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The Effect of Linearly Anisotropic Neutron Scattering on Disadvantage Factor Calculations

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An expression for the disadvantage factor for a two-region heterogeneous cell in plane geometry has been obtained within the structure of one-speed neutron transport theory. The scattering in the fuel region has been assumed isotropic; however, to describe more accurately the scattering in the moderator, a two-term kernel has been used. The analysis is based on Case's normal-mode expansion technique and leads to a set of Fredholm equations for the unknown expansion coefficients. Numerical solutions, obtained by solving iteratively the Fredholm equations for the expansion coefficients, are given for several sets of parameters.

INTRODUCTION

The utility of the disadvantage factor in the calculation of the thermal utilization in heterogeneous reactor cells is well known. For the purpose of reactor design calculations, one would like to know this quantity as accurately as possible. In general, of course, this would require solutions to the energy-dependent neutron transport equation in order to describe accurately the processes involved. Because of the complexity of the general model, the one-speed approximation has found favor in the calculation of fine structure parameters for heterogeneous lattices.

The standard texts[1,2] illustrate the concept of the disadvantage factor; however, the calculations are usually limited to a diffusion-theory treatment. Pomraning and Clark[3] have made calculations based on an improved diffusion-theory model, and Theys[4] has applied the Amouyal-Benoist-Horowitz (A-B-H) technique[5] to plane geometry problems and has also made S-8 calculations. One of the more rigorous analyses in one-speed theory for the slab lattice is that of Ferziger and Robinson,[6] who used Case's method of normal modes[7,8] and solved numerically the ensuing Fredholm equations for the expansion coefficients. More recently, Carlvik[9] has used an integral transport method, which takes into account the effects of anisotropic scattering, to make disadvantage factor calculations. In addition to having merit in several geometries, the method reported by Carlvik appears to be useful for accurate numerical calculations.

It is well known that the assumption of isotropic scattering is especially violated in media containing light elements.[10] This, of course, is precisely the case in a heterogeneous reactor cell, where a fuel region is embedded in a thermalizing moderator. The assumption of isotropic scattering in the heavy fuel region is quite reasonable, but this proposition is certainly not

[1]R. V. MEGHREBLIAN and D. K. HOLMES, *Reactor Analysis*, McGraw-Hill Book Co., New York (1960).

[2]S. GLASSTONE and M. C. EDLUND, *The Elements of Nuclear Reactor Theory*, D. Van Nostrand, Princeton, N.J. (1952).

[3]G. C. POMRANING and M. CLARK, Jr., *Nucl. Sci. Eng.*, **17**, 227 (1963).

[4]M. THEYS, *Nucl. Sci. Eng.*, **7**, 58 (1960).

[5]A. AMOUYAL, P. BENOIST, and J. HOROWITZ, *J. Nucl. Ener.*, **6**, 79 (1957).

[6]J. FERZIGER and A. ROBINSON, *Nucl. Sci. Eng.*, **21**, 382 (1965).

[7]K. M. CASE, *Ann. Phys. (N.Y.)*, **9**, 1 (1960).

[8]K. M. CASE and P. F. ZWEIFEL, *Linear Transport Theory*, Addison-Wesley, Reading, Mass. (1967).

[9]I. CARLVIK, "Calculations of Neutron Flux Distributions by Means of Integral Transport Methods," AE-279, Aktiebolaget Atomenergi, Stockholm (1967).

[10]R. L. MURRAY, *Nuclear Reactor Physics*, Prentice-Hall, Englewood Cliffs, N.J. (1957).

justified in the moderator. The purpose of this paper is thus to extend the technique used by Ferziger and Robinson[6] to include the effect of a two-term scattering kernel in the moderator region.

