Pergamon

J. Quant. Spectrosc. Radiat. Transfer Vol. 57, No. 3, pp. 399–404, 1997

PII: S0022-4073(96)00122-7
0022-4073/97 $17.00 + 0.00

SOME IDENTITIES FOR CHANDRASEKHAR POLYNOMIALS

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(*Received 22 April 1996*)

Abstract—Basic techniques of linear algebra are used to derive some identities involving the Chandrasekhar polynomials that play a vital role in the spherical-harmonics (P_N) solution to basic radiative-transfer problems. ©1997 Elsevier Science Ltd. All rights reserved

1. INTRODUCTION

The normal procedure[1] in solving radiative-transfer problems that lack azimuthal symmetry is to develop a Fourier decomposition of the intensity and then to solve a set of problems defined by the Fourier index m, where $m = 0, 1, 2, \ldots, L$ and where L is the order of a Legendre expansion of the scattering law. Basic to solving these Fourier-component problems by, for example, the spherical-harmonics or the discrete-ordinates method, is a class of polynomials introduced into the field of radiative transfer by Chandrasekhar.[1]

In a recent paper,[2] various identities concerning the Chandrasekhar polynomials were reported; however, in that work errors made in deriving the identities caused many of those results for $m > 0$ to be incorrect.[3] In this work we report a superior way of establishing correct versions of the mentioned identities. First of all, we choose here to base our development of the identities on the *normalized* Chandrasekhar polynomials used by Garcia and Siewert[4] in devising a sound computational scheme for evaluating the polynomials. Secondly, and more importantly, in using elementary linear-algebra techniques to provide an alternative (and more direct) derivation of all of the identities discussed in Ref. 2, we also avoid the need to extend to the cases $m > 0$ some results in a work by Inönü[5] that concerned only $m = 0$.

Following Garcia and Siewert,[4] we define the *normalized* Chandrasekhar polynomials $g_l^m(\xi)$ by the three-term recursion relation

$$(l^2 - m^2)^{1/2} g_{l-1}^m(\xi) - h_l \xi g_l^m(\xi) + [(l+1)^2 - m^2]^{1/2} g_{l+1}^m(\xi) = 0, \tag{1}$$

for $l \geq m$, and the (arbitrary) starting value

$$g_m^m(\xi) = (2m-1)!![(2m)!]^{-1/2}. \tag{2}$$

Here

$$h_l = 2l + 1 - \varpi\beta_l, \quad \text{for} \quad 0 \leq l \leq L, \tag{3a}$$

and

$$h_l = 2l + 1, \quad \text{for} \quad l > L. \tag{3b}$$

In addition, $\varpi \in [0, 1)$ is the albedo for single scattering, and the β_l, with $\beta_0 = 1$, and $|\beta_l| < 2l + 1$ for $0 < l \leq L$, are coefficients in an L-th-order expansion of the scattering law.[1] We note that for the values of ϖ and $\{\beta_l\}$ considered, we have $h_l > 0$ for all l; our work here is also valid for all

$m > 0$ for the special case of $\varpi = 1$, but some minor modifications (discussed later) are required to include the case of $m = 0$ with $\varpi = 1$.

We note that the normalized g polynomials used here correspond (except for the choice of $g_m^m(\xi)$, the arbitrary starting value) to those of Chandrasekhar[1] multiplied by the factor

$$\Delta_{l,m} = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2}. \tag{4}$$

2. THE IDENTITIES

We find it convenient to let

$$y_l^m(\xi) = (h_l)^{1/2} g_l^m(\xi) \tag{5}$$

and

$$W_l^m = (l^2 - m^2)^{1/2}(h_l h_{l-1})^{-1/2} \tag{6}$$

for $l \geq m$, where $W_0^0 \equiv 0$, so we can rewrite Eq. (1) as

$$W_l^m y_{l-1}^m(\xi) - \xi y_l^m(\xi) + W_{l+1}^m y_{l+1}^m(\xi) = 0. \tag{7}$$

