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The F_N method in atmospheric radiative transfer

R.D.M. Garcia [a], C.E. Siewert [b]

[a]Centro Técnico Aeroespacial, Instituto de Estudos Avançados, 12231-970 São José dos Campos, SP, Brazil
[b]Mathematics Department, North Carolina State University, Raleigh, NC, 27695-8205, USA

Abstract

The F_N method for solving atmospheric radiative-transfer problems is reviewed. In particular, a new choice of basis functions and collocation points that was found to perform very well for a class of problems characterized by highly anisotropic scattering (a standing challenge to the method) is reported, along with some improved computational techniques.

1. Introduction

The F_N method was introduced in the field of atmospheric radiative transfer in 1978 by Siewert [1], who solved the azimuthally symmetric equation of transfer for a homogeneous atmospheric layer subject to specified radiation intensities incident on both boundaries.

Essentially, the F_N method, as applied to radiative transfer in homogeneous plane-parallel atmospheres, begins by reducing the integro-differential equation of transfer and associated boundary conditions to a system of singular integral equations and constraints for the radiation intensities that emerge from the top and the bottom of the atmosphere. Next, these unknown exiting intensities are approximated by a set of basis functions, and a resulting system of linear algebraic equations, obtained by collocation, is solved to establish the coefficients of the approximation. Once the exiting intensities become available, a similar procedure can be used to derive a system of singular integral equations and constraints relating the intensity at any position inside the atmosphere to the exiting intensities. Again, the interior radiation intensities are approximated by a set of basis functions, and a system of linear algebraic equations is obtained for the coefficients of the approximation. The related matrix of coefficients turns out to be position independent, and thus one LU factorization is sufficient for computing the intensities at any desired number of interior points [2,3].

The extension of the method for studying inhomogeneous atmospheres modeled by a number, say K, of homogeneous layers in multilayer geometry is straightforward. Of course, in

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addition to satisfying the boundary conditions, here the intensity has to be continuous across the $K-1$ layer interfaces. There are two different computational approaches that can be used to implement the F_N solution for this problem. In the *direct* approach [4], all layers are considered simultaneously, resulting in a system of $2(N+1)K$ linear algebraic equations, to be solved for the coefficients of the F_N approximations to the exiting and interface intensities. In the *iterative* approach [5], the problem is solved one layer at a time, and the results are iterated along the layers by means of a sweep technique. More specifically, initial guesses are made for the unknown intensities incident on the surfaces of each layer, the LU factorizations of K matrices (one matrix per layer) of order $2(N+1)$ are computed and stored, and a sequence of spatial sweeps through the layers is performed in order to update the surface intensities, until convergence within a prescribed tolerance is achieved. Starting with the top layer, a sweep runs downward through all the other layers and then upward, back to the top layer. In terms of computational work, each sweep requires the solution of $2K-1$ linear systems of order $2(N+1)$, using the stored LU factorizations. In both approaches, once the exiting and interface intensities have been determined, the interior intensities for each atmospheric layer can be computed as described for the single-layer case.

In 1980 the F_N method was extended by Devaux and Siewert [6] in order to solve azimuthally asymmetric problems. A generalization of the Fourier decomposition approach of Chandrasekhar [7] was used to reduce the problem to a series of problems without azimuthal dependence [6]. However, since in Ref. [6] the powers μ^α, $\alpha = 0,1,\ldots,N$, multiplied by the factor $(1-\mu^2)^{m/2}$, where m denotes the Fourier index, were used as basis functions, the linear systems turned out to be badly conditioned for $N > 15$, so that only problems defined by few-term phase functions could be adequately solved with this version of the method. This difficulty was overcome in a subsequent work [8], through the use of the shifted-Legendre basis $P_\alpha(2\mu-1)$, $\alpha = 0,1,\ldots,N$, multiplied by $(1-\mu^2)^{m/2}$, and a collocation scheme based on the zeros of the Chebyshev polynomials. In 1985, this version of the method was used to generate benchmark results [9] for various haze and cloud problems posed by the Radiation Commission of the International Association of Meteorology and Atmospheric Physics [10]. However, in order to get good results for the cloud problems, which were defined by a phase function that required 300 terms in a Legendre expansion to be well represented, a linear equation solver based on the singular-value decomposition technique had to be used in place of the standard Gauss elimination technique [9]. More recently, while trying to implement the method for solving azimuthally asymmetric cloud problems (the cloud problems studied in Ref. [9] are azimuthally symmetric), we have discovered that the method, as used in Ref. [9], is not computationally viable for solving that class of problems. In addition, after performing an extensive analysis of newly generated results for the haze problems solved in Ref. [9], we concluded that the accuracy of the method for the intermediate and high order components in a Fourier decomposition of the intensity for these problems, is not as good as thought in the past. Although the magnitude of these components was found to be relatively small, so that the six-figure results reported for the radiation intensity in Ref. [9] are still valid, we were not happy with this situation and decided to reconsider the aspects of the method that were not working as desired. As a result of our study, we report in this paper several improvements that we have introduced in the method, in order to make it work for azimuthally asymmetric radiative-transfer problems with highly anisotropic scattering.

Finally, we would like to mention that the F_N method has also been developed for solving radiative-transfer problems with polarization [11], but this class of problems will not be considered here.

2. Formulation of the problem

We let $I(\tau, \mu, \phi)$ denote the intensity (radiance) of the radiation field and utilize the equation of transfer [7] for a plane-parallel medium to model our atmosphere. We thus write

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu, \phi) + I(\tau, \mu, \phi) = \frac{\varpi}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos \Theta) I(\tau, \mu', \phi') \mathrm{d}\phi' \mathrm{d}\mu' \tag{1}$$

where $\tau \in (0, \tau_0)$ is the optical variable and $\mu \in [-1,1]$ and $\phi \in [0,2\pi]$ are, respectively, the cosine of the polar angle (as measured from the *positive* τ axis) and the azimuthal angle which describe the direction of propagation of the radiation. In addition, $\varpi \in [0,1]$ is the albedo for single scattering, and we assume that the phase function $p(\cos \Theta)$ can be represented by a finite Legendre polynomial expansion in terms of the scattering angle Θ, viz.

$$p(\cos \Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos \Theta), \tag{2}$$

where the coefficients are such that $\beta_0 = 1$ and $|\beta_l| < 2l + 1,\ 0 < l \leq L$.

We also assume that the atmosphere is illuminated uniformly on the top by a solar beam with a direction specified by (μ_0, ϕ_0), and so we seek a solution to Eq. (1) that satisfies the boundary conditions

$$I(0, \mu, \phi) = \pi \delta(\mu - \mu_0) \delta(\phi - \phi_0) \tag{3}$$

and

$$I(\tau_0, -\mu, \phi) = 0, \tag{4}$$

for $\mu \in (0,1]$ and $\phi \in [0,2\pi]$.

Following Chandrasekhar in Ref. [7], we write the intensity as

$$I(\tau, \mu, \phi) = I_0(\tau, \mu, \phi) + I_*(\tau, \mu, \phi), \tag{5}$$

where the unscattered component $I_0(\tau,\mu,\phi)$ is the solution to Eqs. (1), (3) and (4) for the case $\varpi = 0$ and $I_*(\tau, \mu, \phi)$ is the scattered component of the solution. By solving Eqs. (1), (3) and (4) for the case $\varpi = 0$, we find that the unscattered component is given by

$$I_0(\tau, \mu, \phi) = \pi \delta(\mu - \mu_0) \delta(\phi - \phi_0) \mathrm{e}^{-\tau/\mu} \tag{6}$$

and

$$I_0(\tau, -\mu, \phi) = 0, \tag{7}$$

for $\mu \in (0,1]$ and $\phi \in [0,2\pi]$. We now substitute Eq. (5) into Eqs. (1), (3) and (4), and deduce, after noting Eqs. (6) and (7), that the scattered component $I_*(\tau, \mu, \phi)$ must satisfy

$$\mu \frac{\partial}{\partial \tau} I_*(\tau, \mu, \phi) + I_*(\tau, \mu, \phi) = \frac{\varpi}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos \Theta) I_*(\tau, \mu', \phi') \mathrm{d}\phi' \mathrm{d}\mu' + Q(\tau, \mu, \phi), \tag{8}$$

for $\tau \in (0,\tau_0)$, $\mu \in [-1,1]$ and $\phi \in [0,2\pi]$, and the boundary conditions

$$I_*(0, \mu, \phi) = I_*(\tau_0, -\mu, \phi) = 0, \tag{9}$$

for $\mu \in (0,1]$ and $\phi \in [0,2\pi]$. Here, the known inhomogeneous term is given by

$$Q(\tau, \mu, \phi) = \frac{\varpi}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos \Theta) I_0(\tau, \mu', \phi') \mathrm{d}\phi' \mathrm{d}\mu'. \tag{10}$$

Fourier decompositions have long been used [7] as a convenient way of treating the azimuthal dependence of radiative-transfer problems. Here, we can use the *cosine* decomposition

$$I_*(\tau, \mu, \phi) = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) I_*^m(\tau, \mu) \cos[m(\phi - \phi_0)] \tag{11}$$

along with the addition theorem for the Legendre polynomials [12],

$$p(\cos \Theta) = \sum_{m=0}^{L} (2 - \delta_{0,m}) \sum_{l=m}^{L} \beta_l P_l^m(\mu') P_l^m(\mu) \cos[m(\phi' - \phi)], \tag{12}$$

where

$$P_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} (1 - \mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_l(\mu) \tag{13}$$

denotes a *normalized* associated Legendre function, to deduce that the problem formulated by Eqs. (8)–(10) can be reduced to the problem of solving, for $m = 0,1,\ldots,L$,

$$\mu \frac{\partial}{\partial \tau} I_*^m(\tau, \mu) + I_*^m(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu') I_*^m(\tau, \mu') \mathrm{d}\mu' + Q^m(\tau, \mu), \tag{14}$$

where

$$Q^m(\tau, \mu) = \frac{\varpi}{2} \mathrm{e}^{-\tau/\mu_0} \sum_{l=m}^{L} \beta_l P_l^m(\mu_0) P_l^m(\mu), \tag{15}$$

subject to the boundary conditions

$$I_*^m(0, \mu) = I_*^m(\tau_0, -\mu) = 0, \tag{16}$$

for $\mu \in (0,1]$. It is clear that once we solve the problems formulated by Eqs. (14)–(16) for $m = 0,1,\ldots,L$, we can compute the scattered component of the intensity with Eq. (11).

Finally, we note that since our basic formulation of the problem differs from that used in previous works on the subject [6,8,9], we intend to be very explicit in our presentation.

