). \tag{56}$$

Now we divide Eq. (56) by $\sum_j c_{2j}a_j(\eta) \sum_j a_i^{\dagger}(\eta')c_{i2}$ and call $c_{22}(\text{L.H.S.})$ divided by this product $J$. Thus we have

$$J = \frac{\sum_i b_i^{\dagger}(\eta')c_{i2}}{\sum_i a_i^{\dagger}(\eta')c_{i2}} - \frac{\sum_j c_{2j}\, b_j(\eta)}{\sum_j c_{2j}\, a_j(\eta)}. \tag{57}$$

The two terms on the right hand side of Eq. (57) are related by $\eta' \Leftrightarrow \eta$ and $c_{ij} \Leftrightarrow c_{ji}$. A sufficient condition for the theorem to be true ($J = 0$) is that one of these terms, the second for example, be a constant independent of $\eta$ and of the $c_{ij}$. We shall show that this is indeed true. We denote the second term of Eq. (57) by $J_2$. Substituting for $b_j(\eta)$ (Eq. (52)) we obtain

$$J_2 = \frac{\sum_j c_{2j} \int_0^1 \frac{d\mu}{\mu}\, \sigma_j\, \gamma(\mu/\sigma_j) f_j(\eta, \mu)}{\sum c_{2j}\, a_j(\eta)}. \tag{58}$$

Here $f_j(\eta, \mu)$ are the components of any of the $\mathbf{F}(\eta, \mu)$ and can be written in the form (cf. Eq. (14))

$$f_j(\eta, \mu) = \left\{\frac{\eta P}{\sigma_j \eta - \mu} + \delta(\sigma_j \eta - \mu)\lambda_j(\eta)\right\} \sum_{i=1}^{2} c_{ji} \int_{-1}^{1} f_i(\eta, \mu')\, d\mu', \quad (59a)$$

or

$$f_j(\eta, \mu) = \left\{\frac{\eta P}{\sigma_j \eta - \mu} + \delta(\sigma_j \eta - \mu)\lambda_j(\eta)\right\} \sum_{i=1}^{2} c_{ji}\, a_i(\eta). \quad (59b)$$

(This equation holds for all continuum modes as well as the discrete mode, although in various regions the delta function may vanish and the symbol $P$ may be superfluous. This makes no difference to the general form of $f_j(\eta, \mu)$. The function $\lambda_j(\eta)$, of course, must be chosen so that Eq. (59a) is consistent.). If Eq. (59b) is substituted into Eq. (58), and if we replace the product $c_{2j}c_{ji}$ by the equivalent $c_{jj}c_{2i}$, the factor $\sum_j c_{2j}a_j(\eta)$ cancels out of numerator and denominator, and we find

$$J_2 = \sum_j c_{jj} \int_0^1 \frac{d\mu}{\mu}\, \sigma_j\, \gamma\left(\frac{\mu}{\sigma_j}\right) \left\{\frac{\eta P}{\sigma_j \eta - \mu} + \lambda_j(\eta)\delta(\sigma_j \eta - \mu)\right\}. \quad (60)$$

The integration over the delta function can be carried out, and we can write

$$J_2 = \sum_j c_{jj}\, \lambda_j(\eta)\, \frac{\gamma(\eta)}{\eta} + \sum_j c_{jj} \int_0^{1/\sigma_j} d\mu \left\{\frac{1}{\mu} - \frac{P}{\mu - \eta}\right\} \gamma(\mu). \quad (61)$$

The integrals in Eq. (61) can be carried out with the help of Identity I given in Appendix A.
We find

$$J_2 = \frac{\gamma(\eta)}{\eta} \sum_j c_{jj}\, \lambda_j(\eta) + 1 - \frac{\omega(\eta)\gamma(\eta)}{\eta}, \quad (62)$$

($\omega(\eta)$ was defined in Eq. (32); we have also noted that $X(0) = 1$). We should clarify the meaning of Eq. (62). First, we note that $\omega(\eta)$ is a different function in the two continuum regions and in the "discrete region." Similarly, $\lambda_j(\eta)$ takes a different value for each of the three continuum $\mathbf{F}(\eta, \mu)$ and for the discrete mode. That $J_2$ = constant must be verified for each of the four possibilities in order that the theorem be proved. For example, by comparing Eqs. (59) and (18a), and recalling that (18a) was obtained with the normalization

$$\mathbf{A}(\eta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (63)$$

we see that for the mode $\mathbf{F}_1^{(1)}(\eta, \mu)$,

$$\lambda_1(\eta) = \frac{1}{c_{11}} (1 - 2\eta c_{11} T(\sigma_1 \eta)) \tag{64}$$

and

$$\lambda_2(\eta) = \frac{1}{c_{21}} (-2\eta c_{21} T(\eta)). \tag{65}$$

Also, for $\eta \in$ region 1 (cf. Eq. (32))

$$\omega_1(\eta) = 1 - 2\eta c_{11} T(\sigma_1 \eta) - 2\eta c_{22} T(\eta). \tag{66}$$