We note that the N-th-order spherical-harmonics approximation to the solution of a radiative-transfer problem requires the eigenvalues $\{\xi_j\}$ that are defined as the zeros of $g_{M+1}^m(\xi)$, where $M = N + m$. Thus, if we consider Eq. (7) for $l = m, m+1, \ldots, M$ and use the truncation condition $y_{M+1}^m(\xi) = 0$, we find the eigenvalue problem

$$\mathbf{W}\boldsymbol{\Gamma}(\xi) = \xi\boldsymbol{\Gamma}(\xi) \tag{8}$$

where the symmetric tridiagonal matrix $\mathbf{W}$ has all zeros on the diagonal and has the elements

$$W_{m+1}^m, W_{m+2}^m, \ldots, W_M^m$$

on diagonals just above and below the main diagonal. In addition, the vector $\boldsymbol{\Gamma}(\xi)$ has elements

$$y_m^m(\xi), y_{m+1}^m(\xi), \ldots, y_M^m(\xi).$$

Since the tridiagonal matrix $\mathbf{W}$ is symmetric with nonzero elements on the off-diagonals, we know[6] that the eigenvalues of $\mathbf{W}$ are distinct and that the eigenvectors $\boldsymbol{\Gamma}(\xi_j)$ corresponding to the eigenvalues ξ_j, for $j = 1, 2, \ldots, N+1$, are orthogonal. It follows that if we introduce the normalized vectors

$$\mathbf{X}_j = \boldsymbol{\Gamma}(\xi_j)/\eta_j \tag{9}$$

where

$$\eta_j = \left\{\sum_{l=m}^{M} [y_l^m(\xi_j)]^2\right\}^{1/2}, \tag{10}$$

then the matrix $\mathbf{X}$, with columns $\mathbf{X}_j$, for $j = 1, 2, \ldots, N+1$, is such that, if we use superscript T to denote the transpose operation, we can write

$$\mathbf{X}^{-1} = \mathbf{X}^T. \tag{11}$$

The fact that the eigenvectors $\boldsymbol{\Gamma}(\xi_j)$ are orthogonal gives us our first identity, viz.

$$C_j \sum_{l=m}^{M} y_l^m(\xi_i) y_l^m(\xi_j) = 2\delta_{i,j} \tag{12}$$

for $i, j = 1, 2, \ldots, N+1$, and where $\delta_{i,j}$ is the Kronecker delta. Clearly

$$C_j = 2\left\{\sum_{l=m}^{M} [y_l^m(\xi_j)]^2\right\}^{-1} \tag{13}$$

for $j = 1, 2, \dots, N+1$. Of course, we can rewrite Eqs. (12) and (13) in terms of the g polynomials:

$$C_j \sum_{l=m}^{M} h_l g_l^m(\xi_i) g_l^m(\xi_j) = 2\delta_{i,j} \tag{14}$$

and

$$C_j = 2\Big\{ \sum_{l=m}^{M} h_l [g_l^m(\xi_j)]^2 \Big\}^{-1} \tag{15}$$

for $j = 1, 2, \dots, N+1$.

In order to obtain additional identities we now consider a system of linear algebraic equations of the form

$$\sum_{j=1}^{N+1} D_j \gamma_l^m(\xi_j) = f_l \tag{16}$$

for $l = m, m+1, \dots, M$, where, for the moment, the f_l are unspecified. If we let $\mathbf{\Gamma}$ denote the matrix with columns $\mathbf{\Gamma}(\xi_j)$, for $j = 1, 2, \dots, N+1$, then clearly

$$\mathbf{\Gamma} = \mathbf{XN} \tag{17}$$

where $\mathbf{N} = \text{diag}\{\eta_1, \eta_2, \dots, \eta_{N+1}\}$. Defining $\mathbf{D}$ to be a vector with elements D_j and $\mathbf{F}$ as a vector with the f_l as components, we can write Eq. (16) as

$$\mathbf{\Gamma D} = \mathbf{F}, \tag{18}$$

and so, after noting Eqs. (11) and (17), we can write

$$\mathbf{D} = \mathbf{N}^{-1}\mathbf{X}^T\mathbf{F}. \tag{19}$$

If we now select $f_l = \delta_{l,\alpha}$, for $\alpha = m, m+1, \dots, M$, then Eq. (19) yields

$$D_j = \frac{1}{2} C_j \gamma_\alpha^m(\xi_j). \tag{20}$$

Thus, with this choice of f_l, Eq. (16) yields our second identity, viz.