3. Singular integral equations and constraints

We now show how we can reduce the problem formulated by Eqs. (14)–(16), first to a system of singular integral equations and constraints for the exiting Fourier components $I_*^m(0,-\mu)$ and $I_*^m(\tau_0,\mu)$, $\mu \in (0,1]$, and then to a system of singular integral equations and constraints relating $I_*^m(\tau,\mu)$, for $\tau \in (0,\tau_0)$ and $\mu \in [-1,1]$, to the exiting Fourier components $I_*^m(0,-\mu)$ and $I_*^m(\tau_0,\mu)$, $\mu \in (0,1]$. As discussed in two previous reviews of the F_N method [2,3], three different ways of accomplishing this derivation have been reported. The first [13] is based on the Placzek lemma [14] and the singular-eigenfunction technique [15,16], while the second [17] relies only on the singular-eigenfunction technique. Here, we use the third way [18,19], which is based on an integral-transform technique.

We start by changing μ to $-\mu$ in Eq. (14), multiplying the resulting equation by $\exp(-\tau/s)$, where s is a complex parameter, and integrating over τ from a to b, where a and b are arbitrary but obey the restriction $0 \le a < b \le \tau_0$, to obtain

$$s\mu B^m(\mu,s) - (\mu - s)\int_a^b e^{-\tau/s} I_*^m(\tau,-\mu)\,d\tau = \frac{\varpi s}{2}\sum_{l=m}^{L}(-1)^{l-m}\beta_l P_l^m(\mu)[\mathcal{I}_{*,l}^m(s) + s\mu_0 e^{-a/s} e^{-a/\mu_0} S(b-a : s,\mu_0) P_l^m(\mu_0)]. \tag{17}$$

Here we define

$$S(x:\xi,\eta) = \frac{1 - e^{-x/\xi}e^{-x/\eta}}{\xi + \eta}, \tag{18}$$

$$B^m(\mu,s) = I_*^m(a,-\mu)e^{-a/s} - I_*^m(b,-\mu)e^{-b/s} \tag{19}$$

and

$$\mathcal{I}_{*,l}^m(s) = \int_a^b e^{-\tau/s} I_{*,l}^m(\tau)\,d\tau, \tag{20}$$

where

$$I_{*,l}^m(\tau) = \int_{-1}^{1} P_l^m(\mu) I_*^m(\tau,\mu)\,d\mu. \tag{21}$$

Next, restricting s out of the real interval $[-1,1]$, we multiply Eq. (17) by $(\mu - s)^{-1}P_k^m(\mu)$ and integrate over μ from -1 to 1 to obtain, for $k = 0,1,\ldots,$

$$(-1)^{k-m}\mathcal{I}^m_{*,k}(s) = s\int_{-1}^{1} \mu P^m_k(\mu) B^m(\mu, s)\frac{d\mu}{\mu - s} - \frac{\varpi s}{2}\sum_{l=m}^{L}(-1)^{l-m}\beta_l X^m_{l,k}(s)$$
$$\times [\mathcal{I}^m_{*,l}(s) + s\mu_0 e^{-a/s}e^{-a/\mu_0}S(b-a:s,\mu_0)P^m_l(\mu_0)], \tag{22}$$

where

$$X^m_{l,k}(s) = \int_{-1}^{1} P^m_l(\mu)P^m_k(\mu)\frac{d\mu}{\mu - s}. \tag{23}$$

Defining the *normalized* Chandrasekhar polynomials [20] with the starting value

$$g^m_m(\xi) = (2m-1)!![(2m)!]^{-1/2} \tag{24}$$

and the recurrence relation, for $l \geq m$,

$$(l^2 - m^2)^{1/2} g^m_{l-1}(\xi) - h_l \xi g^m_l(\xi) + [(l+1)^2 - m^2]^{1/2} g^m_{l+1}(\xi) = 0, \tag{25}$$

where, for $l = 0,1,\ldots,L$,

$$h_l = 2l + 1 - \varpi\beta_l \tag{26}$$

and, for $l > L$,

$$h_l = 2l + 1, \tag{27}$$

we now multiply Eq. (22) by $\beta_k g^m_k(s)$ and sum the resulting equation from $k = m$ up to $k = L$ to obtain

$$\sum_{l=m}^{L}(-1)^{l-m}\beta_l F^m_l(s)\mathcal{I}^m_{*,l}(s) = s\int_{-1}^{1} \mu G^m(s,\mu)B^m(\mu,s)\frac{d\mu}{\mu - s} - s\mu_0$$
$$\times e^{-a/s}e^{-a/\mu_0}S(b-a:s,\mu_0)\sum_{l=m}^{L}(-1)^{l-m}\beta_l P^m_l(\mu_0)[F^m_l(s) - g^m_l(s)]. \tag{28}$$

Here, we define

$$G^m(s,\mu) = \sum_{l=m}^{L}\beta_l g^m_l(s)P^m_l(\mu) \tag{29}$$

and

$$F^m_l(s) = g^m_l(s) + \frac{\varpi s}{2}\int_{-1}^{1} G^m(s,\mu)P^m_l(\mu)\frac{d\mu}{\mu - s}. \tag{30}$$

By noting that $F^m_l(s)$ satisfies, for $l \geq m$, the same recurrence relation satisfied by the associated Legendre function $P^m_l(s)$ and by working out an explicit result for $F^m_m(s)$, we can show that

$$F^m_l(s) = (1 - s^2)^{-m/2}\Lambda^m(s)P^m_l(s), \tag{31}$$

where

$$\Lambda^m(s) = 1 + s\int_{-1}^{1}\Psi^m(\mu)\frac{d\mu}{\mu - s}, \tag{32}$$

with

$$\Psi^m(\mu) = \frac{\varpi}{2}(1-\mu^2)^{m/2}G^m(\mu,\mu), \tag{33}$$

is the well-known dispersion function [16,21,22]. Considering Eq. (31), we can now write Eq. (28) as

$$(1-s^2)^{-m/2}\Lambda^m(s)\sum_{l=m}^{L}(-1)^{l-m}\beta_l P_l^m(s)\mathcal{I}_{*,l}^m(s) = s\int_{-1}^{1}\mu G^m(s,\mu)B^m(\mu,s)\frac{d\mu}{\mu - s}$$
$$- s\mu_0 e^{-a/s}e^{-a/\mu_0}S(b-a:s,\mu_0)\sum_{l=m}^{L}(-1)^{l-m}\beta_l P_l^m(\mu_0)D_l^m(s), \tag{34}$$

where

$$D_l^m(s) = (1-s^2)^{-m/2}\Lambda^m(s)P_l^m(s) - g_l^m(s). \tag{35}$$

In order to avoid the occurrence of essential singularities at the origin of the complex plane, we find it convenient to multiply Eq. (34) by $\exp(a/s)$ and consider the resulting equation only for $\mathcal{R}s \geq 0$. Similarly, we multiply Eq. (34) with s changed to $-s$ by $\exp(-b/s)$ and consider the resulting equation only for $\mathcal{R}s \geq 0$. We thus find, for $\mathcal{R}s \geq 0$,

$$(1-s^2)^{-m/2}\Lambda^m(s)I^m(s) = \int_{-1}^{1}\mu G^m(s,\mu)[I_*^m(a,-\mu) - I_*^m(b,-\mu)e^{-(b-a)/s}]\frac{d\mu}{\mu - s}$$
$$- \mu_0 e^{-a/\mu_0}S(b-a:s,\mu_0)\sum_{l=m}^{L}(-1)^{l-m}\beta_l P_l^m(\mu_0)D_l^m(s), \tag{36}$$

and

$$(1-s^2)^{-m/2}\Lambda^m(s)J^m(s) = \int_{-1}^{1}\mu G^m(s,\mu)[I_*^m(b,\mu) - I_*^m(a,\mu)e^{-(b-a)/s}]\frac{d\mu}{\mu - s}$$
$$- \mu_0 e^{-a/\mu_0}C(b-a:s,\mu_0)\sum_{l=m}^{L}\beta_l P_l^m(\mu_0)D_l^m(s), \tag{37}$$

where

$$I^m(s) = \frac{1}{s}\sum_{l=m}^{L}(-1)^{l-m}\beta_l P_l^m(s)\int_a^b e^{-(\tau-a)/s}I_{*,l}^m(\tau)d\tau, \tag{38}$$

$$J^m(s) = \frac{1}{s}\sum_{l=m}^{L} \beta_l P_l^m(s) \int_a^b \mathrm{e}^{-(b-\tau)/s} I_{*,l}^m(\tau)\mathrm{d}\tau \tag{39}$$

and

$$C(x:\xi,\eta) = \frac{\mathrm{e}^{-x/\xi} - \mathrm{e}^{-x/\eta}}{\xi - \eta}. \tag{40}$$

We note that the dispersion function $\Lambda^m(s)$ is analytic in the complex plane cut from -1 to 1 along the real axis, and that it has $\aleph^m$ pairs of zeros $\pm v_\beta^m$, $\beta = 0,1,\ldots,\aleph^m-1$, which are all real, bounded numbers [21,23] with $|v_\beta^m| \geq 1$, except when $\bar{\omega} = 1$ and $m = 0$, for which case one pair of zeros becomes unbounded [24]. In the following, we consider that $\varpi \neq 1$ when $m = 0$; the modifications required in our general development to handle the special case $\varpi = 1$ and $m = 0$ will be discussed in a section of this paper specifically devoted to this matter. Thus, if we now use Eqs. (36) and (37) for $s = v_\beta^m$, $\beta = 0,1,\ldots,\aleph^m-1$, we obtain

$$\int_{-1}^{1} \mu G^m(v_\beta^m,\mu)\left[I_*^m(a,-\mu) - I_*^m(b,-\mu)\mathrm{e}^{-(b-a)/v_\beta^m}\right]\frac{\mathrm{d}\mu}{v_\beta^m - \mu} = \mu_0 \mathrm{e}^{-a/\mu_0} S(b-a:v_\beta^m,\mu_0) G^m(-v_\beta^m,\mu_0) \tag{41}$$

and

$$\int_{-1}^{1} \mu G^m(v_\beta^m,\mu)\left[I_*^m(b,\mu) - I_*^m(a,\mu)\mathrm{e}^{-(b-a)/v_\beta^m}\right]\frac{\mathrm{d}\mu}{v_\beta^m - \mu} = \mu_0 \mathrm{e}^{-a/\mu_0} C(b-a:v_\beta^m,\mu_0) G^m(v_\beta^m,\mu_0). \tag{42}$$

We can also let $s \to v \in [0,1]$ in Eqs. (36) and (37) and use the Plemelj formulas [25] to obtain

$$(1-v^2)^{-m/2}\lambda^m(v)\left[I_*^m(a,-v) - I_*^m(b,-v)\mathrm{e}^{-(b-a)/v}\right] + \frac{\varpi}{2}⨍_{-1}^{1} \mu G^m(v,\mu)\left[I_*^m(a,-\mu) - I_*^m(b,-\mu)\mathrm{e}^{-(b-a)/v}\right]\frac{\mathrm{d}\mu}{v-\mu} = \frac{\varpi\mu_0}{2}\mathrm{e}^{-a/\mu_0} S(b-a:v,\mu_0) G^m(-v,\mu_0), \tag{43}$$

and

$$(1-v^2)^{-m/2}\lambda^m(v)\left[I_*^m(b,v) - I_*^m(a,v)\mathrm{e}^{-(b-a)/v}\right] + \frac{\varpi}{2}⨍_{-1}^{1} \mu G^m(v,\mu)\left[I_*^m(b,\mu) - I_*^m(a,\mu)\mathrm{e}^{-(b-a)/v}\right]\frac{\mathrm{d}\mu}{v-\mu} = \frac{\varpi\mu_0}{2}\mathrm{e}^{-a/\mu_0} C(b-a:v,\mu_0) G^m(v,\mu_0), \tag{44}$$

where

$$\lambda^m(v) = 1 + v⨍_{-1}^{1} \Psi^m(\mu)\frac{\mathrm{d}\mu}{\mu - v} \tag{45}$$

and the symbol ⨍ indicates that the integration is to be evaluated in the Cauchy principal-value sense.