Inserting these values into Eq. (62), we find

$$J_2 = 1. \tag{67}$$

In the same way, we verify that $J_2 = 1$ when the eigensolutions $\mathbf{F}_2^{(1)}(\eta, \mu)$ and $\mathbf{F}^{(2)}(\eta, \mu)$ are used (in the latter case, of course,

$$\omega_2(\eta) = 1 - 2\eta c_{11} T\left(\frac{1}{\sigma_1 \eta}\right) - 2\eta c_{22} T(\eta) \tag{68}$$

must be used in Eq. (62)). For the discrete mode, $\lambda = 0$. But $\Omega^+(\eta_0) = \Omega^-(\eta_0) = \Omega(\eta_0) = 0$, by definition, and we see immediately that $J_2 = 1$. Thus we obtain

$$J_2 \equiv 1 \tag{69}$$

and the theorem is proved, i.e.,

$$\left(\frac{1}{\eta'} - \frac{1}{\eta}\right) \int_0^1 \tilde{\mathbf{\Phi}}^\dagger(\eta', \mu) \mathbf{W}(\mu) \mathbf{\Phi}(\eta, \mu) \, d\mu = 0. \tag{70}$$

Our method of proving Theorem II has several advantages. We mention two:

1. If the form of $\mathbf{W}(\mu)$ had not already been suggested, a similar procedure to the above could be used to find $\mathbf{W}(\mu)$.

2. The method can be generalized to the $N$-group problem, in which the above procedure serves as a guide.

For the normalization integrals, we return to the $\mathbf{\Phi}(\eta, \mu)$ rather than the $\mathbf{F}(\eta, \mu)$, since we will expand in terms of the $\mathbf{\Phi}(\eta, \mu)$.

The degenerate eigensolutions can be orthogonalized using a Schmidt type procedure. First we present the normalization integrals. Defining

$$(\mathbf{E}, \mathbf{G}) \triangleq \int_0^1 \tilde{\mathbf{E}}^\dagger(\eta', \mu) \mathbf{W}(\mu) \mathbf{G}(\eta, \mu) \, d\mu, \tag{71}$$

we have

$$(\mathbf{\Phi}_+, \mathbf{\Phi}_+) = N_+ \tag{72a}$$

$$(\mathbf{\Phi}_1, \mathbf{\Phi}_1) = N_1(\eta) \delta(\eta - \eta') \tag{72b}$$

$$(\mathbf{\Phi}_2^{(1)}, \mathbf{\Phi}_2^{(1)}) = N_2^{(1)}(\eta)\delta(\eta - \eta') \tag{72c}$$

$$(\mathbf{\Phi}_2^{(2)}, \mathbf{\Phi}_2^{(2)}) = N_2^{(2)}(\eta)\delta(\eta - \eta') \tag{72d}$$

$$(\mathbf{\Phi}_1, \mathbf{\Phi}_2^{(1)}) = M_{12}(\eta)\delta(\eta - \eta') \tag{72e}$$

$$(\mathbf{\Phi}_2^{(1)}, \mathbf{\Phi}_1) = M_{21}(\eta)\delta(\eta - \eta'). \tag{72f}$$

Here

$$N_+ = 2(\sigma_1 w_1 + w_2) \tag{73a}$$

$$N_1(\eta) = \gamma(\eta)\,\frac{c_{11} + c_{22}}{c_{11}^2\, c_{22}} \tag{73b}$$

$$N_2^{(1)}(\eta) = \gamma(\eta)\{\pi^2\eta^2 c_{22}(c_{11} + c_{22}) + (1 - 2\eta c_{22}\, T(\eta))^2 + 4\eta^2 c_{11}\, c_{22}\, T^2(\sigma_1\, \eta)\} \tag{73c}$$

$$N_2^{(2)}(\eta) = \gamma(\eta)\Omega_2^+(\eta)\Omega_2^-(\eta) \tag{73d}$$

$$M_{ij}(n) = \frac{-c_{ij}}{c_{11}\, c_{22}}\,\gamma(\eta)\{1 - 2\eta c_{22}\, T(\eta) + 2\eta c_{22}\, T(\sigma_1\,\eta)\}. \tag{73e}$$

Explicitly, we write (cf. Eqs. (33))

$$\Omega_1^+(\eta)\Omega_1^-(\eta) = (1 - 2\eta c_{11} T(\sigma_1\eta) - 2\eta c_{22} T(\eta))^2 + \pi^2\eta^2(c_{11} + c_{22})^2 \tag{74a}$$

and

$$\Omega_2^+(\eta)\Omega_2^-(\eta) = (1 - 2\eta c_{11} T(1/\sigma_1\eta) - 2\eta c_{22} T(\eta))^2 + \pi^2\eta^2 c_{22}^2\,. \tag{74b}$$