BASIC FORMALISM

If we denote the fuel and moderator regions of the reactor cell by 1 and 2, respectively, then the appropriate transport equations are

$$\mu \frac{\partial}{\partial z} \psi_1(z,\mu) + \sigma_1\psi_1(z,\mu) = \frac{\sigma_1 c_1}{2} \int_{-1}^{1} \psi_1(z,\mu')d\mu', \quad 0 \leq z \leq \alpha, \tag{1a}$$

and

$$\mu \frac{\partial}{\partial z} \psi_2(z,\mu) + \sigma_2\psi_2(z,\mu) = \frac{\sigma_2 c_2}{2} \int_{-1}^{1} (1+\omega\mu\mu')\psi_2(z,\mu')d\mu' + S, \quad \alpha \leq z \leq \beta, \tag{1b}$$

where

α and β = the dimensions of the cell

$\psi_1(z,\mu)$ and $\psi_2(z,\mu)$ = the neutron angular densities

σ_1 and σ_2 = the total cross sections

c_1 and c_2 = the mean numbers of secondary neutrons per collision (no fission)

S = the source (assumed constant and isotropic)

z = the position coordinate (in cm)

μ = the direction cosine of the neutron velocity vector (as measured from the positive z axis)

ω = the Legendre coefficient in a two-term expansion of the scattering kernel.

We thus seek solutions to Eqs. (1) subject to the boundary conditions:

a) $\psi_1(\alpha,\mu) = \psi_2(\alpha,\mu)$

b) $\psi_1(0,-\mu) = \psi_1(0,\mu)$

c) $\psi_2(\beta,-\mu) = \psi_2(\beta,\mu)$.

The first boundary condition states that the angular density should be continuous at the interface between the two media. The second and third conditions reflect the symmetry of the problem.

For the sake of economy in notation, we prefer to write Eqs. (1) in terms of an optical variable, x, defined as

$$x = \sigma_1 z, \quad 0 \leq z \leq \alpha, \tag{2a}$$

$$x = \sigma_2 z + \alpha(\sigma_1 - \sigma_2), \quad \alpha \leq z \leq \beta. \tag{2b}$$

Thus, we obtain

$$\mu \frac{\partial}{\partial x} \psi_i(x,\mu) + \psi_i(x,\mu) = \frac{c_i}{2} \int_{-1}^{1} (1+\omega\mu\mu'\delta_{i2})\psi_i(x,\mu')d\mu' + \frac{S}{\sigma_2}\delta_{i2}, \quad i = 1, 2, \tag{3}$$

and the boundary conditions

$$\psi_1(a,\mu) = \psi_2(a,\mu), \quad \mu \in (-1,1) \tag{4a}$$

$$\psi_2(b,\mu) = \psi_2(b,-\mu), \quad \mu \in (-1,1) \tag{4b}$$

$$\psi_1(0,\mu) = \psi_1(0,-\mu), \quad \mu \in (-1,1), \tag{4c}$$

where

$$a \triangleq \sigma_1\alpha$$

$$b \triangleq a + \sigma_2(\beta-\alpha).$$

ANALYSIS

Following the work of Case and Zweifel[8] for isotropic scattering and that of McCormick and Kuščer[11] for anisotropic scattering, we write the solutions to Eq. (3) in the forms

$$\psi_1(x,\mu) = A_+\phi_+(\mu)\exp(-x/\nu_0) + A_-\phi_-(\mu)\exp(x/\nu_0) + \int_{-1}^{1} A(\nu)\phi_\nu(\mu)\exp(-x/\nu)d\nu \tag{5a}$$

$$\psi_2(x,\mu) = B_+\chi_+(\mu)\exp[(b-x)/\eta_0] + B_-\chi_-(\mu) \times \exp[-(b-x)/\eta_0] + \int_{-1}^{1} B(\eta)\chi_\eta(\mu) \times \exp[(b-x)/\eta]d\eta + \frac{S}{\sigma_2(1-c_2)}. \tag{5b}$$

Here, the discrete solutions are

$$\varphi_\pm(\mu) = \frac{c_1\nu_0}{2} \frac{1}{\nu_0 \mp \mu} \tag{6a}$$

$$\chi_\pm(\mu) = \frac{c_2\eta_0}{2} \frac{d(\pm\eta_0\mu)}{\eta_0 \mp \mu}, \tag{6b}$$

where ν_0 and η_0 are the positive zeros of

$$\Lambda_1(z) = 1 - c_1 z \tanh^{-1}\frac{1}{z} \tag{7a}$$

$$\Lambda_2(z) = d(z^2)\left(1 - c_2 z \tanh^{-1}\frac{1}{z}\right) - \omega(1-c_2)^2 z^2, \tag{7b}$$

[11] N. J. McCORMICK and I. KUŠČER, *J. Math. Phys.*, **6**, 1939 (1965).