Having derived a system of singular integral equations [Eqs. (43) and (44)] and constraints [Eqs. (41) and (42)] relating the Fourier component $I_*^m(\tau,\mu)$, $\mu\in[-1,1]$, at two arbitrary positions $\tau = a$ and $\tau = b$ with $a < b$, we can now obtain our desired system of singular integral equations and constraints for $I_*^m(0,-\mu)$ and $I_*^m(\tau_0,\mu)$, $\mu\in(0,1]$, by simply letting $a = 0$ and $b = \tau_0$ in Eqs. (41)–(44) and by making use of the boundary conditions expressed by Eq. (16). We find, for $\beta = 0,1,\dots,\aleph^m-1$,

$$\int_0^1 \mu G^m(v_\beta^m,\mu)I_*^m(0,-\mu)\frac{\mathrm{d}\mu}{v_\beta^m-\mu}+\mathrm{e}^{-\tau_0/v_\beta^m}\int_0^1 \mu G^m(-v_\beta^m,\mu)I_*^m(\tau_0,\mu)\frac{\mathrm{d}\mu}{v_\beta^m+\mu}=E^m(0,-v_\beta^m) \tag{46}$$

and

$$\int_0^1 \mu G^m(v_\beta^m,\mu)I_*^m(\tau_0,\mu)\frac{\mathrm{d}\mu}{v_\beta^m-\mu}+\mathrm{e}^{-\tau_0/v_\beta^m}\int_0^1 \mu G^m(-v_\beta^m,\mu)I_*^m(0,-\mu)\frac{\mathrm{d}\mu}{v_\beta^m+\mu}=E^m(\tau_0,v_\beta^m), \tag{47}$$

where, in general,

$$E^m(x,-\xi)=\mu_0\mathrm{e}^{-x/\mu_0}S(\tau_0-x:\xi,\mu_0)G^m(-\xi,\mu_0) \tag{48}$$

and

$$E^m(x,\xi)=\mu_0C(x:\xi,\mu_0)G^m(\xi,\mu_0). \tag{49}$$

Similarly, we find, for $v\in[0,1]$,

$$(1-v^2)^{-m/2}\lambda^m(v)I_*^m(0,-v)+\frac{\varpi}{2}⨍_0^1 \mu G^m(v,\mu)I_*^m(0,-\mu)\frac{\mathrm{d}\mu}{v-\mu}+\frac{\varpi}{2}\mathrm{e}^{-\tau_0/v}\int_0^1 \mu G^m(-v,\mu)I_*^m(\tau_0,\mu)\frac{\mathrm{d}\mu}{v+\mu}=\frac{\varpi}{2}E^m(0,-v) \tag{50}$$

and

$$(1-v^2)^{-m/2}\lambda^m(v)I_*^m(\tau_0,v)+\frac{\varpi}{2}⨍_0^1 \mu G^m(v,\mu)I_*^m(\tau_0,\mu)\frac{\mathrm{d}\mu}{v-\mu}+\frac{\varpi}{2}\mathrm{e}^{-\tau_0/v}\int_0^1 \mu G^m(-v,\mu)I_*^m(0,-\mu)\frac{\mathrm{d}\mu}{v+\mu}=\frac{\varpi}{2}E^m(\tau_0,v). \tag{51}$$

Assuming that Eqs. (46), (47), (50) and (51) have been solved for $I_*^m(0,-\mu)$ and $I_*^m(\tau_0,\mu)$, $\mu\in(0,1]$, in the manner to be shown in Section 4, we now proceed to derive a system of singular integral equations and constraints relating the Fourier component $I_*^m(\tau,\mu)$, $\mu\in[-1,1]$, at any interior point of the atmosphere to $I_*^m(0,-\mu)$ and $I_*^m(\tau_0,\mu)$, $\mu\in(0,1]$. We start by letting $a = \tau$ and $b = \tau_0$ in Eq. (41) and $a = 0$ and $b = \tau$ in Eq. (42) to find, for $\beta = 0,1,\dots,\aleph^m-1$,

$$\int_{-1}^{1} \mu G^m(\nu_\beta^m, \mu) I_*^m(\tau, -\mu) \frac{d\mu}{\nu_\beta^m - \mu} = E^m(\tau, -\nu_\beta^m) - e^{-(\tau_0-\tau)/\nu_\beta^m} \int_0^1 \mu G^m(-\nu_\beta^m, \mu) I_*^m(\tau_0, \mu) \frac{d\mu}{\nu_\beta^m + \mu} \tag{52}$$

and

$$\int_{-1}^{1} \mu G^m(\nu_\beta^m, \mu) I_*^m(\tau, \mu) \frac{d\mu}{\nu_\beta^m - \mu} = E^m(\tau, \nu_\beta^m) - e^{-\tau/\nu_\beta^m} \int_0^1 \mu G^m(-\nu_\beta^m, \mu) I_*^m(0, -\mu) \frac{d\mu}{\nu_\beta^m + \mu}. \tag{53}$$

Finally we let $a = \tau$ and $b = \tau_0$ in Eq. (43) and $a = 0$ and $b = \tau$ in Eq. (44) to find, for $\nu \in [0,1]$,

$$(1-\nu^2)^{-m/2} \lambda^m(\nu) I_*^m(\tau, -\nu) + \frac{\varpi}{2} \int_{-1}^{1} \mu G^m(\nu, \mu) I_*^m(\tau, -\mu) \frac{d\mu}{\nu - \mu}$$
$$= \frac{\varpi}{2} E^m(\tau, -\nu) - \frac{\varpi}{2} e^{-(\tau_0-\tau)/\nu} \int_0^1 \mu G^m(-\nu, \mu) I_*^m(\tau_0, \mu) \frac{d\mu}{\nu + \mu} \tag{54}$$

and

$$(1-\nu^2)^{-m/2} \lambda^m(\nu) I_*^m(\tau, \nu) + \frac{\varpi}{2} \int_{-1}^{1} \mu G^m(\nu, \mu) I_*^m(\tau, \mu) \frac{d\mu}{\nu - \mu}$$
$$= \frac{\varpi}{2} E^m(\tau, \nu) - \frac{\varpi}{2} e^{-\tau/\nu} \int_0^1 \mu G^m(-\nu, \mu) I_*^m(0, -\mu) \frac{d\mu}{\nu + \mu}. \tag{55}$$

Eqs. (52)–(55) constitute our system of singular equations and constraints to be solved for $I_*^m(\tau,\mu)$, $\tau \in (0,\tau_0)$ and $\mu \in [-1,1]$.

4. The F_N solution

The F_N method is an approximate, but concise and accurate, way of solving systems of singular integral equations and constraints of the type formulated in Section 3. We note that in this section we continue to assume that $\varpi \neq 1$ when $m = 0$; our discussion of the modifications required to handle this special case is deferred to the next section of this paper.

We begin our presentation of the method by discussing its use for solving the system of singular integral equations and constraints given by Eqs. (46), (47), (50) and (51) for the boundary unknowns $I_*^m(0, -\mu)$ and $I_*^m(\tau_0, \mu)$, $\mu \in (0,1]$. First, we introduce the approximations, for $\mu \in (0,1]$,

$$I_*^m(0, -\mu) = \frac{\varpi}{2} \sum_{\alpha=0}^{N} a_\alpha^m \Phi_\alpha^m(\mu) \tag{56}$$

and

$$I_*^m(\tau_0, \mu) = \frac{\varpi}{2} \sum_{\alpha=0}^{N} b_\alpha^m \Phi_\alpha^m(\mu), \tag{57}$$

where $\{a_\alpha^m\}$ and $\{b_\alpha^m\}$ are unknown coefficients, and $\{\Phi_\alpha^m(\mu)\}$ is a set of basis functions that will be specified in a forthcoming section on computational methods, into Eqs. (46), (47), (50) and (51) to obtain

$$\sum_{\alpha=0}^{N} a_\alpha^m B_\alpha^m(\xi) + e^{-\tau_0/\xi} \sum_{\alpha=0}^{N} b_\alpha^m A_\alpha^m(\xi) = 2E^m(0, -\xi) \tag{58}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha^m B_\alpha^m(\xi) + e^{-\tau_0/\xi} \sum_{\alpha=0}^{N} a_\alpha^m A_\alpha^m(\xi) = 2E^m(\tau_0, \xi), \tag{59}$$

for $\xi = \{v_\beta^m\} \cup [0,1]$. Here, we define the A functions as

$$A_\alpha^m(\xi) = \varpi \int_0^1 \mu G^m(-\xi, \mu) \Phi_\alpha^m(\mu) \frac{d\mu}{\xi + \mu}, \tag{60}$$

for $\xi = \{v_\beta^m\} \cup [0,1]$, and the B functions as

$$B_\alpha^m(v_\beta^m) = \varpi \int_0^1 \mu G^m(v_\beta^m, \mu) \Phi_\alpha^m(\mu) \frac{d\mu}{v_\beta^m - \mu} \tag{61}$$

and, for $v \in [0,1]$,

$$B_\alpha^m(v) = 2(1 - v^2)^{-m/2} \lambda^m(v) \Phi_\alpha^m(v) + \varpi \fint_0^1 \mu G^m(v, \mu) \Phi_\alpha^m(\mu) \frac{d\mu}{v - \mu}. \tag{62}$$

Next, we note that there are $2(N + 1)$ unknowns in Eqs. (58) and (59), i.e. the coefficients a_α^m and b_α^m, for $\alpha = 0,1,\ldots,N$, but the number of equations can be thought of as infinite, since ξ can take on any value in [0,1]. To overcome this difficulty we use collocation, i.e. we impose that Eqs. (58) and (59) be satisfied for $\{\xi_\beta^m\}$, a set of $N + 1$ points composed of $\aleph^m$ points that are the positive zeros of the dispersion function and $N + 1 - \aleph^m$ points in [0,1]. A discussion of collocation schemes, in particular the one we intend to use here, will be presented in the section of this paper devoted to computational methods. Thus, on using Eqs. (58) and (59) for $\xi = \xi_\beta^m$, $\beta = 0,1,\ldots,N$, we obtain two coupled linear systems of order $N + 1$ for the unknowns $\{a_\alpha^m\}$ and $\{b_\alpha^m\}$, *viz.*

$$\mathbf{Ba} + \mathbf{D}(\tau_0)\mathbf{Ab} = 2\mathbf{E}_1(0) \tag{63}$$

and

$$\mathbf{Bb} + \mathbf{D}(\tau_0)\mathbf{Aa} = 2\mathbf{E}_2(\tau_0). \tag{64}$$