Since $\mathbf{\Phi}_1$ and $\mathbf{\Phi}_2^{(1)}$ are not orthogonal for $\eta = \eta'$, we introduce two new functions $\chi_1^{(1)}$ and $\chi_2^{(1)}$ defined such that $\chi_1^{(1)}$ is orthogonal to $\mathbf{\Phi}_2^{(1)}$ and $\chi_2^{(1)}$ is orthogonal to $\mathbf{\Phi}_1$ (clearly both $\chi_j^{(1)}$ are orthogonal to $\mathbf{\Phi}_2^{(2)}$). Therefore

$$\chi_1^{(1)}(\eta, \mu) \triangleq \frac{c_{11}^2}{\gamma(\eta)}\,\{N_2^{(1)}(\eta)\mathbf{\Phi}_1(\eta, \mu) - M_{12}(\eta)\mathbf{\Phi}_2^{(1)}(\eta, \mu)\} \tag{75a}$$

and

$$\chi_2^{(1)}(\eta, \mu) \triangleq \frac{c_{11}^2}{\gamma(\eta)}\,\{N_1(\eta)\mathbf{\Phi}_2^{(1)}(\eta, \mu) - M_{21}(\eta)\mathbf{\Phi}_1(\eta, \mu\}\,. \tag{75b}$$

We find

$$(\chi_1^{(1)}, \mathbf{\Phi}_1) = \gamma(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta)\delta(\eta - \eta'), \tag{76a}$$

$$(\chi_1^{(1)}, \mathbf{\Phi}_2^{(1)}) = 0, \tag{76b}$$

$$(\chi_2^{(1)}, \mathbf{\Phi}_2^{(1)}) = \gamma(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta)\delta(\eta - \eta'), \tag{76c}$$

and

$$(\chi_2^{(1)}, \mathbf{\Phi}_1) = 0. \tag{76d}$$

With the formalism developed in this section, typical half-space problems can be solved immediately. We consider, in the next section, the Milne problem.

## VI. MILNE PROBLEM

We seek the angular energy density, $\boldsymbol{\Psi}_M(x, \mu)$, in the source-free half-space subject to the boundary conditions (7):

(a) $\boldsymbol{\Psi}_M(0, \mu) = 0, \mu \geqq 0$ (zero re-entrant radiation)
(b) $\boldsymbol{\Psi}_M(x, \mu) \sim \boldsymbol{\Psi} - (x, \mu)$ for large $x$.

The second condition specifies that $\boldsymbol{\Psi}_M(x, \mu)$ diverges no more rapidly than the slowest diverging mode $\boldsymbol{\Psi}_-(x, \mu)$.

The solution can be constructed from the normal modes of the transport equation. Condition (b) requires that no $\boldsymbol{\Psi}(\eta, x, \mu)$ be included for $\eta \in [-1, 0]$, Thus we write

$$\begin{aligned}\boldsymbol{\Psi}_M(x, \mu) = A_+\boldsymbol{\Psi}_+(x, \mu) + A_- \boldsymbol{\Psi}_-(x, \mu) + \int_{①'} \alpha_1(\eta)\, e^{-x/\eta}\boldsymbol{\Phi}_1(\eta, \mu)\, d\eta \\ + \int_{①'} \alpha_2(\eta)\, e^{-x/\eta}\boldsymbol{\Phi}_2^{(1)}(\eta, \mu)\, d\eta + \int_{②'} \alpha_2(\eta)\, e^{-x/\eta}\, \boldsymbol{\Phi}_2^{(2)}(\eta, \mu)\, d\eta.\end{aligned} \tag{77}$$

The coefficient $A_-$ we leave arbitrary (it depends upon the normalization). The other coefficients are obtained from condition (a). Setting $x = 0$ in Eq. (77), we have

$$\begin{aligned}- A_- \boldsymbol{\Psi}_-(0, \mu) = A_+ \boldsymbol{\Phi}_+ + \int_{①'} \alpha_1(\eta)\boldsymbol{\Phi}_1(\eta, \mu)\, d\eta + \int_{①'} \alpha_2(\eta)\boldsymbol{\Phi}_2^{(1)}(\eta,\mu)\, d\eta \\ + \int_{②'} \alpha_2(\eta)\, \boldsymbol{\Phi}_2^{(2)}(\eta, \mu)\, d\eta, \qquad \mu \geqq 0.\end{aligned} \tag{78}$$

Thus the coefficients are merely the half-range expansion coefficients for the function

$$- A_- \boldsymbol{\Psi}_-(0, \mu) = A_- \begin{pmatrix} w_1/\sigma_1 \\ w_2 \end{pmatrix} \mu. \tag{79}$$