respectively. In addition, the continuum modes are given by

$$\phi_\nu(\mu) = \frac{c_1\nu}{2}\,\frac{P}{\nu-\mu} + \lambda_1(\nu)\,\delta(\nu-\mu) \tag{8a}$$

$$\chi_\eta(\mu) = \frac{c_2\eta}{2}\,d(\eta\mu)\,\frac{P}{\eta-\mu} + \lambda_2(\eta)\,\delta(\eta-\mu)\,, \tag{8b}$$

where

$$\lambda_1(\nu) = 1 - c_1\nu\tanh^{-1}\nu \tag{9a}$$

$$\lambda_2(\eta) = d(\eta^2)(1 - c_2\eta\tanh^{-1}\eta) - \omega(1-c_2)^2\eta^2\,. \tag{9b}$$

Also,

$$d(\eta\mu) = 1 + \omega(1-c_2)\eta\mu\,. \tag{10}$$

Finally, the symbol P indicates that the various integrals are to be considered in the Cauchy principal-value sense, and $\delta(x)$ denotes the Dirac delta function.

Case[7] showed that the $\phi_\nu(\mu)$ are orthogonal, with respect to weight function μ, over the full range $-1 < \mu < 1$. The required integrals are[7]

$$\int_{-1}^{1} \mu\,\phi_\xi(\mu)\,\phi_{\xi'}(\mu)\,d\mu = 0, \qquad \xi \neq \xi' \tag{11}$$

$$\int_{-1}^{1} \mu\,\phi_\pm^2(\mu)\,d\mu = \pm N_{1+} = \pm\frac{c_1\nu_0^3}{2}\left(\frac{c_1}{\nu_0^2-1} - \frac{1}{\nu_0^2}\right) \tag{12}$$

$$\int_{-1}^{1} \mu\,\phi_\nu(\mu)\,\phi_{\nu'}(\mu)\,d\mu = N_1(\nu)\,\delta(\nu-\nu') = \nu\left[\lambda_1^2(\nu) + \left(\frac{\pi c_1\nu}{2}\right)^2\right]\delta(\nu-\nu')\,. \tag{13}$$

The corresponding forms for the $\chi_\eta(\mu)$, developed by Mika,[12] are

$$\int_{-1}^{1} \mu\,\chi_\xi(\mu)\,\chi_{\xi'}(\mu)\,d\mu = 0\,, \qquad \xi' \neq \xi \tag{14}$$

$$\int_{-1}^{1} \mu\,\chi_\pm^2(\mu)\,d\mu = \pm N_{2+} = \pm\frac{c_2\eta_0}{2} \times\left[\frac{c_2\,d^2(\eta_0^2)}{\eta_0^2-1} - (1-c_2)\,d(3\eta_0^2)\right] \tag{15}$$

$$\int_{-1}^{1} \mu\,\chi_\eta(\mu)\,\chi_{\eta'}(\mu)\,d\mu = N_2(\eta)\,\delta(\eta-\eta') = \eta\left\{\lambda_2^2(\eta) + \left[\frac{\pi c_2\eta\,d(\eta^2)}{2}\right]^2\right\}\delta(\eta-\eta')\,. \tag{16}$$

In addition, we shall need the "cross-product" integrals, which may be summarized as

$$\int_{-1}^{1} \mu\,\phi_\xi(\mu)\,\chi_{\xi'}(\mu)\,d\mu = \xi\xi'\,F(\xi\xi')\,\frac{P}{\xi'-\xi} + M(\xi)\,\xi\,\delta(\xi-\xi'),$$

$$\xi = \pm\nu_0 \text{ or } \xi = \nu\ \varepsilon\ (-1,1) \text{ and } \xi' = \pm\eta_0 \text{ or } \xi'\ \varepsilon\ (-1,1)\,, \tag{17}$$

where

$$F(\xi\xi') = \frac{c_2-c_1}{2} + \frac{c_2}{2}\,(1-c_1)(1-c_2)\,\omega\,\xi\xi' \tag{18}$$

$$M(\xi) = \lambda_1(\xi)\,\lambda_2(\xi) + \frac{\pi^2}{4}\,c_1c_2\,\xi^2\,d(\xi^2)\,. \tag{19}$$