$$\sum_{j=1}^{N+1} C_j \gamma_\alpha^m(\xi_j) \gamma_l^m(\xi_j) = 2\delta_{\alpha,l} \tag{21}$$

for $\alpha, l = m, m+1, \dots, M$ or in terms of the g polynomials,

$$h_l \sum_{j=1}^{N+1} C_j g_\alpha^m(\xi_j) g_l^m(\xi_j) = 2\delta_{\alpha,l} \tag{22}$$

for $\alpha, l = m, m+1, \dots, M$. If we now select

$$f_l = W_\alpha^m \delta_{l,\alpha-1} + W_{\alpha+1}^m \delta_{l,\alpha+1} \tag{23}$$

for $\alpha = m, m+1, \dots, M$, then Eq. (19) yields, after we note Eq. (7),

$$D_j = \frac{1}{2} C_j \xi_j \gamma_\alpha^m(\xi_j). \tag{24}$$

Thus, with this choice of f_l, Eq. (16) yields our third identity, viz.

$$\sum_{j=1}^{N+1} C_j \xi_j \gamma_\alpha^m(\xi_j) \gamma_l^m(\xi_j) = 2[W_\alpha^m \delta_{l,\alpha-1} + W_{\alpha+1}^m \delta_{l,\alpha+1}] \tag{25}$$

or

$$h_l h_\alpha \sum_{j=1}^{N+1} C_j \xi_j g_\alpha^m(\xi_j) g_l^m(\xi_j) = 2[U_\alpha^m \delta_{l,\alpha-1} + U_{\alpha+1}^m \delta_{l,\alpha+1}] \tag{26}$$

for $\alpha, l = m, m+1, \dots, M$, where

$$U_l^m = (l^2 - m^2)^{1/2}, \quad \text{for} \quad l \geq m. \tag{27}$$

It is clear that other choices for the f_l will lead to additional identities concerning the g polynomials.

3. THE IDENTITIES FOR THE CASE OF N ODD

Rather than seeking more identities, we now consider that N is odd, so that the eigenvalues occur in $\pm$ pairs. We let ξ_k, for $k = 1, 2, \dots, J = (N+1)/2$, denote the positive eigenvalues, and since $g_l^m(-\xi) = (-1)^{l-m} g_l^m(\xi)$, it is clear that restricting N to be odd allows us to deduce from Eq. (14) that

$$C_j \sum_{l=m}^{M} h_l g_l^m(\xi_i) g_l^m(\xi_j) = 2\delta_{i,j} \tag{28}$$

and

$$\sum_{l=m}^{M} (-1)^{l-m} h_l g_l^m(\xi_i) g_l^m(\xi_j) = 0 \tag{29}$$

for $i, j = 1, 2, \dots, J$. It follows that if we wish only the C_k, for $k = 1, 2, \dots, J$, that correspond to positive eigenvalues, then we can use Eq. (29) to rewrite Eq. (15) as

$$C_j = \left\{ \sum_{k=1}^{J} h_{m+2k-2} [g_{m+2k-2}^m(\xi_j)]^2 \right\}^{-1} \tag{30a}$$

or

$$C_j = \left\{ \sum_{k=1}^{J} h_{m+2k-1} [g_{m+2k-1}^m(\xi_j)]^2 \right\}^{-1} \tag{30b}$$

for $j = 1, 2, \dots, J$. Also, we can rewrite Eqs. (22) and (26) as

$$[1 + (-1)^{\alpha+l}] h_l \sum_{j=1}^{J} C_j g_\alpha^m(\xi_j) g_l^m(\xi_j) = 2\delta_{\alpha,l} \tag{31}$$

and

$$[1 - (-1)^{\alpha+l}] h_l h_\alpha \sum_{j=1}^{J} C_j \xi_j g_\alpha^m(\xi_j) g_l^m(\xi_j) = 2[U_\alpha^m \delta_{l,\alpha-1} + U_{\alpha+1}^m \delta_{l,\alpha+1}] \tag{32}$$

for $\alpha, l = m, m+1, \dots, M$.