They are found immediately from the orthogonality relations:

$$\frac{A_+}{A_-} = \frac{1}{N_+} \int_0^1 \tilde{\boldsymbol{\Phi}}_+^\dagger\, \mathbf{W}(\mu) \begin{pmatrix} w_1/\sigma_1 \\ w_2 \end{pmatrix} \mu\, d\mu, \tag{80a}$$

or

$$\frac{A_+}{A_-} = \frac{1}{N_+}\left\{ w_2 \int_0^1 \gamma(\mu)\mu\, d\mu + \sigma_1^2\, w_1 \int_0^{1/\sigma_1} \gamma(\mu)\mu\, d\mu\right\}. \tag{80b}$$

This expression can be put in terms of the $X$-function by the use of Identity

IV, Appendix A. We find

$$\frac{A_+}{A_-} = \frac{3\sigma_1}{2(\sigma_1 w_2 + w_1)} \left\{ w_2 \int_0^1 \frac{\mu^2\, d\mu}{x(-\mu)} + \sigma_1^2 w_1 \int_0^{1/\sigma_1} \frac{\mu^2}{x(-\mu)}\, d\mu \right\}, \tag{81}$$

where the explicit form of $N_+$ (Eq. (73a)) has been used. Similarly,

$$\frac{\alpha_1(\eta)}{A_-} = \frac{\int_0^1 \tilde{\chi}_1^{(1)}(\eta, \mu)\mathbf{W}(\mu) \begin{pmatrix} w_1/\sigma_1 \\ w_2 \end{pmatrix} \mu\, d\mu}{\gamma(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta)}. \tag{82}$$

This integration is somewhat lengthy, but completely straightforward. We find

$$\frac{\alpha_1(\eta)}{A_-} = -\frac{c_{11}\, c_{12}}{c_{22}} \frac{\eta w_2 \{1 - 2\eta\, c_{22}\, T(\eta) + 2\eta\, c_{22}\, T(\sigma_1 \eta)\}}{\gamma(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta)}. \tag{83}$$

Similarly

$$\frac{\alpha_2(\eta)}{A_-} = -\frac{\eta\{w_2 + \sigma_1^2 w_1 \Theta_1(\mu)\}}{\gamma(\eta)\Omega_2^+(\eta)\Omega_2^-(\eta)}. \tag{84}$$

The expansion coefficients are now determined, so the problem is solved. Note that all results are expressed in terms of the two functions $T(z)$ and $X(-\mu)$ ($\Omega^{\pm}(\mu)$ are given in terms of $T$-functions, while $\gamma(\mu)$ is expressed in terms of $X(-\mu)$, $0 \leqq \mu \leqq 1$ in Appendix A, Identity IV.).

The customary normalization (*2*) is to set

$$-2\pi \int_{-1}^1 \mu\, d\mu \int_0^\infty d\nu\, \psi_\nu(x, \mu) = \sigma T_e^4 \tag{85}$$

where $T_e$ is the "effective temperature". This leads to

$$A_- = \frac{3\sigma_1\, \sigma T_e^4}{4\pi(\sigma_1 w_2 + w_1)}. \tag{86}$$

The energy density

$$\mathfrak{F}(x) \triangleq 2\pi \int_{-1}^1 d\mu \int_0^\infty d\nu\, \psi_\nu(x, \mu) \tag{87}$$

is given by

$$\mathfrak{F}(x) = 2\pi \widetilde{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \int_{-1}^1 \mathbf{\Psi}_M(x, \mu)\, d\mu \tag{88}$$

or finally

$$\frac{\mathfrak{F}(x)}{4\pi} = (A_+ + xA_-)(w_1 + w_2) + \frac{1}{2}\int_{①'} \left\{\frac{\alpha_1(\eta)}{c_{11}} - \frac{\alpha_2(\eta)}{c_{12}}\right\} e^{-x/\eta}\, d\eta + \frac{1}{2}\int_{②'} \alpha_2(\eta)\, e^{-x/\eta}\{1 + 2(c_{12} - c_{11})\,\eta T(1/\sigma_1 \eta)\}\, d\eta. \tag{89}$$

The extrapolated endpoint, $x_0$, is defined as the distance from the boundary at which the asymptotic density extrapolates to zero.
It is

$$x_0 = \frac{A_+}{A_-} \tag{90}$$

which is given already by Eq. (81).