Clearly, the symmetry conditions, Eqs. (4b) and (4c), are satisfied if $A_+ = A_-$, $B_+ = B_-$, $A(\nu) = A(-\nu)$, and $B(\eta) = B(-\eta)$. Thus, the expansion coefficients, A_+, B_+, $A(\nu)$, and $B(\eta)$, are to be determined from the continuity condition, Eq. (4a), i.e.,

$$A_+[\phi_+(\mu)\exp(-a/\nu_0) + \phi_-(\mu)\exp(a/\nu_0)] + \int_{-1}^{1} A(\nu)\exp(-a/\nu)\,\phi_\nu(\mu)\,d\nu = 1 + B_+[\chi_+(\mu)\exp(\Delta/\eta_0) + \chi_-(\mu)\exp(-\Delta/\eta_0)] + \int_{-1}^{1} B(\eta)\exp(\Delta/\eta)\,\chi_\eta(\mu)\,d\eta\,, \quad \mu\ \varepsilon\ (-1,1)\,, \tag{20}$$

where $\Delta = b - a$; we have, without loss of generality, taken $S = \sigma_2(1 - c_2)$. That Eq. (20) has a solution is a consequence of the completeness theorems proved for the eigenfunctions, $\phi_\nu(\mu)$ and $\chi_\eta(\mu)$, by Case[7] and Mika.[12] All of the unknown expansion coefficients can be determined from Eq. (20), and this, of course, is the basic problem; however, we digress for a moment to formulate the expression for the disadvantage factor.

Since the eigenfunctions are normalized to unity,[8] the neutron densities are obtained immediately by integration of Eqs. (5). Keeping in mind the symmetry of the expansion coefficients and the source normalization, we write

$$\rho_1(x) = 2A_+\cosh x/\nu_0 + 2\int_0^1 A(\nu)\cosh x/\nu\,d\nu \tag{21a}$$

$$\rho_2(x) = 2B_+\cosh\left(\frac{b-x}{\eta_0}\right) + 2\int_0^1 B(\eta)\times\cosh\left(\frac{b-x}{\eta}\right)d\eta + 2\,. \tag{21b}$$

Thus, the disadvantage factor,

$$\zeta \triangleq \frac{\langle\rho_2\rangle}{\langle\rho_1\rangle} = \frac{a}{\Delta}\,\frac{\int_a^b \rho_2(x)\,dx}{\int_0^a \rho_1(x)\,dx}\,, \tag{22}$$

becomes

$$\zeta = \frac{a}{\Delta}\left[\frac{\eta_0B_+\sinh\Delta/\eta_0 + \int_0^1 \eta B(\eta)\sinh\Delta/\eta\,d\eta + \Delta}{\nu_0A_+\sinh a/\nu_0 + \int_0^1 \nu A(\nu)\sinh a/\nu\,d\nu}\right]. \tag{23}$$

We write Eq. (23) in a more convenient form by eliminating the discrete coefficients A_+ and B_+,

[12] J. R. MIKA, *Nucl. Sci. Eng.*, **11**, 415 (1961).

following the procedure used by Ferziger and Robinson,[6] i.e., by taking the zeroth and first moments of Eq. (20) with respect to μ, we can express A_+ and B_+ in terms of $A(\nu)$ and $B(\eta)$. When this is done and the results substituted into Eq. (23), there results

$$\zeta = \frac{\frac{a}{\Delta}\left\{\frac{(1-c_1)}{(1-c_2)}\left[\frac{\Delta}{\eta_0}\coth\frac{\Delta}{\eta_0} + K - 1\right] + \frac{\Delta}{\nu_0}\coth\frac{a}{\nu_0}\right\}}{1-K}, \quad (24)$$

where

$$K = A_0 - B_0 - \frac{A_1}{\nu_0}\coth\frac{a}{\nu_0} + \frac{B_1}{\eta_0}\coth\frac{\Delta}{\eta_0} \quad (25)$$

$$A_0 = \int_0^1 A(\nu)\cosh\frac{a}{\nu}\,d\nu \quad (26a)$$

$$A_1 = \int_0^1 A(\nu)\,\nu\sinh\frac{a}{\nu}\,d\nu \quad (26b)$$

$$B_0 = \int_0^1 B(\eta)\cosh\frac{\Delta}{\eta}\,d\eta \quad (26c)$$

$$B_1 = \int_0^1 B(\eta)\,\eta\sinh\frac{\Delta}{\eta}\,d\eta\,. \quad (26d)$$

Clearly, the lowest order solution for ζ is obtained by taking $K = 0$; however, for a more rigorous calculation, we must find $A(\nu)$ and $B(\eta)$. Though we cannot obtain these expansion coefficients explicitly, Fredholm integral equations for them can be generated, and by iterating these Fredholm equations, we can obtain the solutions for $A(\nu)$ and $B(\eta)$ to any desired degree of accuracy.