4. THE IDENTITIES FOR THE CASE OF $\varpi = 1$

The one special case for which our identities must be modified corresponds to the case of $\varpi = 1$ with $m = 0$. For this case we see, first of all, that the factor W_1^0 that appears (twice) in the **W** matrix is unbounded, and so two of the eigenvalues coalesce at infinity. Considering the finite eigenvalues $\{\xi_j\}$, for $j = 2, 3, \dots, N+1$, we see that $g_1^0(\xi_j) = 0$ and $g_2^0(\xi_j) = -1/2$; it follows that if we let **V** denote the matrix obtained from the **W** matrix by deleting the first two rows and the first two columns, then we can deduce from Eq. (8) that

$$\mathbf{V}\boldsymbol{\Gamma}(\xi) = \xi \boldsymbol{\Gamma}(\xi) \tag{33}$$

where now the components of $\boldsymbol{\Gamma}(\xi)$ are

$$y_2^0(\xi), y_3^0(\xi), \dots, y_N^0(\xi).$$

It follows that for the considered case, we find new versions of Eqs. (14) and (15), viz.

$$C_j \sum_{l=2}^{N} h_l g_l^0(\xi_i) g_l^0(\xi_j) = 2\delta_{i,j} \tag{34}$$

for $i, j = 2, 3, \dots, N+1$ and

$$C_j = 2\left\{ \sum_{l=2}^{N} h_l [g_l^0(\xi_j)]^2 \right\}^{-1} \tag{35}$$

for $j = 2, 3, \ldots, N+1$. It is clear that similar modifications to Eqs. (22) and (26) yield

$$h_l \sum_{j=2}^{N+1} C_j g_\alpha^0(\xi_j) g_l^0(\xi_j) = 2\delta_{\alpha,l} \tag{36}$$

and

$$h_l h_\alpha \sum_{j=2}^{N+1} C_j \xi_j g_\alpha^0(\xi_j) g_l^0(\xi_j) = 2[\alpha\delta_{l,\alpha-1} + (\alpha+1)\delta_{l,\alpha+1}] \tag{37}$$

for $\alpha, l = 2, 3, \ldots, N$.

If we now consider N to be odd, then we find from Eqs. (28)-(32) the special forms required here, viz.

$$C_j \sum_{l=2}^{N} h_l g_l^0(\xi_i) g_l^0(\xi_j) = 2\delta_{i,j} \tag{38}$$

and

$$\sum_{l=2}^{N} (-1)^l h_l g_l^0(\xi_i) g_l^0(\xi_j) = 0 \tag{39}$$

for $i, j = 2, 3, \ldots, J$, and

$$C_j = \left\{ \sum_{k=2}^{J} h_{2k-2}[g_{2k-2}^0(\xi_j)]^2 \right\}^{-1} \tag{40a}$$

and

$$C_j = \left\{ \sum_{k=2}^{J} h_{2k-1}[g_{2k-1}^0(\xi_j)]^2 \right\}^{-1} \tag{40b}$$

for $j = 2, 3, \ldots, J$, and

$$[1 + (-1)^{\alpha+l}] h_l \sum_{j=2}^{J} C_j g_\alpha^0(\xi_j) g_l^0(\xi_j) = 2\delta_{\alpha,l} \tag{41}$$

and

$$[1 - (-1)^{\alpha+l}] h_l h_\alpha \sum_{j=2}^{J} C_j \xi_j g_\alpha^0(\xi_j) g_l^0(\xi_j) = 2[\alpha\delta_{l,\alpha-1} + (\alpha+1)\delta_{l,\alpha+1}] \tag{42}$$

for $\alpha, l = 2, 3, \ldots, N$.

5. CONCLUDING REMARKS

There is a vast literature concerning polynomials defined by three-term recursion formulas, and so the identities reported here are likely included as special cases of existing identities developed in a more general setting. In a similar vein, the theory of symmetric tridiagonal matrices with non-vanishing off-diagonal elements is also a highly developed[6] area of mathematics, and so again the identities reported here can probably be found as special cases of a more general theory. Still, we believe the explicit forms for the identities we have reported can be useful in radiative transfer where, in addition to possible analytical uses, the identities may be of value in trying to evaluate the accuracy of numerical calculations of the Chandrasekhar polynomials.

Acknowledgements— One of the authors (CES) would like to express his thanks to L. B. Barichello and A. H. Karp for some helpful discussions concerning this work.

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