Pergamon

J. Quant. Spectrosc. Radiat. Transfer Vol. 60, No. 2, pp. 261–276, 1998

Printed in Great Britain
PII: S0022-4073(97)00171-4
0022-4073/98 $19.00 + 0.00

THE F_N METHOD FOR SPECTRAL-LINE FORMATION BY COMPLETELY NONCOHERENT SCATTERING

L. B. BARICHELLO[a] and C. E. SIEWERT[b]

[a]Instituto de Matemática, Universidade Federal do Rio Grande do Sul, 91509-900 Porto Alegre, RS, Brasil and [b]Mathematics Department, North Carolina State University, Raleigh, NC 27695-8205, U.S.A.

(Received 28 April 1997)

Abstract—The F_N method is used to develop a solution to a class of nongrey problems in the theory of radiative transfer. The model considered allows for scattering with complete frequency redistribution (completely noncoherent scattering) and continuum absorption. In addition to a general formulation, a specific solution is developed for an inhomogeneous source term (Planck function) that varies linearly with optical depth in a semi-infinite medium. Test problems based on Doppler and Lorentz profiles of the line-scattering coefficient are considered, and numerical results (thought to be correct to five significant figures) are given for the frequency-dependent intensity exiting the medium and for the source function within the medium. For comparison purposes, a previously reported solution that is expressed in terms of Chandrasekhar's H function is evaluated numerically.

1. INTRODUCTION

Some 25 years or so ago, McCormick and Siewert[1] used an expansion in terms of singular eigenfunctions[2] to solve analytically a class of nongrey problems in radiative transfer that was based on the equation of transfer written, after Hummer,[3] as

$$\mu \frac{\partial}{\partial \tau} I_x(\tau, \mu) = [\phi(x) + \beta][S_x(\tau) - I_x(\tau, \mu)] \tag{1}$$

where $S_x(\tau)$ is the source function,

$$[\phi(x) + \beta]S_x(\tau) = \frac{1}{2}\varpi\phi(x)\int_{-\infty}^{\infty}\phi(x')\int_{-1}^{1} I_{x'}(\tau, \mu')\mathrm{d}\mu'\mathrm{d}x' + [\rho\beta + (1 - \varpi)\phi(x)]B(\tau), \tag{2}$$

and $B(\tau)$ is the Planck function. We note that $\tau \geq 0$ is the optical variable and $\mu \in [-1, 1]$ is the *cosine* of the polar angle (as measured from the *positive* τ axis) that describes the direction of propagation of the radiation. In addition, $\varpi \in [0, 1]$ is the albedo for single scattering, $\beta > 0$ is the ratio of the continuum absorption coefficient to the average line coefficient, ρ is the ratio of the continuum source function to the Planck function and $\phi(x)$ is the line-scattering profile. We note that while some of the quantities we compute here remain valid in the limit $\beta \to 0$, other quantities, as we shall see, do not. In fact the case $\beta = 0$ requires special attention since for some specific line-scattering profiles certain integrals we use fail to exist for this case.

For a specified Planck function $B(\tau)$ we seek a solution of Eq. (1) subject to the boundary condition that no radiation is entering the medium, i.e.

$$I_x(0, \mu) = 0 \tag{3}$$

for $\mu \in (0,1]$.

In addition to providing a formulation to the problem we consider here, Hummer[3] developed some asymptotic results, reported a solution based on a discrete-ordinates method and provided some of the first numerical results for this challenging class of problems. We note also that Ivanov[4] and colleagues have reported numerous works devoted to analytical and computational aspects of this problem; Ref. 4 in particular provides an excellent entry into this body of work.

In this paper we first report an H-function calculation and an evaluation of the analytical results given in Ref. 1 for $I_x(0,-\mu)$, $\mu\in(0, 1]$, and $S_x(0)$. Then, in order to establish some numerical results valid on the surface and within the medium, we use the F_N method to compute the radiation intensity exiting the surface, $I_x(0,-\mu)$ for $\mu\in(0, 1]$ and the source function $S_x(\tau)$ for $\tau\geq 0$.