Here we use the definitions

$$\mathbf{B} = \begin{pmatrix} B_0^m(\xi_0^m) & B_1^m(\xi_0^m) & \cdots & B_N^m(\xi_0^m) \\ B_0^m(\xi_1^m) & B_1^m(\xi_1^m) & \cdots & B_N^m(\xi_1^m) \\ \vdots & \vdots & \ddots & \vdots \\ B_0^m(\xi_N^m) & B_1^m(\xi_N^m) & \cdots & B_N^m(\xi_N^m) \end{pmatrix}, \tag{65}$$

$$\mathbf{A} = \begin{pmatrix} A_0^m(\xi_0^m) & A_1^m(\xi_0^m) & \cdots & A_N^m(\xi_0^m) \\ A_0^m(\xi_1^m) & A_1^m(\xi_1^m) & \cdots & A_N^m(\xi_1^m) \\ \vdots & \vdots & \ddots & \vdots \\ A_0^m(\xi_N^m) & A_1^m(\xi_N^m) & \cdots & A_N^m(\xi_N^m) \end{pmatrix}, \tag{66}$$

$$\mathbf{a} = \begin{pmatrix} a_0^m \\ a_1^m \\ \vdots \\ a_N^m \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_0^m \\ b_1^m \\ \vdots \\ b_N^m \end{pmatrix}. \tag{67}$$

We also define, in general,

$$\mathbf{D}(x) = \operatorname{diag}\{e^{-x/\xi_0^m}, e^{-x/\xi_1^m}, \ldots, e^{-x/\xi_N^m}\}, \tag{68}$$

$$\mathbf{E}_1(x) = \begin{pmatrix} E^m(x, -\xi_0^m) \\ E^m(x, -\xi_1^m) \\ \vdots \\ E^m(x, -\xi_N^m) \end{pmatrix} \quad \text{and} \quad \mathbf{E}_2(x) = \begin{pmatrix} E^m(x, \xi_0^m) \\ E^m(x, \xi_1^m) \\ \vdots \\ E^m(x, \xi_N^m) \end{pmatrix}. \tag{69}$$

In addition, we note that, for simplicity, we have omitted in our notation of the quantities defined by Eqs. (65)–(69) the superscript m that characterizes the Fourier index.

We now report our way of solving the coupled systems for **a** and **b** expressed by Eqs. (63) and (64). In Ref. [9], by adding and subtracting Eqs. (63) and (64), we were able to reformulate this problem as the problem of solving two decoupled systems of order $N + 1$ for $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$. However, if we analyze the behavior of the equations that represent the result of these operations [9] in the limit of $\tau_0 \to \infty$, we can conclude that, for a sufficiently thick layer, the approach of Ref. [9] is not capable of yielding accurate results for the exiting intensities at the bottom of the atmosphere. Of course, an alternative would be to reformulate the systems given by Eqs. (63) and (64) as one system of order $2(N + 1)$, which was the approach used before Ref. [9] appeared. Nevertheless, in order to keep the advantage of solving two linear systems of order $N + 1$ instead of one of order $2(N + 1)$, we prefer to introduce an approach that we have devised in order to overcome this difficulty. To explain our new way of solving Eqs. (63) and (64), we first assume that $\mathbf{B}$ is invertible. We note that no proof is available of this assumption, but all numerical evidence collected so far indicates that this is true. We then multiply Eqs. (63) and (64) on the left by $\mathbf{B}^{-1}$ to obtain

$$\mathbf{a} = 2\mathbf{B}^{-1}\mathbf{E}_1(0) - \mathbf{B}^{-1}\mathbf{D}(\tau_0)\mathbf{A}\mathbf{b} \tag{70}$$

and

$$\mathbf{b} = 2\mathbf{B}^{-1}\mathbf{E}_2(\tau_0) - \mathbf{B}^{-1}\mathbf{D}(\tau_0)\mathbf{A}\mathbf{a}. \tag{71}$$

If we now substitute Eqs. (70) and (71) into each other and define, in general,

$$\mathbf{C}(x) = \mathbf{B}^{-1}\mathbf{D}(x)\mathbf{A}, \tag{72}$$

we find

$$\mathbf{M}(\tau_0)\mathbf{a} = \mathbf{B}^{-1}\mathbf{E}_1(0) - \mathbf{C}(\tau_0)\mathbf{B}^{-1}\mathbf{E}_2(\tau_0) \tag{73}$$

and

$$\mathbf{M}(\tau_0)\mathbf{b} = \mathbf{B}^{-1}\mathbf{E}_2(\tau_0) - \mathbf{C}(\tau_0)\mathbf{B}^{-1}\mathbf{E}_1(0), \tag{74}$$

where, in general,

$$\mathbf{M}(x) = \frac{1}{2}[\mathbf{I} - \mathbf{C}^2(x)]. \tag{75}$$

Clearly, both of the Eqs. (73) and (74) are well behaved for $\tau_0 \to \infty$ and so, having confirmed that our proposed scheme is computationally sound for various $\tau_0 \in (0,\infty)$ and $\varpi \in (0,1)$, even for ϖ very close to 1, say $1 - \varpi = 10^{-8}$, we recommend its adoption as an alternative to the scheme of Ref. [9].

Once Eqs. (73) and (74) are solved for $\mathbf{a}$ and $\mathbf{b}$, Eqs. (56) and (57) could, in principle, be used to compute our F_N approximations to $I_*^m(0, -\mu)$ and $I_*^m(\tau_0, \mu)$, $\mu \in (0,1]$. However, we have found that by post processing Eqs. (56) and (57) with the singular integral equations expressed by Eqs. (50) and (51) we obtain improved approximations for $I_*^m(0, -\mu)$ and $I_*^m(\tau_0, \mu)$, $\mu \in (0,1]$, which converge faster and more uniformly in μ as N increases [26]. A simple derivation of the post processed formulas is in order. First, we take $\nu = \mu \in (0,1]$ in Eqs. (50) and (51) and rearrange the resulting equations to obtain

$$I_*^m(0,-\mu) = [1-\lambda^m(\mu)]I_*^m(0,-\mu) + \frac{\varpi}{2}(1-\mu^2)^{m/2}\bigg[E^m(0,-\mu) - \fint_0^1 \mu' G^m(\mu,\mu')I_*^m(0,-\mu')\frac{\mathrm{d}\mu'}{\mu-\mu'} - \mathrm{e}^{-\tau_0/\mu}\int_0^1 \mu' G^m(-\mu,\mu')I_*^m(\tau_0,\mu')\frac{\mathrm{d}\mu'}{\mu+\mu'}\bigg] \tag{76}$$

and

$$I_*^m(\tau_0,\mu) = [1-\lambda^m(\mu)]I_*^m(\tau_0,\mu) + \frac{\varpi}{2}(1-\mu^2)^{m/2}\bigg[E^m(\tau_0,\mu) - \fint_0^1 \mu' G^m(\mu,\mu')I_*^m(\tau_0,\mu')\frac{\mathrm{d}\mu'}{\mu-\mu'} - \mathrm{e}^{-\tau_0/\mu}\int_0^1 \mu' G^m(-\mu,\mu')I_*^m(0,-\mu')\frac{\mathrm{d}\mu'}{\mu+\mu'}\bigg]. \tag{77}$$

Then, we can use the approximations expressed by Eqs. (56) and (57) for $I_*^m(0,-\mu)$ and $I_*^m(\tau_0,\mu)$, $\mu \in (0,1]$, in the right-hand sides of Eqs. (76) and (77) and the definitions given by Eqs. (60) and (62) to find our post processed results, *viz.*

$$I_*^m(0,-\mu) = \frac{\varpi}{2}\sum_{\alpha=0}^{N} a_\alpha^m \Phi_\alpha^m(\mu) + \frac{\varpi}{4}(1-\mu^2)^{m/2} \times \left[2E^m(0,-\mu) - \sum_{\alpha=0}^{N} a_\alpha^m B_\alpha^m(\mu) - \mathrm{e}^{-\tau_0/\mu}\sum_{\alpha=0}^{N} b_\alpha^m A_\alpha^m(\mu)\right] \tag{78}$$

and

$$I_*^m(\tau_0,\mu) = \frac{\varpi}{2}\sum_{\alpha=0}^{N} b_\alpha^m \Phi_\alpha^m(\mu) + \frac{\varpi}{4}(1-\mu^2)^{m/2} \times \left[2E^m(\tau_0,\mu) - \sum_{\alpha=0}^{N} b_\alpha^m B_\alpha^m(\mu) - \mathrm{e}^{-\tau_0/\mu}\sum_{\alpha=0}^{N} a_\alpha^m A_\alpha^m(\mu)\right], \tag{79}$$

for $\mu \in (0,1]$. If we compare Eqs. (78) and (79) to Eqs. (56) and (57) and recall that the method requires that Eqs. (58) and (59) be satisfied at the collocation points $\{\xi_\beta^m\}$, it is clear that the post processed results are simply the non post processed results added to correction terms that vanish for the values of μ that coincide with the collocation points embedded in the continuum.