The usual technique for isolating the expansion coefficients, namely, employing the orthogonality of the eigenfunctions, yields a pair of coupled singular integral equations for $A(\nu)$ and $B(\eta)$. Robinson[13] was able to circumvent the singular nature of these equations by judicious algebraic manipulation. We take a slightly different tack, which yields similar results, by utilizing an integral operator $H(\alpha,\beta)$; if $G(\mu)$ is an arbitrary function defined for $\mu\ \varepsilon(-1,1)$, then

$$H(\alpha,\beta)G(\mu) \triangleq \int_{-1}^1 \mu\left[\Omega_\alpha(\mu)\exp(-\beta/\alpha) + \Omega_{-\alpha}(\mu) \times \exp(\beta/\alpha)\right] G(\mu)\,d\mu\,, \quad (27)$$

where $\Omega_\alpha(\mu)$ represents either set $\phi_\nu(\mu)$ or $\chi_\eta(\mu)$. [We will always associate the parameter ν with $\phi_\nu(\mu)$ and η with $\chi_\eta(\mu)$.]

By operating on Eq. (20) with $H(\nu_0,\Delta)$, $H(\eta_0,-a)$, $H(\nu,\Delta)$, and $H(\eta,-a)$ and then rearranging the resulting equations, we obtain, respectively, the following:

$$\tilde{A}_+ N_{+1} T(\nu_0) = Q(\nu_0,c_1,\Delta) + \tilde{B}_+ R(\eta_0,\nu_0,\Delta) + \int_0^1 \tilde{B}(\eta)\,R(\eta,\nu_0,\Delta)\,d\eta \quad (28)$$

$$-\tilde{B}_+ N_{+2} T(\eta_0) = Q(\eta_0,c_2,a) + \tilde{A}_+ R(\nu_0,\eta_0,a) + \int_0^1 \tilde{A}(\nu)\,R(\nu,\eta_0,a)\,d\nu \quad (29)$$

$$\tilde{A}(\nu)N_1(\nu)T(\nu) = Q(\nu,c_1,\Delta) + \tilde{B}_+ R(\eta_0,\nu,\Delta) + \int_0^1 \tilde{B}(\eta)\,R(\eta,\nu,\Delta)\,d\eta \quad (30)$$

$$-\tilde{B}(\eta)N_2(\eta)T(\eta) = Q(\eta,c_2,a) + \tilde{A}_+ R(\nu_0,\eta,a) + \int_0^1 \tilde{A}(\nu)\,R(\nu,\eta,a)\,d\nu\,, \quad (31)$$

where

$$\tilde{A}_+ = A_+\cosh a/\nu_0 \quad (32a)$$

$$\tilde{A}(\nu) = A(\nu)\cosh a/\nu \quad (32b)$$

$$\tilde{B}_+ = B_+\cosh \Delta/\eta_0 \quad (33a)$$

$$\tilde{B}(\eta) = B(\eta)\cosh \Delta/\eta_0\,, \quad (33b)$$

$$T(x) = \tanh\frac{a}{x} + \tanh\frac{\Delta}{x} \quad (34)$$

$$Q(x,y,z) = x(1-y)\tanh z/x \quad (35)$$

$$R(x,y,z) = \frac{2xy}{x^2-y^2}\left[xF(y^2)\tanh\frac{z}{y} - yF(x^2)\tanh\frac{z}{x}\right]. \quad (36)$$

We note that the function $R(x,y,z)$ is nonsingular, and thus standard quadrature formulae may be used to evaluate the integrals appearing in Eqs. (28) through (31). The numerical solution of these four coupled equations is considered in the following section.