The temperature distribution is found from Eq. (9), i.e.,

$$\frac{\sigma T^4(x)}{\pi} = \frac{1}{2(\sigma_1 w_1 + w_2)} \widetilde{\begin{pmatrix}\sigma_1 \\ 1\end{pmatrix}} \int_{-1}^{1} \boldsymbol{\Psi}_M(x, \mu)\, d\mu. \tag{91}$$

We find

$$\frac{T^4(x)}{T_e^4} = \frac{3\sigma_1}{4(\sigma_1 w_2 + w_1)} \left\{ x + x_0 - (c_{11} + c_{22}) \int_0^{1/\sigma_1} \frac{\eta e^{-x/\eta}}{\gamma(\eta)\Omega_1^+(\eta)\Omega_1^-(\eta)}\, d\eta - c_{22} \int_{1/\sigma_1}^{1} \frac{\eta e^{-x/\eta}}{\gamma(\eta)\Omega_2^+(\eta)\Omega_2^-(\eta)}\, d\eta \right\}. \tag{92a}$$

Writing this in terms of $X(-\mu)$ we have

$$\frac{T^4(x)}{T_e^4} = \frac{3\sigma_1}{4(\sigma_1 w_2 + w_1)} \left\{ x + x_0 - \frac{(c_{11} + c_{22})\,(\sigma_1 w_2 + w_1)}{3\sigma_1(\sigma_1 w_1 + w_2)} \cdot \int_0^{1/\sigma_1} \frac{X(-\eta) e^{-x/\eta}\, d\eta}{\Omega_1^+(\eta)\Omega_1^-(\eta)} - \frac{c_{22}(\sigma_1 w_2 + w_1)}{3\sigma_1(\sigma_1 w_1 + w_2)} \cdot \int_{1/\sigma_1}^{1} \frac{X(-\eta)\, e^{-x/\eta}}{\Omega_2^+(\eta)\Omega_2^-(\eta)}\, d\eta \right\}. \tag{92b}$$

Now that the temperature distribution is known, one can obtain the frequency dependent angular density, $\psi_\nu(z, \mu)$, simply by integrating Eq. (1) (after transforming it to the optical variable $x$). The frequency dependent law of darkening is then given by $\psi_\nu(0, \mu)$. The law of darkening for the integrated quantities is found from $\boldsymbol{\Psi}_M(0, \mu)$. Just as in the gray problem, if we consider $\boldsymbol{\Psi}_M(0, \mu)$, $\mu < 0$ the integrals over $d\eta$ can be performed by extending the technique of ref. *21*. We thus write

$$\boldsymbol{\Psi}_M(0, -\mu) = A_+ \boldsymbol{\Phi}_+ + A_- \boldsymbol{\Psi}_-(0, -\mu) + \int_0^1 \alpha_2(\eta) \begin{pmatrix} \dfrac{c_{12}}{\sigma_1 \eta + \mu} \\ \dfrac{c_{22}}{\eta + \mu} \end{pmatrix} \eta\, d\eta, \mu > 0. \tag{93a}$$

Here the continuum part of the expansion becomes particularly simple because by restricting $\mu$ to be negative, the eigen solutions ($\eta > 0$) are no longer singular. We find

$$\boldsymbol{\Psi}_{\mathrm{M}}(0, -\mu) = \frac{3\sigma_1 \, \sigma T_e^4}{4\pi(\sigma_1 \, w_2 + w_1)} \begin{Bmatrix} \dfrac{w_1}{X(-\mu/\sigma_1)} \\ \dfrac{w_2}{X(-\mu)} \end{Bmatrix}, \quad \mu > 0. \tag{93b}$$

It is clear how other half-space problems could be solved. For example, consider the albedo problem. Here we have a source-free half-space with incident distribution[8]

$$\boldsymbol{\Psi}_{\mathrm{inc}}(\mu) = \begin{pmatrix} \delta(\mu - \mu_1) \\ \delta(\mu - \mu_2) \end{pmatrix}, \qquad \mu_i \geqq 0. \tag{94}$$

Here the solution must not diverge at infinity, so we set

$$\begin{aligned} \boldsymbol{\Psi}_{\mathrm{a}}(x, \mu) &= A_+ \boldsymbol{\Phi}_+ + \int_{①'} \alpha_1(\eta) \, e^{-x/\eta} \, \boldsymbol{\Phi}_1(\eta, \mu) \, d\eta \\ &\quad + \int_{①'} \alpha_2(\eta) e^{-x/\eta} \, \boldsymbol{\Phi}_2^{(1)}(\eta, \mu) \, d\eta + \int_{②'} \alpha_2(\eta) e^{-x/\eta} \, \boldsymbol{\Phi}_2^{(2)}(\eta, \mu) \, d\eta. \end{aligned} \tag{95}$$

Since

$$\boldsymbol{\Psi}_{\mathrm{a}}(0, \mu) = \boldsymbol{\Psi}_{\mathrm{inc}}(\mu), \qquad \mu \geqq 0 \tag{96}$$

the expansion coefficients are found as integrals of the adjoint functions times delta functions. We omit any details of this or other possible half-space problems except for the half-space Green's function, discussed briefly in Appendix B.