We now turn our attention to the solution of the system of singular integral equations and constraints expressed by Eqs. (52)–(55) for the Fourier component $I_*^m(\tau,\mu)$, $\tau \in (0,\tau_0)$ and $\mu \in [-1,1]$. We begin by using the approximations, for $\mu \in [0,1]$,

$$I_*^m(\tau,-\mu) = \frac{\varpi}{2}\sum_{\alpha=0}^{N} c_\alpha^m(\tau)\Phi_\alpha^m(\mu) \tag{80}$$

and

$$I_*^m(\tau, \mu) = \frac{\varpi}{2} \sum_{\alpha=0}^{N} d_\alpha^m(\tau) \Phi_\alpha^m(\mu) \tag{81}$$

in Eqs. (52)–(55), to obtain

$$\sum_{\alpha=0}^{N} c_\alpha^m(\tau) B_\alpha^m(\xi) - \sum_{\alpha=0}^{N} d_\alpha^m(\tau) A_\alpha^m(\xi) = 2E^m(\tau, -\xi) - \mathrm{e}^{-(\tau_0-\tau)/\xi} \sum_{\alpha=0}^{N} b_\alpha^m A_\alpha^m(\xi) \tag{82}$$

and

$$\sum_{\alpha=0}^{N} d_\alpha^m(\tau) B_\alpha^m(\xi) - \sum_{\alpha=0}^{N} c_\alpha^m(\tau) A_\alpha^m(\xi) = 2E^m(\tau, \xi) - \mathrm{e}^{-\tau/\xi} \sum_{\alpha=0}^{N} a_\alpha^m A_\alpha^m(\xi), \tag{83}$$

for $\xi = \{\nu_\beta^m\} \cup [0,1]$. Following the approach used for solving Eqs. (58) and (59), i.e. requiring that Eqs. (82) and (83) be satisfied at the set of collocation points $\{\xi_\beta^m\}$, we find the linear systems of order $N + 1$:

$$\mathbf{Bc}(\tau) - \mathbf{Ad}(\tau) = 2\mathbf{F}_1(\tau) \tag{84}$$

and

$$\mathbf{Bd}(\tau) - \mathbf{Ac}(\tau) = 2\mathbf{F}_2(\tau), \tag{85}$$

where we define the matrices **B** and **A** as in Eqs. (65) and (66), the vectors of unknowns as

$$\mathbf{c}(\tau) = \begin{pmatrix} c_0^m(\tau) \\ c_1^m(\tau) \\ \vdots \\ c_N^m(\tau) \end{pmatrix} \quad \text{and} \quad \mathbf{d}(\tau) = \begin{pmatrix} d_0^m(\tau) \\ d_1^m(\tau) \\ \vdots \\ d_N^m(\tau) \end{pmatrix}, \tag{86}$$

and the right-hand-side vectors as

$$\mathbf{F}_1(\tau) = \mathbf{E}_1(\tau) - \tfrac{1}{2}\mathbf{D}(\tau_0 - \tau)\mathbf{Ab} \tag{87}$$

and

$$\mathbf{F}_2(\tau) = \mathbf{E}_2(\tau) - \tfrac{1}{2}\mathbf{D}(\tau)\mathbf{Aa}. \tag{88}$$

Clearly, we can decouple the linear systems given by Eqs. (84) and (85) in much the same way as we did for the systems given by Eqs. (63) and (64). We obtain the linear systems of order $N + 1$,

$$\mathbf{M}(0)\mathbf{c}(\tau) = \mathbf{B}^{-1}\mathbf{F}_1(\tau) + \mathbf{C}(0)\mathbf{B}^{-1}\mathbf{F}_2(\tau) \tag{89}$$

and

$$\mathbf{M}(0)\mathbf{d}(\tau) = \mathbf{B}^{-1}\mathbf{F}_2(\tau) + \mathbf{C}(0)\mathbf{B}^{-1}\mathbf{F}_1(\tau), \tag{90}$$

where $\mathbf{C}(0)$, according to Eqs. (68) and (72), is simply $\mathbf{B}^{-1}\mathbf{A}$ and $\mathbf{M}(0)$ is given by Eq. (75) with $x = 0$. As $\mathbf{M}(0)$ does not depend on τ and we consider that a LU factorization of $\mathbf{B}$ that allows us to compute, in a straightforward manner, all the products involving $\mathbf{B}^{-1}$ in Eqs. (89) and (90) has been performed while solving Eqs. (73) and (74), it is clear that the solution of the linear systems given by Eqs. (89) and (90) for $\mathbf{c}(\tau)$ and $\mathbf{d}(\tau)$, for any number of interior points, requires only one extra LU factorization, that of $\mathbf{M}(0)$.

Once the solutions to Eqs. (89) and (90) for specified interior points are obtained, Eqs. (80) and (81) could be used to compute our F_N approximations to the Fourier components at these interior points. However, as done for the exiting Fourier components, here we also wish to perform a post processing of Eqs. (80) and (81) in order to improve our results. We obtain, from Eqs. (54) and (55) with $\nu = \mu \in [0,1]$, Eqs. (56), (57), (80) and (81),

$$I_*^m(\tau, -\mu) = \frac{\varpi}{2}\sum_{\alpha=0}^{N} c_\alpha^m(\tau)\Phi_\alpha^m(\mu) + \frac{\varpi}{4}(1-\mu^2)^{m/2}\Big[2E^m(\tau,-\mu) - \sum_{\alpha=0}^{N} c_\alpha^m(\tau)B_\alpha^m(\mu) + \sum_{\alpha=0}^{N} d_\alpha^m(\tau)A_\alpha^m(\mu) - \mathrm{e}^{-(\tau_0-\tau)/\mu}\sum_{\alpha=0}^{N} b_\alpha^m A_\alpha^m(\mu)\Big] \tag{91}$$

and

$$I_*^m(\tau, \mu) = \frac{\varpi}{2}\sum_{\alpha=0}^{N} d_\alpha^m(\tau)\Phi_\alpha^m(\mu) + \frac{\varpi}{4}(1-\mu^2)^{m/2}\Big[2E^m(\tau,\mu) - \sum_{\alpha=0}^{N} d_\alpha^m(\tau)B_\alpha^m(\mu) + \sum_{\alpha=0}^{N} c_\alpha^m(\tau)A_\alpha^m(\mu) - \mathrm{e}^{-\tau/\mu}\sum_{\alpha=0}^{N} a_\alpha^m A_\alpha^m(\mu)\Big], \tag{92}$$

for $\mu \in [0,1]$. Again, we see that our post processed results can be written as the non post processed results plus correction terms that vanish for the values of μ that coincide with the collocation points in the continuum.

5. The special case $\varpi = 1$ and $m = 0$

As discussed in Section 3, the dispersion function $\Lambda^m(s)$ has one pair of unbounded zeros when $\varpi = 1$ and $m = 0$, and so in this section we describe the modifications that are required in our general development to handle this special case [9]. In order to simplify our notation, we do not use in this section the superscript m that characterizes the Fourier index, with the understanding that by default it is equal to 0.

We first consider that the zeros of the dispersion function are ordered such that $\nu_0 > \nu_1 > \ldots > \nu_{\aleph-1}$. Next we note that, since for $\bar{\omega} = 1$ and $m = 0$ the Chandrasekhar polynomials $g_l(\xi) \to 0$ as $\xi \to \infty$, for $l > 0$, we can show that both Eqs. (41) and (42) reduce, for $\beta = 0$, to

$$\int_{-1}^{1} \mu[I_*(b,\mu) - I_*(a,\mu)]\mathrm{d}\mu = \mu_0[\mathrm{e}^{-a/\mu_0} - \mathrm{e}^{-b/\mu_0}]. \tag{93}$$

It is thus clear that one additional equation is needed to replace the missing equation. To this end, we multiply Eq. (14) with $\varpi = 1$ and $m = 0$ by μ and integrate over μ from -1 to 1 to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{-1}^{1} \mu^2 I_*(\tau,\mu)\mathrm{d}\mu + \frac{h_1}{3}\int_{-1}^{1} \mu I_*(\tau,\mu)\mathrm{d}\mu = \frac{\beta_1}{3}\mu_0\mathrm{e}^{-\tau/\mu_0}. \tag{94}$$

In addition, if we integrate Eq. (14) with $\varpi = 1$ and $m = 0$ over μ from -1 to 1 we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\int_{-1}^{1} \mu I_*(\tau,\mu)\mathrm{d}\mu + \mu_0\mathrm{e}^{-\tau/\mu_0}\right] = 0. \tag{95}$$

Clearly, the term between brackets in Eq. (95) must be a constant, and so, denoting this constant as F, we are allowed to write, for any $\tau \in [0,\tau_0]$,

$$F = \mu_0\mathrm{e}^{-\tau/\mu_0} + \int_{-1}^{1} \mu I_*(\tau,\mu)\mathrm{d}\mu, \tag{96}$$

which is just a mathematical statement of the constant flux condition in a conservative atmosphere. In particular, if we note the boundary conditions expressed by Eq. (16) we see that Eq. (96) yields, for $\tau = 0$,

$$F = \mu_0 - \int_{0}^{1} \mu I_*(0,-\mu)\mathrm{d}\mu, \tag{97}$$

and, for $\tau = \tau_0$,

$$F = \mu_0\mathrm{e}^{-\tau_0/\mu_0} + \int_{0}^{1} \mu I_*(\tau_0,\mu)\mathrm{d}\mu. \tag{98}$$

Using the result expressed by Eq. (96), we can now integrate Eq. (94) over τ from a to b to obtain

$$\int_{-1}^{1} \mu^2[I_*(b,\mu) - I_*(a,\mu)]\mathrm{d}\mu + \frac{h_1}{3}(b-a)F = \mu_0^2[\mathrm{e}^{-a/\mu_0} - \mathrm{e}^{-b/\mu_0}], \tag{99}$$

an equation that can be used, along with Eq. (93), to replace Eqs. (41) and (42) for $\beta = 0$ when $\varpi = 1$ and $m = 0$.

We now let $a = 0$ and $b = \tau_0$ in Eqs. (93) and (99) and note Eq. (16) to obtain a pair of constraints that we can use as a replacement of Eqs. (46) and (47) for $\beta = 0$ when $\varpi = 1$ and $m = 0$, *viz.*

$$\int_{0}^{1} \mu[I_*(0,-\mu) + I_*(\tau_0,\mu)]\mathrm{d}\mu = \mu_0[1 - \mathrm{e}^{-\tau_0/\mu_0}] \tag{100}$$

and

$$\int_0^1 \mu^2[I_*(\tau_0,\mu) - I_*(0,-\mu)]\mathrm{d}\mu = \mu_0^2[1-\mathrm{e}^{-\tau_0/\mu_0}] - \frac{h_1}{3}\tau_0 F, \tag{101}$$

with F represented either by Eq. (97) or by Eq. (98). However, instead of using Eqs. (100) and (101), we prefer to use here two linear combinations of these equations, in order to preserve the cross symmetry exhibited by Eqs. (46) and (47) for the general case. We thus multiply Eq. (100) by $h_1\tau_0/3$, use Eq. (97) to represent F, and subtract Eq. (101) from the resulting equation to obtain

$$\int_0^1 \mu\left(\mu + \frac{h_1}{3}\tau_0\right) I_*(0,-\mu)\mathrm{d}\mu - \int_0^1 \mu^2 I_*(\tau_0,\mu)\mathrm{d}\mu = \frac{h_1}{3}\tau_0\mu_0 - \mu_0^2[1-\mathrm{e}^{-\tau_0/\mu_0}]. \tag{102}$$

Similarly, we multiply Eq. (100) by $h_1\tau_0/3$, use Eq. (98) to represent F, and add the resulting equation to Eq. (101) to obtain

$$\int_0^1 \mu\left(\mu + \frac{h_1}{3}\tau_0\right) I_*(\tau_0,\mu)\mathrm{d}\mu - \int_0^1 \mu^2 I_*(0,-\mu)\mathrm{d}\mu = \mu_0^2[1-\mathrm{e}^{-\tau_0/\mu_0}] - \frac{h_1}{3}\tau_0\mu_0\mathrm{e}^{-\tau_0/\mu_0}. \tag{103}$$