Although the development in Ref. 1 is general in regard to the given Planck function and line-scattering profile $\phi(x)$, we focus our numerical work here on a linear form of the Planck function

$$B(\tau)=B_0+B_1\tau \tag{4}$$

where the constants $\{B_0, B_1\}$ are considered given, and the Doppler

$$\phi(x)=\frac{1}{\sqrt{\pi}}\mathrm{e}^{-x^2} \tag{5a}$$

and the Lorentz

$$\phi(x)=\frac{1}{\pi(1+x^2)} \tag{5b}$$

scattering profiles.

2. SUMMARY OF A PREVIOUS WORK

As the considered problem was solved explicitly in Ref. 1, we wish here simply to summarize the basic elements of Ref. 1 that serve as a starting point for the current work. First of all, a change of the angular variable by $\xi=\mu\gamma_x$ with

$$\gamma_x=[\phi(x)+\beta]^{-1} \tag{6}$$

allows us to rewrite Eq. (1) as

$$\xi\frac{\partial}{\partial\tau}I_x(\tau,\xi)+I_x(\tau,\xi)=\frac{1}{2}\varpi\phi(x)\int_{-\gamma}^{\gamma}\int_{M_{\xi'}}\phi(x')I_{x'}(\tau,\xi')\mathrm{d}x'\mathrm{d}\xi'+Q_x(\tau) \tag{7}$$

where

$$Q_x(\tau)=[\rho\beta+(1-\varpi)\phi(x)]B(\tau) \tag{8}$$

is an inhomogeneous source term, and where $\gamma=\sup\gamma_x$. Since the considered profiles $\phi(x)$ vanish at infinity it is clear that $\gamma=1/\beta$. In addition, the set M_ξ is defined such that $x\in M_\xi$ if and only if $[\phi(x)+\beta]|\xi|\leq 1$.

As was pointed out in Ref. 1, the desired solution can be written as

$$I_x(\tau,\xi)=I_x^p(\tau,\xi)+\phi(x)G(\tau,\xi)-\left[I_x^p(0,\xi)+\phi(x)G(0,\xi)\right]\mathrm{e}^{-\tau/\xi} \tag{9a}$$

and

$$I_x(\tau,-\xi)=I_x^p(\tau,-\xi)+\phi(x)G(\tau,-\xi) \tag{9b}$$

for $\xi\in(0,\gamma)$. Here $I_x^p(\tau,\xi)$ is a particular solution of Eq. (7) corresponding to a specified inhomogeneous source term $Q_x(\tau)$, and the function $G(\tau,\xi)$ is a solution (that vanishes as τ tends to infinity) of

$$\xi\frac{\partial}{\partial\tau}G(\tau,\xi)+G(\tau,\xi)=\int_{-\gamma}^{\gamma}\Psi(\xi')G(\tau,\xi')\mathrm{d}\xi' \tag{10}$$

such that

$$\Psi(\xi)G(0,\xi)=\Gamma(\xi),\qquad \xi\in(0,\gamma). \tag{11}$$

Here

$$\Psi(\xi) = \frac{\varpi}{2}\int_{M_\xi} \phi^2(x)\mathrm{d}x \tag{12}$$

and

$$\Gamma(\xi) = -\frac{\varpi}{2}\int_{M_\xi} \phi(x) I_x^p(0,\xi)\mathrm{d}x. \tag{13}$$

In Ref. 1, McCormick and Siewert used classical eigenfunction methods and an H function similar to those discussed by Chandrasekhar[5] to establish the desired solution of the G problem defined by Eqs. (10) and (11); in particular, they expressed the emerging intensity and the source function on the surface as

$$I_x(0,-\xi) = I_x^p(0,-\xi) + \phi(x)G(0,-\xi) \tag{14}$$

and

$$S_x(0) = \gamma_x \lim_{\xi\to 0} I_x(0,-\xi) \tag{15}$$

where

$$G(0,-\xi) = H(\xi)\int_0^\gamma \xi'\Gamma(\xi')H(\xi')\frac{\mathrm{d}\xi'}{\xi'+\xi}. \tag{16}$$

Here the function $H(\xi)$ satisfies, in addition to a singular-integral equation, the nonlinear integral equation

$$H(z) = 1 + zH(z)\int_0^\gamma \Psi(z')H(z')\frac{\mathrm{d}z'}{z'+z}, \quad z \in [0,\gamma]. \tag{17}$$

We note that our notation here differs slightly from that of Ref. 1 in that we have included a factor ϖ with the previous definitions of $G(\tau,\xi)$ and $\Gamma(\xi)$.