NUMERICAL SOLUTION

While one may partially decouple Eqs. (28) through (31) by repeated substitution, there appears to be no particular merit in doing so, since the resulting equations must still be solved numerically. There is, of course, no general proof that Eqs. (28) through (31) have a *unique* solution; this point, however, can be circumvented by invoking the uniqueness theorems established by Case and Zweifel[8] for the complete solution $\psi_i(x,\mu)$. The procedure used here to solve Eqs. (28) through (31) is to replace all integral terms by an appropriate quadrature formula and to solve by iteration the ensuing equations.

We initiate the solution scheme by neglecting

[13] A. ROBINSON, "Transport Calculation of the Disadvantage Factor," PhD thesis, Stanford University (1965).

the continuum contribution in Eqs. (28) and (29); the lowest order estimates for the discrete coefficients $\tilde{A}_+$ and $\tilde{B}_+$ are thus readily available. These values are then entered into Eqs. (30) and (31), and initial estimates of $\tilde{A}(\nu)$ and $\tilde{B}(\eta)$ are found by ignoring the integral terms. We now proceed through each of the four equations, obtaining new estimates of the coefficients from the last iterated values. This process is repeated until the difference in successive iterations is less than some preassigned number.

This scheme has been used to compute the disadvantage factor for a number of slab cells and for several values of the anisotropy factor ω. The results of our calculations are shown in Table I. There, the basic cells are those introduced by Theys[4] and studied by others.[3,6,9,13]

In generating the results given in Table I, we have employed an 81-point improved Gaussian quadrature formula[14] for the numerical integrations, and we have performed the computations in double-precision arithmetic on an IBM 360/75 computer. A need for a relatively high-order quadrature formula was suggested by preliminary results which indicated a sharp peak, generally occurring for ν between 0.8 and unity, in the coefficient $A(\nu)$. A number of test cases showed that increasing the quadrature nodal density beyond the 81-point scheme failed to alter the disadvantage factor within the significant figures reported here.

[14]A. S. KRONROD, *Nodes and Weights of Quadrature Formulas*, Consultants Bureau, Inc., New York (1965).

The iterative procedure outlined above was terminated when successive iterations showed a relative difference of $< 10^{-10}$. This is, of course, no assurance that the coefficients thus determined reflect this degree of precision; as will be discussed, additional measures are needed to increase confidence in the calculations. It is clear that the proposed solutions, Eqs. (5), satisfy the given transport equations. In addition, the symmetry properties of the expansion coefficients guarantee that the reflection conditions, Eqs. (4), are met rigorously. Thus, the only possible error is measured by how accurately we can meet the interface condition, Eq. (20). This boundary condition is, of course, precisely the one that led to the coupled set of integral equations for the unknown expansion coefficients. Having obtained the expansion coefficients numerically, we should now like to estimate the degree of accuracy with which our solution meets the remaining boundary condition.

Ideally, one would like to compute $\psi_1(a,\mu)$ and $\psi_2(a,\mu)$ explicitly for many values of μ to ensure that the continuity condition is satisfied. This would, however, necessitate the numerical evaluation of principal-value integrals; we prefer to avoid this by comparing numerous moments of the continuity condition rather than $\psi_1(a,\mu)$ and $\psi_2(a,\mu)$ directly. Defining

$$M_k = \int_{-1}^{1} \psi_1(a,\mu)\mu^k d\mu \Big/ \int_{-1}^{1} \psi_2(a,\mu)\mu^k d\mu \,, \qquad (37)$$

we note that it can be written in terms of the

TABLE I

The Disadvantage Factor of Slab Cells with Linearly Anisotropic Scattering

Calculational Model	Anisotropy Coefficient ω	ξ, the Disadvantage Factor			
		Cell 1[a] α = 0.10 cm β = 0.45 cm	Cell 2[a] α = 0.20 cm β = 0.90 cm	Cell 3[a] α = 0.30 cm β = 1.35 cm	Cell 4[a] α = 0.40 cm β = 1.80 cm
P_1 theory	0.0	1.028	1.113	1.253	1.447
Converged solution	0.0	1.0978	1.2317	1.4077	1.6284
P_1 theory	0.1	1.027	1.110	1.245	1.433
Converged solution	0.1	1.0970	1.2283	1.4001	1.6151
P_1 theory	0.3	1.026	1.103	1.230	1.407
Converged solution	0.3	1.0953	1.2215	1.3849	1.5885
P_1 theory	0.6	1.023	1.093	1.207	1.366
Converged solution	0.6	1.0927	1.2113	1.3621	1.5485
P_1 theory	0.9	1.021	1.082	1.184	1.326
Converged solution	0.9	1.0901	1.2010	1.3392	1.5083