Eqs. (102) and (103) along with Eqs. (46) and (47) for $\beta = 1,2,\ldots,\aleph-1$ are the constraints that we impose on $I_*(0,-\mu)$ and $I_*(\tau_0,\mu)$, $\mu \in (0,1]$, when $\varpi = 1$ and $m = 0$. By introducing the F_N approximations given by Eqs. (56) and (57) in Eqs. (102) and (103), we find the equations that replace Eqs. (58) and (59) for $\xi = \nu_0$, *viz.*

$$\sum_{\alpha=0}^{N} a_\alpha \mathcal{B}_\alpha(\tau_0) + \sum_{\alpha=0}^{N} b_\alpha \mathcal{A}_\alpha(0) = 2\mathcal{E}_1(0) \tag{104}$$

and

$$\sum_{\alpha=0}^{N} b_\alpha \mathcal{B}_\alpha(\tau_0) + \sum_{\alpha=0}^{N} a_\alpha \mathcal{A}_\alpha(0) = 2\mathcal{E}_2(\tau_0). \tag{105}$$

Here we define, in general,

$$\mathcal{A}_\alpha(x) = -\int_0^1 \mu\left(\mu - \frac{h_1}{3}x\right)\Phi_\alpha(\mu)\mathrm{d}\mu, \tag{106}$$

$$\mathcal{B}_\alpha(x) = \int_0^1 \mu\left(\mu + \frac{h_1}{3}x\right)\Phi_\alpha(\mu)\mathrm{d}\mu, \tag{107}$$

$$\mathcal{E}_1(x) = \mu_0\left(\mu_0 - \frac{h_1}{3}x\right)\mathrm{e}^{-\tau_0/\mu_0} - \mu_0\left(\mu_0 - \frac{h_1}{3}\tau_0\right)\mathrm{e}^{-x/\mu_0} \tag{108}$$

and

$$\mathcal{E}_2(x) = \mu_0\left[\mu_0 + \frac{h_1}{3}(\tau_0 - x)\right][1 - e^{-x/\mu_0}] - \frac{h_1}{3}x\mu_0 e^{-x/\mu_0}. \tag{109}$$

It is clear that if we define, for the special case we are considering here,

$$\mathbf{B} = \begin{pmatrix} \mathcal{B}_0(\tau_0) & \mathcal{B}_1(\tau_0) & \cdots & \mathcal{B}_N(\tau_0) \\ B_0(\xi_1) & B_1(\xi_1) & \cdots & B_N(\xi_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_0(\xi_N) & B_1(\xi_N) & \cdots & B_N(\xi_N) \end{pmatrix}, \tag{110}$$

$$\mathbf{A} = \begin{pmatrix} \mathcal{A}_0(0) & \mathcal{A}_1(0) & \cdots & \mathcal{A}_N(0) \\ A_0(\xi_1) & A_1(\xi_1) & \cdots & A_N(\xi_1) \\ \vdots & \vdots & \ddots & \vdots \\ A_0(\xi_N) & A_1(\xi_N) & \cdots & A_N(\xi_N) \end{pmatrix}, \tag{111}$$

$$\mathbf{D}(x) = \text{diag}\{1, e^{-x/\xi_1}, \ldots, e^{-x/\xi_N}\}, \tag{112}$$

$$\mathbf{E}_1(x) = \begin{pmatrix} \mathcal{E}_1(x) \\ E(x, -\xi_1) \\ \vdots \\ E(x, -\xi_N) \end{pmatrix} \quad \text{and} \quad \mathbf{E}_2(x) = \begin{pmatrix} \mathcal{E}_2(x) \\ E(x, \xi_1) \\ \vdots \\ E(x, \xi_N) \end{pmatrix}, \tag{113}$$

then our linear systems of order $N + 1$ for the coefficients $\{a_\alpha\}$ and $\{b_\alpha\}$ can be written here in the same form as in Section 4 [Eqs. (63) and (64)]. We thus conclude that the procedure developed in Section 4 for solving Eqs. (63) and (64) can also be used in this case.

We now develop our alternatives to the two constraints on the interior Fourier component $I_*(\tau, \mu)$, $\mu \in [-1,1]$, expressed by Eqs. (52) and (53) for $\beta = 0$. We first let $a = \tau$ and $b = \tau_0$ in Eq. (93) and use Eqs. (16) and (98) to write our first constraint as

$$\int_{-1}^{1} \mu I_*(\tau, -\mu)d\mu = \mu_0 e^{-\tau/\mu_0} - F. \tag{114}$$

Next, we let $a = 0$ and $b = \tau$ in Eq. (99) and use Eq. (16) to write our second constraint as

$$\int_{-1}^{1} \mu^2 I_*(\tau, \mu)d\mu = \mu_0^2[1 - e^{-\tau/\mu_0}] - \frac{h_1}{3}\tau F + \int_0^1 \mu^2 I_*(0, -\mu)d\mu. \tag{115}$$

Here, we also prefer to work with linear combinations of Eqs. (114) and (115), so that we are allowed to use an approach similar to that of Section 4. We thus add Eq. (115) to (114) multiplied by $h_1\tau_0/3$ to obtain, after we use Eq. (98) to define F in the resulting expression,

$$\int_{-1}^{1}\mu\left(\mu+\frac{h_1}{3}\tau_0\right)I_*(\tau,-\mu)\mathrm{d}\mu=\mathcal{E}_1(\tau)+\int_0^1\mu\left(\mu-\frac{h_1}{3}\tau\right)I_*(\tau_0,\mu)\mathrm{d}\mu. \tag{116}$$

Similarly, we subtract Eq. (114) multiplied by $h_1\tau_0/3$ from Eq. (115) to obtain, after we use Eq. (97) to define F in the resulting expression,

$$\int_{-1}^{1}\mu\left(\mu+\frac{h_1}{3}\tau_0\right)I_*(\tau,\mu)\mathrm{d}\mu=\mathcal{E}_2(\tau)+\int_0^1\mu\left[\mu-\frac{h_1}{3}(\tau_0-\tau)\right]I_*(0,-\mu)\mathrm{d}\mu. \tag{117}$$

Eqs. (116) and (117) along with Eqs. (52) and (53) for $\beta = 1,2,\ldots,\aleph-1$ are the constraints we impose on $I_*(\tau,\mu)$, $\mu\in[-1,1]$, when $\varpi = 1$ and $m = 0$. By introducing the F_N approximations given by Eqs. (80) and (81) in Eqs. (116) and (117), we find the equations that replace Eqs. (82) and (83) for $\xi = \nu_0$, *viz.*

$$\sum_{\alpha=0}^{N}c_\alpha(\tau)\mathcal{B}_\alpha(\tau_0)-\sum_{\alpha=0}^{N}d_\alpha(\tau)\mathcal{A}_\alpha(\tau_0)=2\mathcal{E}_1(\tau)-\sum_{\alpha=0}^{N}b_\alpha\mathcal{A}_\alpha(\tau) \tag{118}$$

and

$$\sum_{\alpha=0}^{N}d_\alpha(\tau)\mathcal{B}_\alpha(\tau_0)-\sum_{\alpha=0}^{N}c_\alpha(\tau)\mathcal{A}_\alpha(\tau_0)=2\mathcal{E}_2(\tau)-\sum_{\alpha=0}^{N}a_\alpha\mathcal{A}_\alpha(\tau_0-\tau). \tag{119}$$

It is now apparent that if we denote as $\mathbf{A}(x)$ the matrix that we obtain by changing from 0 to x the argument in the first row of the matrix $\mathbf{A}$ defined by Eq. (111), we can, for the case $\varpi = 1$ and $m = 0$, write our systems of linear algebraic equations for the coefficients $\{c_\alpha(\tau)\}$ and $\{d_\alpha(\tau)\}$ as:

$$\mathbf{Bc}(\tau)-\mathbf{A}(\tau_0)\mathbf{d}(\tau)=2\mathbf{F}_1(\tau) \tag{120}$$

and

$$\mathbf{Bd}(\tau)-\mathbf{A}(\tau_0)\mathbf{c}(\tau)=2\mathbf{F}_2(\tau), \tag{121}$$

where $\mathbf{B}$ is defined as in Eq. (110), and the right-hand-side vectors are defined as

$$\mathbf{F}_1(\tau)=\mathbf{E}_1(\tau)-\tfrac{1}{2}\mathbf{D}(\tau_0-\tau)\mathbf{A}(\tau)\mathbf{b} \tag{122}$$

and

$$\mathbf{F}_2(\tau)=\mathbf{E}_2(\tau)-\tfrac{1}{2}\mathbf{D}(\tau)\mathbf{A}(\tau_0-\tau)\mathbf{a}. \tag{123}$$

Clearly, Eqs. (120) and (121) can be decoupled as done for Eqs. (84) and (85) in Section 4. The result can be written in the form of Eqs. (89) and (90) except that $\mathbf{A}(\tau_0)$ replaces $\mathbf{A}$ in the definitions of $\mathbf{C}(0)$ and $\mathbf{M}(0)$ and, in addition, $\mathbf{A}(\tau)$ and $\mathbf{A}(\tau_0-\tau)$ replace $\mathbf{A}$ in the definitions of $\mathbf{F}_1(\tau)$ and $\mathbf{F}_2(\tau)$, respectively. As for the general case, considering that a LU factorization of $\mathbf{B}$ has been performed in order to solve the linear systems for $\mathbf{a}$ and $\mathbf{b}$, we need to perform only one additional LU factorization to be able to solve the linear systems for $\mathbf{c}(\tau)$ and $\mathbf{d}(\tau)$, for any number of interior points. Finally, here we have also confirmed computationally, for various $\tau_0 \in (0,\infty)$, that our scheme for solving the resulting linear systems is well behaved for $\tau_0 \to \infty$.

6. Computational methods

In this section, we wish to discuss some computational aspects relevant to our reported F_N solution. We begin by specifying our choice of the basis functions $\{\Phi_\alpha^m(\mu)\}$ used in the F_N approximations given by Eqs. (56), (57), (80) and (81). As discussed in the Introduction, two sets of basis functions widely used in the past, namely the power basis and the shifted-Legendre basis, did not perform well for the class of problems we are addressing in this paper. After some searching, we found that the even associated Legendre functions could be used with confidence for all required values of the Fourier index, and so we write, for $\alpha = 0,1,\ldots,N$,

$$\Phi_\alpha^m(\mu) = P_{m+2\alpha}^m(\mu). \tag{124}$$

Another difficulty showed up when we had to select a collocation scheme for solving Eqs. (58), (59), (82) and (83). From all the schemes that have been reported in the literature [2], only one was successful in producing well-conditioned systems of linear equations for the coefficients of the F_N approximations, when used in conjunction with the basis functions defined by Eq. (124). Our selected collocation scheme is a slight variant of a scheme introduced by McCormick and Sanchez [27] and consists of the *positive* discrete eigenvalues ν_β^m, $\beta = 0,1,\ldots,\aleph^m - 1$ and the *positive* zeros of the Chandrasekhar polynomial $g_{m+2N^*+2}^m(\xi)$ with magnitudes of less than one. The value of N^* that defines the order of the Chandrasekhar polynomial is, in principle, given by $N^* = N$, but we can have $N^* < N$, since sometimes more than $N + 1 - \aleph^m$ zeros of $g_{m+2N+2}^m(\xi)$ appear in (0,1); when this occurs the value of N^* has to be reduced successively by 1 until exactly $N + 1 - \aleph^m$ collocation points are obtained in (0,1). We should also mention that, in order to obtain well-conditioned systems of equations for intermediate and large values of the Fourier index, we introduced a scaling that consists in dividing each equation by the Euclidean norm of the corresponding row of the associated **B** matrix. Thus, our prescription for an accurate determination of the coefficients of the F_N approximations involves a mix of three ingredients: a basis composed of the even associated Legendre functions, a collocation scheme that uses the zeros of a Chandrasekhar polynomial in the continuum and a scaling of the linear algebraic equations.