## VII. THE CASE OF GENERAL C

As has been discussed in the introduction, the case of general **C** has interest both in neutron transport theory and in radiative transfer. In the latter, if we had included scattering in the resonance lines ($\epsilon \neq 1$ (*1*)) we would have been led to this case. We discuss this situation briefly in the present section. All results are presented without proof.

The eigenfunctions for region 1 have the same form as for det $\mathbf{C} = 0$. They are given by Eqs. (18). For region 2, however, we find

$$\mathbf{F}^{(2)}(\eta, \mu) = \begin{Bmatrix} \dfrac{c_{12} \, \eta}{\sigma_1 \, \eta - \mu} \\ \dfrac{\eta f(\eta) P}{\eta - \mu} + \delta(\eta - \mu) \, \lambda(\eta) \end{Bmatrix} \tag{97}$$

[8] If the albedo solution is to be used to generate the solution to a problem involving arbitrary incident distribution, then two albedo problems must be defined, just as two Green's functions are defined; cf. Appendix B.

where

$$\lambda(\eta) = 1 - 2\eta c_{22}T(\eta) - 2\eta c_{11}T(1/\sigma_1\eta) + 4\eta^2 det\mathbf{C}T(\eta)T(1/\sigma_1\eta) \tag{98}$$

and

$$f(\eta) = c_{22} - 2\eta \det \mathbf{C}T(1/\sigma_1\eta). \tag{99}$$

The discrete solutions are

$$\mathbf{F}_{i\pm}(\mu) = \begin{pmatrix} \dfrac{c_{12}\,\eta_i}{\sigma_1\,\eta_i \mp \mu} \\ \dfrac{c_{22} - 2\det\mathbf{C}\;\eta_i T(1/\sigma_1\,\eta_i)}{\eta_i \mp \mu}\eta_i \end{pmatrix} \tag{100}$$

where $\eta_i$ are positive roots of the dispersion function

$$\Omega(z) = 1 - 2zc_{11}T(1/\sigma_1 z) - 2zc_{22}T(1/z) + 4z^2 \det \mathbf{C}T(1/z)T(1/\sigma_1 z). \tag{101}$$

It can be shown (*20*) that for det $\mathbf{C} \leqq 0$, $\Omega(z)$ has only two zeros while for det $\mathbf{C} > 0$ it may have two or four depending upon the relative magnitude of the $c_{ij}$ and $\sigma_1$. Note that $\Omega(z) = \Omega(-z)$; hence the roots occur in $\pm$ pairs, explaining the notation of Eq. (100).

THEOREM III. *The function* $\mathbf{F}_i^{(1)}(\eta, \mu)$, $\mathbf{F}^{(2)}(\eta, \mu)$ *and* $\mathbf{F}_i \pm (\mu)$ *form a complete set in the sense that an arbitrary function* $\mathbf{\Psi}(\mu)$ *defined on the interval* $[-1, 1]$ *can be expanded in the form*

$$\mathbf{\Psi}(\mu) = \sum_i A_{i+}\,\mathbf{F}_{i+}(\mu) + \sum_i A_{i-}\,\mathbf{F}_{i-}(\mu) + \int_{①} \alpha_1(\eta)\,\mathbf{F}_1^{(1)}(\eta, \mu)\,d\eta + \int_{①} \alpha_2(\eta)\,\mathbf{F}_2^{(1)}(\eta, \mu)\,d\eta + \int_{②} \alpha_3(\eta)\,\mathbf{F}^{(2)}(\eta, \mu)\,d\eta. \tag{102}$$

The proof of this theorem follows in analogous fashion to that of Theorem I except that here it is unnecessary to introduce the $X$ function since $N(z)$ and $\Omega(z)$ have the same branch cut.

THEOREM IV. *The functions* $\mathbf{F}(\eta, \mu)$ *are orthogonal to the adjoint functions,* $\mathbf{F}^{\dagger}(\eta, \mu)$, *on the interval* $[-1, 1]$ *with weight function* $\mu$. *This applies to both continuum and discrete modes.*[9]

This theorem is proved, in the usual manner, directly from the equations obeyed by $\mathbf{F}(\eta, \mu)$ and $\mathbf{F}^{+}(\eta, \mu)$.

In attempting to prove the half-range theorem, analogous to Theorem I but for det $\mathbf{C} \neq 0$, one proceeds in exactly the same manner as for Theorem I. However, instead of arriving at Eq. (37), which is solved by Wiener-Hopf factoriza-

[9] It is possible even for the case det $\mathbf{C} \neq 0$ that there is a degenerate double root at infinity (*20*). In this situation $\mathbf{\Psi} - (0, \mu)$ is not included in the orthogonal set.

tion, one finds an equation which cannot be simply factored. Actually, a proof of this theorem is given in ref. *3*, but not in a form which makes it possible to obtain the expansion coefficients in any simple way. While we have remedied this for the full-range case through Theorems III and IV, we have thus far been unable to discover a similar scheme for the half-range problem.