[a] $\Sigma_a^{\text{fuel}} = 0.32/\text{cm}$ $\Sigma_T^{\text{fuel}} = 0.717/\text{cm}$

$\Sigma_a^{\text{mod}} = 0.0195/\text{cm}$ $\Sigma_T^{\text{mod}} = 2.33/\text{cm}$.

already calculated expansion coefficients,

$$\bar{\phi}_k(\nu) \triangleq \int_{-1}^{1} \phi_\nu(\mu)\mu^k d\mu \tag{38a}$$

and

$$\bar{\chi}_k(\eta) \triangleq \int_{-1}^{1} \chi_\eta(\mu)\mu^k d\mu \, . \tag{38b}$$

It is a simple matter to show that $\bar{\phi}_k(\nu)$ and $\bar{\chi}_k(\eta)$ satisfy, respectively, the following recurrence relations:

$$\bar{\phi}_k(\nu) = \nu\bar{\phi}_{k-1}(\nu) - \frac{c_1\nu}{2} J_{k-1} \, , \quad k \geq 1 \, , \tag{39a}$$

$$\bar{\chi}_k(\eta) = \eta\bar{\chi}_{k-1}(\eta) - \frac{c_2\eta}{2} J_{k-1} - \frac{\omega c_2(1-c_2)}{2} \eta^2 J_k \, , \quad k \geq 1 \, , \tag{39b}$$

where

$$J_k = \frac{1 + (-1)^k}{k+1} \tag{40}$$

$$\bar{\phi}_0(\nu) = \bar{\chi}_0(\eta) = 1 \, . \tag{41}$$

Should the coefficients be precisely determined, then clearly M_k would equal unity for all k. In the present work, we have calculated the first ten moments of the interface condition for each of the cells shown in Table I. In the worst case, our results showed $|M_k - 1|_{\max} < 2 \times 10^{-6}$. For this reason, we believe our results for the disadvantage factor to be accurate to as many figures as reported.

In Table II, we have compared our results to previously reported calculations for the special case of isotropic scattering ($\omega = 0$). The values of Ferziger and Robinson[6] shown in the table are also based on Case's method. Their procedure, while more rapidly convergent than ours, required numerical estimation of principal-value integrals and utilized lower-order quadrature.

Inspection of Table I reveals that, for the cases considered here, the disadvantage factor decreases with moderator anisotropy, an effect that we would have anticipated from the P_1-theory results. It should be noted that only positive values of the anisotropy factor ω have been investigated. For the most frequently used scattering law, i.e., isotropic scattering in the center-of-mass system, this will always be the case.

TABLE II

The Disadvantage Factor of Isotropically Scattering Slab Cells by Various Computational Methods

Computational Method	Cell 1	Cell 2	Cell 3	Cell 4
P_1 theory	1.028	1.113	1.253	1.447
Pomraning and Clark[3] Asymptotic diffusion theory	1.06	1.18	1.34	1.56
Theys[4] Modified A-B-H method	1.08	1.20	1.36	1.58
Theys[4] S_8 calculation	1.09	1.23	1.43	1.64
Perkins[6] S_8 calculation	1.069	1.203	1.382	1.605
Lathrop[13] S_8 calculation	1.090	1.231	1.410	1.632
Carlvik[9] Integral transport theory	1.0979	1.2318	1.408	1.629
Ferziger and Robinson[a] Case's method	1.094	1.227	1.401	1.623
Converged solution	1.0978	1.2317	1.4077	1.6284

[a]A. H. Robinson (Private Communication) has established these as the correct values rather than those reported in Ref. 6.

Certainly there are a number of other methods, usually utilizing discrete ordinates and/or difference-equation techniques, for treating anisotropic scattering in slab cells. We believe our results to be of sufficient accuracy to serve as *bench marks* in assessing the relative merits of these various approximations.

In addition, we have further illustrated that Case's normal-mode expansion technique can be conveniently used in reactor computations. For practical applications, one would probably elect to reduce the quadrature order below that which we have used, sacrificing but a small degree of accuracy for considerable savings in computer time.

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