We now turn our attention to the computation of the functions $A_\alpha^m(\xi)$ and $B_\alpha^m(\xi)$ defined by Eqs. (60)–(62). First we note that particularly accurate and efficient methods are reported in the literature for computing the zeros of the dispersion function [23] and the zeros of the Chandrasekhar polynomials [28] that define the set of collocation points for which we need to compute these functions. Taking into account our choice of basis functions given by Eq. (124), we find that the functions $A_\alpha^m(\xi)$ and $B_\alpha^m(\xi)$ are given here by

$$A_\alpha^m(\xi) = \varpi \int_0^1 \mu G^m(-\xi,\mu) P_{m+2\alpha}^m(\mu) \frac{\mathrm{d}\mu}{\xi+\mu}, \tag{125}$$

for $\xi = \{\nu_\beta^m\} \cup [0,1]$,

$$B_{\alpha}^{m}(\nu_{\beta}^{m}) = \varpi \int_{0}^{1} \mu G^{m}(\nu_{\beta}^{m}, \mu) P_{m+2\alpha}^{m}(\mu) \frac{d\mu}{\nu_{\beta}^{m} - \mu} \tag{126}$$

and

$$B_{\alpha}^{m}(\nu) = 2(1-\nu^2)^{-m/2}\lambda^{m}(\nu) P_{m+2\alpha}^{m}(\nu) + \varpi \fint_{0}^{1} \mu G^{m}(\nu, \mu) P_{m+2\alpha}^{m}(\mu) \frac{d\mu}{\nu - \mu}, \tag{127}$$

for $\nu \in [0,1]$. If we now consider $s = \nu_{\beta}^{m}$, $\beta = 0,1,\ldots,\aleph^{m}-1$, in Eqs. (30) and (31), we find that

$$g_{l}^{m}(\nu_{\beta}^{m}) = \frac{\varpi \nu_{\beta}^{m}}{2} \int_{-1}^{1} G^{m}(\nu_{\beta}^{m}, \mu) P_{l}^{m}(\mu) \frac{d\mu}{\nu_{\beta}^{m} - \mu}, \tag{128}$$

for $l \geq m$. Similarly, if we let $s \to \nu \in [0,1]$ in Eqs. (30) and (31) and use the Plemelj formulas, we find that

$$g_{l}^{m}(\nu) = (1-\nu^2)^{-m/2}\lambda^{m}(\nu) P_{l}^{m}(\nu) + \frac{\varpi \nu}{2} \fint_{-1}^{1} G^{m}(\nu, \mu) P_{l}^{m}(\mu) \frac{d\mu}{\nu - \mu} \tag{129}$$

for $l \geq m$. Using Eqs. (128) and (129) for $l = m + 2\alpha$ and noting Eqs. (125)–(127), we can readily show that the A and B functions are related by

$$B_{\alpha}^{m}(\xi) = A_{\alpha}^{m}(\xi) + 2\left(\frac{h_{m+2\alpha}}{2m + 4\alpha + 1}\right) g_{m+2\alpha}^{m}(\xi), \tag{130}$$

for $\xi = \{\nu_{\beta}^{m}\} \cup [0,1]$. In this work, we have implemented two different Gaussian integration techniques to compute the required A functions. With the A functions available, we computed the Chandrasekhar polynomials as discussed below and then used Eq. (130) to obtain the required B functions.

Our first and more conventional technique for computing the A functions consists in using a standard Gauss–Legendre quadrature [29] in the interval [0,1] for performing the integral in Eq. (125). An accuracy study was done in order to determine the number of quadrature points to be used in the calculation. Our second technique is based on the use of the linear-divisor modification algorithm [30] to generate the quadrature rule associated with the weighting function $1/(\xi + \mu)$ that appears in the integrand of Eq. (125). Since the computation of the A functions with this rule amounts to integrate the polynomial $\mu G^{m}(-\xi, \mu) P_{m+2\alpha}^{m}(\mu)$ for $\alpha = 0,1,\ldots,N$, the required number of quadrature points for performing these integrals exactly (aside from round-off/truncation errors) is known in advance. It turns out that our first integration technique usually requires more quadrature points than the second to attain a desired degree of accuracy, but it has the advantage that the same rule can be used for all of the collocation points while the second technique requires a specific rule for each collocation point.

In regard to the computation of the Chandrasekhar polynomials, we note that some years ago we proposed a method [20] that we believed to be accurate in all situations for computing these polynomials in high order and high degree. However, for some of the Fourier components of the test problem that is described in Section 7, we have detected the occurrence of zeros of the Chandrasekhar polynomials in (0,1) that change only slightly as the order of the polynomial is increased. As we have found that the method of Ref. [20] is not capable of computing accurately the Chandrasekhar polynomials for these zeros, we had to devise a modified procedure (discussed below) that works well even in this situation. Since the behavior of these zeros is very similar to the behavior of the zeros of the Chandrasekhar polynomials with magnitudes greater than one that approach the discrete eigenvalues as the order of the polynomial approaches infinity, we call these nearly stationary zeros in (0,1) *pseudo discrete eigenvalues*. We plan to report on this subject in more detail at a later date. Noting that in this work we require $g_l^m(\xi)$ for $l = m, m+1, \ldots, M = \max\{L, m+2N\}$, where the argument ξ belongs to the set of collocation points or is any $\mu \in [0,1]$, we now discuss our modified procedure for computing these polynomials. First, we note that when ξ is a discrete eigenvalue our original procedure [20] works well; therefore modifications are required only for $\xi \in [0,1]$.

We begin by considering the case $M \le m + 2N^* + 2$, where $N^* \le N$ as discussed in the beginning of this section. When ξ is a collocation point, i.e. one of the positive zeros of $g_{m+2N^*+2}^m(\xi)$, we use basically the same procedure used in Ref. [20] for computing the Chandrasekhar polynomials when the argument is a discrete eigenvalue. The only difference is that here the backward calculation for the ratios $G_l^m(\xi) = g_{l+1}^m(\xi)/g_l^m(\xi)$ is started with $G_{m+2N^*+1}^m(\xi) = 0$ and is stopped when the ratios computed by perturbed and unperturbed calculations performed in parallel differ by more than a specified tolerance two times consecutively. Then the calculation is switched to forward recursion of Eq. (25) as in Ref. [20]. This calculation is also performed in perturbed and unperturbed modes and is stopped if the difference between the results of the two modes exceeds the specified tolerance three times consecutively. When this occurs, the calculation is completed by solving a linear system as in Ref. [20]. Having finished the calculation for all of the collocation points in (0,1), we can compute the Chandrasekhar polynomials for any $\mu \in [0,1]$, as required in Eqs. (78), (79), (91) and (92), by using the Darboux formula reported in Ref. [20] to relate the Chandrasekhar polynomials for μ to the Chandrasekhar polynomials for the collocation point in (0,1) closest to μ.

To complete our description of the modifications required in the procedure of Ref. [20], we now consider the case $M > m + 2N^* + 2$. We first define $L^* = 2[(M + 1 - m)/2]$, where we use $[x]$ to denote the integer part of x, and find the *positive* zeros of $g_{m+L^*}^m(\eta)$, which we denote as η_j^m, $j = 1,2,\ldots,L^*/2$. Then, for any $\xi \in [0,1]$ we use the Darboux formula reported in Ref. [20] to relate the Chandrasekhar polynomials for ξ to the Chandrasekhar polynomials for the value of η_j^m closest to ξ, that we assume have been previously computed using the procedure defined in the foregoing paragraph.

Finally, to close this section, we recall that for the case $\varpi = 1$ and $m = 0$ we require the functions $\mathcal{A}_\alpha(x)$ and $\mathcal{B}_\alpha(x)$ defined by Eqs. (106) and (107). With the choice of basis functions given by Eq. (124) and the definition

$$\Delta_{\alpha,\beta} = \int_0^1 P_{2\alpha}(\mu)\mu^{\beta+1}\mathrm{d}\mu, \tag{131}$$

we can write Eqs. (106) and (107) as

Table 1
The scattered component $I_*(\tau,\mu,\phi)$ for $\phi = 0$

μ	$\tau = 0$	$\tau = 3.2$	$\tau = 6.4$	$\tau = 12.8$	$\tau = 32$	$\tau = 48$	$\tau = 64$
−1.0	1.5693(−2)	4.3109(−3)	1.9173(−3)	4.1771(−4)	4.6929(−6)	1.1176(−7)	
−0.9	3.1461(−2)	4.4831(−3)	1.9566(−3)	4.3218(−4)	4.8971(−6)	1.1667(−7)	
−0.8	5.5820(−2)	5.1302(−3)	2.1107(−3)	4.5717(−4)	5.1688(−6)	1.2317(−7)	
−0.7	9.4452(−2)	6.0688(−3)	2.3347(−3)	4.9108(−4)	5.5193(−6)	1.3153(−7)	
−0.6	1.5445(−1)	7.3441(−3)	2.6338(−3)	5.3502(−4)	5.9630(−6)	1.4209(−7)	
−0.5	2.4801(−1)	9.0382(−3)	3.0209(−3)	5.9062(−4)	6.5166(−6)	1.5527(−7)	
−0.4	3.9563(−1)	1.1271(−2)	3.5139(−3)	6.5996(−4)	7.2002(−6)	1.7154(−7)	
−0.3	6.3352(−1)	1.4211(−2)	4.1372(−3)	7.4564(−4)	8.0377(−6)	1.9146(−7)	
−0.2	1.0302	1.8099(−2)	4.9226(−3)	8.5090(−4)	9.0581(−6)	2.1572(−7)	
−0.1	1.7320	2.3293(−2)	5.9120(−3)	9.7976(−4)	1.0296(−5)	2.4516(−7)	
−0.0	2.3726	3.0378(−2)	7.1611(−3)	1.1372(−3)	1.1795(−5)	2.8077(−7)	
0.0		3.0378(−2)	7.1611(−3)	1.1372(−3)	1.1795(−5)	2.8077(−7)	3.3135(−9)
0.1		4.0795(−2)	8.7454(−3)	1.3295(−3)	1.3606(−5)	3.2381(−7)	5.9127(−9)
0.2		6.6734(−2)	1.0774(−2)	1.5644(−3)	1.5793(−5)	3.7578(−7)	7.5577(−9)
0.3		9.3951(−2)	1.3434(−2)	1.8517(−3)	1.8437(−5)	4.3856(−7)	9.3298(−9)
0.4		1.1178(−1)	1.6980(−2)	2.2036(−3)	2.1634(−5)	5.1449(−7)	1.1342(−8)
0.5		1.2305(−1)	2.1367(−2)	2.6351(−3)	2.5510(−5)	6.0650(−7)	1.3687(−8)
0.6		1.2015(−1)	2.5786(−2)	3.1619(−3)	3.0218(−5)	7.1827(−7)	1.6467(−8)
0.7		1.0525(−1)	2.8953(−2)	3.7887(−3)	3.5957(−5)	8.5450(−7)	1.9805(−8)
0.8		8.3308(−2)	2.9775(−2)	4.4833(−3)	4.2974(−5)	1.0212(−6)	2.3849(−8)
0.9		5.8123(−2)	2.7511(−2)	5.1401(−3)	5.1579(−5)	1.2261(−6)	2.8790(−8)
1.0		2.4544(−2)	1.8366(−2)	5.2268(−3)	6.2035(−5)	1.4792(−6)	3.4874(−8)