## APPENDIX A. X-FUNCTION IDENTITIES

IDENTITY I

$$X(z) = c_{22}\int_0^1 \frac{\gamma(\mu)}{\mu - z}\, d\mu + c_{11}\int_0^{1/\sigma_1} \frac{\gamma(\mu)}{\mu - z}\, d\mu. \tag{A.1}$$

The proof follows by writing Cauchy's theorem for $X(z)$, i.e.,

$$X(z) = \frac{1}{2\pi i}\oint \frac{X(z')}{z' - z}\, dz' \tag{A.2}$$

where the contour can be shrunk down to include only the branch cut (the integrand vanishes at infinity). Thus we write

$$X(z) = \frac{1}{2\pi i}\int_0^1 \{X^+(\mu) - X^-(\mu)\}\, \frac{d\mu}{\mu - z}\, . \tag{A.3}$$

Using Eqs. (34b) and (38) we obtain

$$X^+(\mu) - X^-(\mu) = 2\pi i\gamma(u)\{c_{22} + c_{11}\Theta_1(\mu)\}. \tag{A.4}$$

Entering Eq. (A.3) with this result gives Identity I.

IDENTITY II

$$X(z)\, X(-z) = \frac{3\sigma_1(\sigma_1 w_1 + w_2)}{\sigma_1 w_2 + w_1}\, \Omega(z). \tag{A.5}$$

For the proof, we note that the function

$$F(z) \triangleq \frac{X(z)\, X(-z)\, (\sigma_1 w_2 + w_1)}{3\sigma_1(\sigma_1 w_1 + w_2)\, \Omega(z)} \tag{A.6}$$

is an entire function because it is analytic in the cut plane and its discontinuity across the cut vanishes. It is thus a constant.

Letting $z$ approach infinity we find

$$\lim_{z\to\infty} F(z) = 1 \tag{A.7}$$

because in the same limit

$$X(z)X(-z) \sim -1/z^2 \tag{A.8}$$

and

$$\Omega(z) \sim -\frac{1}{z^2}\frac{\sigma_1 w_2 + w_1}{3\sigma_1(\sigma_1 w_1 + w_2)}\, . \tag{A.9}$$

This proves Identity II.

IDENTITY III

$$X(z) = \frac{3\sigma_1}{2(\sigma_1 w_2 + w_1)} \left\{ w_2 \int_0^1 \frac{\mu \, d\mu}{X(-\mu)(\mu - z)} + \sigma_1^2 w_1 \int_0^{1/\sigma_1} \frac{\mu \, d\mu}{X(-\mu)(\mu - z)} \right\}. \tag{A.10}$$

This is a nonlinear integral equation for $X(z)$ which can be used by the ambitious to evaluate $X(z)$ numerically. It is obtained by combining Identities I and II and noting the trivial,

IDENTITY IV

$$\gamma(\mu) = \frac{3\sigma_1 \mu(\sigma_1 w_1 + w_2)}{(\sigma_1 w_2 + w_1)X(-\mu)} \tag{A.11}$$

obtained from Eq. (A.5) by taking boundary values of both sides.

## APPENDIX B. THE HALF-SPACE GREEN'S FUNCTION

For the case, det $\mathbf{C} = 0$, there is no solution to the infinite medium Green's function which is finite everywhere. However, we can define a "pseudo Green's function" for the infinite medium that goes to a constant at plus infinity and diverges at minus infinity, which can then be used to construct the half-space Green's function in the following way.

We wish to construct a solution to the homogeneous transport equation, Eq. (11), which approaches a constant bounded value at plus infinity, obeys the "jump conditions" at the source position, and has zero incident distribution. We note that there are two Green's functions, $\mathbf{G}_1$ and $\mathbf{G}_2$, corresponding to sources

$$\mathbf{q}_1 = \delta(x - x_0)\, \delta(\mu - \mu_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{B.1a}$$

$$\mathbf{q}_2 = \delta(x - x_0)\, \delta(\mu - \mu_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{B.1b}$$

respectively. Each is found in a similar way, so we simply discuss $\mathbf{G}_1$.

Consider an artificial infinite medium Green's function of the form

$$\begin{aligned} \mathbf{G}_{1\infty}(x_0, \mu_0 \to x, \mu) = A_{1+} \boldsymbol{\emptyset}_+ &+ \int_0^{1/\sigma_1} \alpha_{11}(\eta) e^{-(x-x_0)/\eta} \boldsymbol{\emptyset}_1(\eta, \mu) \, d\eta \\ &+ \int_0^{1/\sigma_1} \alpha_{12}(\eta) e^{-(x-x_0)/\eta} \boldsymbol{\emptyset}_2^{(1)}(\eta, \mu) \, d\eta \\ &+ \int_{1/\sigma_1}^{1} \alpha_{12}(\eta) e^{-(x-x_0)/\eta} \boldsymbol{\emptyset}_2^{(2)}(\eta, \mu) \, d\eta, \qquad x > x_0 . \end{aligned} \tag{B.2a}$$

$$\mathbf{G}_{1\infty}(x_0, \mu_0 \to x, \mu) = -A_{1-} \mathbf{\Psi}_- (x - x_0, \mu) - \int_{-1/\sigma_1}^{0} \alpha_{11}(\eta) e^{(x_0-x)/\eta} \boldsymbol{\phi}_1(\eta, \mu)\, d\eta - \int_{-1/\sigma_1}^{0} \alpha_{12}(\eta) e^{(x_0-x)/\eta} \boldsymbol{\phi}_2^{(1)}(\eta, \mu)\, d\eta - \int_{-1}^{-1/\sigma_1} \alpha_{12}(\eta) e^{(x_0-x)/\eta} \boldsymbol{\phi}_2^{(2)}(\eta, \mu)\, d\eta, \quad x < x_0. \tag{B.2b}$$

Here the coefficients are found in the usual manner; i.e., we set

$$\mathbf{G}_{1\infty}(x_0, \mu_0 \to x_0^+, \mu) - \mathbf{G}_{1\infty}(x_0, \mu_0 \to x_0^-, \mu) = 2\pi\mu\delta(\mu - \mu_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{B.3}$$

which leads to

$$2\pi\mu\delta(\mu - \mu_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A_{1+} \boldsymbol{\phi}_+ + A_{1-} \mathbf{\Psi}_-(0, \mu) + \int_{①} \alpha_{11}(\eta) \boldsymbol{\phi}_1(\eta, \mu)\, d\eta + \int_{①} \alpha_{12}(\eta) \boldsymbol{\phi}_2^{(1)}(\eta, \mu)\, d\eta + \int_{②} \alpha_{12}(\eta) \boldsymbol{\phi}_2^{(2)}(\eta, \mu)\, d\eta. \tag{B.4}$$

Theorems III and IV apply to the above and thus we can find $A_{1+}$, $A_{1-}$, $\alpha_{11}(\eta)$, and $\alpha_{12}(\eta)$. Note that $\mathbf{G}_{1\infty}$ obeys the first two boundary conditions; i.e., it goes to a constant at $+\infty$ and it obeys the "jump condition," Eq. (B.3). That it diverges at $-\infty$ is not important here. We now must fix the zero incident distribution condition. To this end, we write

$$\mathbf{G}_{1_{h.s.}}(x_0, \mu_0 \to x, \mu) = \mathbf{G}_{1\infty}(x_0, \mu_0 \to x, \mu) - A_{1+}' \boldsymbol{\phi}_+ - \int_0^{1/\sigma_1} \alpha_{11}'(\eta) e^{-x/\eta} \boldsymbol{\phi}_1(\eta, \mu)\, d\eta - \int_0^{1/\sigma_1} \alpha_{12}'(\eta) e^{-x/\eta} \boldsymbol{\phi}_2^{(1)}(\eta, \mu)\, d\eta - \int_{1/\sigma_1}^{1} \alpha_{12}'(\eta) e^{-x/\eta} \boldsymbol{\phi}_2^{(2)}(\eta, \mu)\, d\eta, \quad x > 0. \tag{B.5}$$

Setting $x = 0$, we obtain the condition for zero incident distribution; namely,

$$\mathbf{G}_{1\infty}(x_0, \mu_0 \to x, \mu) = A_{1+}' \boldsymbol{\phi}_+ + \int_0^{1/\sigma_1} \alpha_{11}'(\eta) \boldsymbol{\phi}_1(\eta, \mu)\, d\eta + \int_0^{1/\sigma_1} \alpha_{12}'(\eta) \boldsymbol{\phi}_2^{(1)}(\eta, \mu)\, d\eta + \int_{1/\sigma_1}^{1} \alpha_{12}'(\eta) \boldsymbol{\phi}_2^{(2)}(\eta, \mu)\, d\eta, \quad \mu \geqq 0. \tag{B.6}$$

Since the left-hand side of Eq. (B.6) is known, we can determine the unknown coefficients, $A'_{1+}$, $\alpha'_{11}(\eta)$, and $\alpha'_{12}(\eta)$, with the aid of Theorems I and II. The half-space Green's function corresponding to source $\mathbf{q}_1$ is thus known and given by Eq. (B.5). In exactly the same way, $\mathbf{G}_{2_{h.s.}}$ can be found.

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