Table 2
The scattered component $I_*(\tau,\mu,\phi)$ for $\phi = \pi/2$

μ	$\tau = 0$	$\tau = 3.2$	$\tau = 6.4$	$\tau = 12.8$	$\tau = 32$	$\tau = 48$	$\tau = 64$
−1.0	1.5693(−2)	4.3109(−3)	1.9173(−3)	4.1771(−4)	4.6929(−6)	1.1176(−7)	
−0.9	1.8133(−2)	4.6440(−3)	2.0201(−3)	4.3649(−4)	4.8983(−6)	1.1668(−7)	
−0.8	2.1015(−2)	5.0558(−3)	2.1548(−3)	4.6141(−4)	5.1700(−6)	1.2317(−7)	
−0.7	2.4440(−2)	5.5644(−3)	2.3263(−3)	4.9350(−4)	5.5202(−6)	1.3153(−7)	
−0.6	2.8399(−2)	6.1899(−3)	2.5408(−3)	5.3403(−4)	5.9631(−6)	1.4210(−7)	
−0.5	3.3005(−2)	6.9544(−3)	2.8059(−3)	5.8449(−4)	6.5156(−6)	1.5527(−7)	
−0.4	3.8125(−2)	7.8841(−3)	3.1310(−3)	6.4670(−4)	7.1976(−6)	1.7154(−7)	
−0.3	4.3707(−2)	9.0094(−3)	3.5274(−3)	7.2284(−4)	8.0331(−6)	1.9145(−7)	
−0.2	4.9155(−2)	1.0365(−2)	4.0084(−3)	8.1554(−4)	9.0507(−6)	2.1571(−7)	
−0.1	5.2813(−2)	1.1991(−2)	4.5902(−3)	9.2799(−4)	1.0285(−5)	2.4515(−7)	
−0.0	3.6615(−2)	1.3926(−2)	5.2918(−3)	1.0641(−3)	1.1779(−5)	2.8076(−7)	
0.0		1.3926(−2)	5.2918(−3)	1.0641(−3)	1.1779(−5)	2.8076(−7)	3.3135(−9)
0.1		1.6204(−2)	6.1348(−3)	1.2284(−3)	1.3584(−5)	3.2379(−7)	5.9127(−9)
0.2		1.8815(−2)	7.1431(−3)	1.4269(−3)	1.5764(−5)	3.7575(−7)	7.5576(−9)
0.3		2.1553(−2)	8.3390(−3)	1.6663(−3)	1.8398(−5)	4.3852(−7)	9.3297(−9)
0.4		2.3931(−2)	9.7301(−3)	1.9551(−3)	2.1583(−5)	5.1444(−7)	1.1342(−8)
0.5		2.5576(−2)	1.1284(−2)	2.3032(−3)	2.5442(−5)	6.0643(−7)	1.3687(−8)
0.6		2.6419(−2)	1.2916(−2)	2.7212(−3)	3.0130(−5)	7.1818(−7)	1.6467(−8)
0.7		2.6571(−2)	1.4521(−2)	3.2191(−3)	3.5844(−5)	8.5438(−7)	1.9804(−8)
0.8		2.6203(−2)	1.6004(−2)	3.8034(−3)	4.2834(−5)	1.0210(−6)	2.3849(−8)
0.9		2.5483(−2)	1.7297(−2)	4.4747(−3)	5.1424(−5)	1.2259(−6)	2.8790(−8)
1.0		2.4544(−2)	1.8366(−2)	5.2268(−3)	6.2035(−5)	1.4792(−6)	3.4874(−8)

Table 3
The scattered component $I_*(\tau,\mu,\phi)$ for $\phi = \pi$

μ	$\tau = 0$	$\tau = 3.2$	$\tau = 6.4$	$\tau = 12.8$	$\tau = 32$	$\tau = 48$	$\tau = 64$
−1.0	1.5693(−2)	4.3109(−3)	1.9173(−3)	4.1771(−4)	4.6929(−6)	1.1176(−7)	
−0.9	2.1743(−2)	5.1121(−3)	2.1061(−3)	4.4106(−4)	4.8994(−6)	1.1668(−7)	
−0.8	3.7822(−2)	5.6234(−3)	2.2545(−3)	4.6649(−4)	5.1712(−6)	1.2317(−7)	
−0.7	4.6109(−2)	6.1326(−3)	2.4165(−3)	4.9756(−4)	5.5211(−6)	1.3153(−7)	
−0.6	4.8097(−2)	6.6837(−3)	2.5999(−3)	5.3563(−4)	5.9633(−6)	1.4210(−7)	
−0.5	5.5274(−2)	7.2900(−3)	2.8107(−3)	5.8212(−4)	6.5147(−6)	1.5527(−7)	
−0.4	6.6011(−2)	7.9645(−3)	3.0552(−3)	6.3860(−4)	7.1952(−6)	1.7153(−7)	
−0.3	9.2217(−2)	8.7247(−3)	3.3401(−3)	7.0695(−4)	8.0285(−6)	1.9145(−7)	
−0.2	1.6293(−1)	9.5903(−3)	3.6723(−3)	7.8939(−4)	9.0434(−6)	2.1571(−7)	
−0.1	1.3044(−1)	1.0586(−2)	4.0600(−3)	8.8855(−4)	1.0274(−5)	2.4514(−7)	
−0.0	8.6109(−2)	1.1742(−2)	4.5125(−3)	1.0075(−3)	1.1764(−5)	2.8074(−7)	
0.0		1.1742(−2)	4.5125(−3)	1.0075(−3)	1.1764(−5)	2.8074(−7)	3.3135(−9)
0.1		1.3093(−2)	5.0411(−3)	1.1501(−3)	1.3563(−5)	3.2376(−7)	5.9127(−9)
0.2		1.4704(−2)	5.6589(−3)	1.3207(−3)	1.5735(−5)	3.7572(−7)	7.5576(−9)
0.3		1.6540(−2)	6.3808(−3)	1.5248(−3)	1.8359(−5)	4.3848(−7)	9.3297(−9)
0.4		1.8276(−2)	7.2201(−3)	1.7690(−3)	2.1532(−5)	5.1439(−7)	1.1342(−8)
0.5		1.8952(−2)	8.1616(−3)	2.0615(−3)	2.5375(−5)	6.0636(−7)	1.3687(−8)
0.6		1.8127(−2)	9.1387(−3)	2.4125(−3)	3.0043(−5)	7.1809(−7)	1.6467(−8)
0.7		1.6980(−2)	1.0121(−2)	2.8352(−3)	3.5732(−5)	8.5427(−7)	1.9804(−8)
0.8		1.5915(−2)	1.1172(−2)	3.3500(−3)	4.2696(−5)	1.0209(−6)	2.3849(−8)
0.9		1.5655(−2)	1.2610(−2)	4.0040(−3)	5.1273(−5)	1.2258(−6)	2.8790(−8)
1.0		2.4544(−2)	1.8366(−2)	5.2268(−3)	6.2035(−5)	1.4792(−6)	3.4874(−8)

$$\mathcal{A}_\alpha(x) = \frac{h_1}{3} x\Delta_{\alpha,0} - \Delta_{\alpha,1} \tag{132}$$

and

$$\mathcal{B}_\alpha(x) = \frac{h_1}{3} x\Delta_{\alpha,0} + \Delta_{\alpha,1}. \tag{133}$$

It is a simple matter to show, with the help of some elementary properties of the Legendre polynomials, that the integrals required in Eqs. (132) and (133) are given explicitly by

$$\Delta_{\alpha,0} = (-1)^{\alpha+1}\left[\frac{1}{(2\alpha-1)(2\alpha+1)}\right]\left[\frac{(2\alpha+1)!!}{(2\alpha+2)!!}\right] \tag{134}$$

and

$$\Delta_{\alpha,1} = \tfrac{1}{3}\left(\delta_{0,\alpha} + \tfrac{2}{5}\delta_{1,\alpha}\right). \tag{135}$$

7. Numerical results and concluding remarks

In order to give some specific numerical results, we report in Tables 1–3 our F_N results for the scattered component $I_*(\tau, \mu, \phi)$ for a problem defined by $\varpi = 0.9$, $\tau_0 = 64$, $\mu_0 = 0.2$, $\phi_0 = 0$ and coefficients β_l, $l = 0,1,\ldots,L = 299$, tabulated in Refs. [9, 28] for the cloud C_1 phase function defined in Ref. [10]. The numerical results reported in Tables 1–3 were obtained by using $N = 699$ for the Fourier components $0 \le m \le 25$, $N = 599$ for $26 \le m \le 50$, $N = 499$ for $51 \le m \le 70$, $N = 399$ for $71 \le m \le 100$, $N = 299$ for $101 \le m \le 150$ and $N = 199$ for $151 \le m \le 299$, and are thought to be accurate in general to within ± 1 in the last figure shown; moreover these results were confirmed using an improved version of the spherical-harmonics (P_N) method with Mark boundary conditions [31]. It is clear from these tables that as the scattered radiation propagates deeper into the cloud layer it becomes more and more azimuthally symmetric.

Finally, to conclude this work, we report a couple of observations we have made in regard to the performance of the method. First, we have confirmed our previous observation [3] that the F_N method converges faster (i.e. with fewer terms in the approximation) and more uniformly in μ than the P_N method, for a given level of precision. On the other hand, we note that the current version of the F_N method is computationally less efficient, in high order, than the P_N method. This is in a great part because of the time spent by the code in numerical integrations used to compute the A functions defined by Eq. (125). For example, when an approximation of order $N \ge 299$ is used for solving the considered problem, about 90% of the total execution time is spent in the calculation of these functions. Clearly, this aspect of the method needs to be improved. Our second observation has to do with the fact that, for some other problems that we tried, we found that the post processing technique defined by Eqs. (78), (79), (91) and (92) failed to give accurate results. We were able to resolve this problem by reformulating our post processing technique, but since there are some specific points of the new technique that we still need to sort out, we plan to report our improved post processing prescription at a later date.

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