It is clear that to complete the definition of the desired solution we require a particular solution that corresponds to the inhomogeneous source term defined by Eqs. (4) and (8). In Ref. 1 an appropriate particular solution was expressed as

$$I_x^p(\tau,\xi) = \left[B_0 + B_1(\tau-\xi)\right][L\phi(x) + \rho\beta] \tag{18}$$

where

$$L = \varpi K + 1 - \varpi \tag{19a}$$

and

$$\varpi K = \frac{1}{\Lambda(\infty)}\int_{-\gamma}^{\gamma}[\rho\beta\Delta(\xi) + (1-\varpi)\Psi(\xi)]\mathrm{d}\xi \tag{19b}$$

with

$$\Lambda(\infty) = 1 - \int_{-\gamma}^{\gamma}\Psi(\xi)\mathrm{d}\xi \tag{20}$$

and

$$\Delta(\xi) = \frac{\varpi}{2}\int_{M_\xi}\phi(x)\mathrm{d}x. \tag{21}$$

Next we can substitute Eq. (18) into Eq. (13) to find

$$\Gamma(\xi) = [L\Psi(\xi) + \rho\beta\Delta(\xi)][B_1\xi - B_0] \tag{22}$$

which can then be used in Eq. (16) so that we can express the required solution for $G(0, -\xi)$ as

$$G(0, -\xi) = B_1\big[L\Upsilon_1(\xi) + \rho\beta\Xi_1(\xi)\big] - B_0\big[L\Upsilon_0(\xi) + \rho\beta\Xi_0(\xi)\big] \tag{23}$$

for $\xi \in (0, \gamma)$. Here the functions $\Upsilon(\xi)$ and $\Xi(\xi)$ are independent of ρ, B_0 and B_1. Clearly

$$\Upsilon_0(\xi) = H(\xi)\int_0^\gamma \xi'\Psi(\xi')H(\xi')\frac{\mathrm{d}\xi'}{\xi'+\xi}, \tag{24}$$

$$\Upsilon_1(\xi) = H(\xi)\int_0^\gamma \xi'^2\Psi(\xi')H(\xi')\frac{\mathrm{d}\xi'}{\xi'+\xi}, \tag{25}$$

$$\Xi_0(\xi) = H(\xi)\int_0^\gamma \xi'\Delta(\xi')H(\xi')\frac{\mathrm{d}\xi'}{\xi'+\xi} \tag{26}$$

and

$$\Xi_1(\xi) = H(\xi)\int_0^\gamma \xi'^2\Delta(\xi')H(\xi')\frac{\mathrm{d}\xi'}{\xi'+\xi}. \tag{27}$$

We note that the integrals defining the Υ functions can be expressed in terms of the H function and moments of the H function; however, since we intend eventually to evaluate the Ξ functions by numerical integration, we choose to evaluate the Υ functions in the same way. We proceed now to consider two specific choices of the line-scattering profile and to report some numerical results.

3. NUMERICAL RESULTS FOR THE DOPPLER AND LORENTZ PROFILES

As we now intend to report some numerical results, we focus our attention on the two special choices of the scattering-line profile listed in Eqs. (5), and so to continue we require the functions $\Psi(\xi)$ and $\Delta(\xi)$ defined by Eq. (12) and Eq. (21) and the constant L defined by Eqs. (19).

Considering first the case of the Doppler profile, we find we can substitute Eq. (5a) into Eq. (12) to obtain

$$\Psi(\xi) = \begin{cases} \Psi_0, & \xi \in [0, \gamma_0], \\ \Psi_0 \mathrm{erfc}[\sqrt{2}m(\xi)], & \xi \in [\gamma_0, \gamma), \end{cases} \tag{28}$$

where $\mathrm{erfc}(z)$ is the complementary error function, and where

$$m(\xi) = \sqrt{\ln\left[\frac{\xi}{\sqrt{\pi}(1-\beta\xi)}\right]}, \qquad \xi \in [\gamma_0, \gamma). \tag{29}$$

In addition

$$\Psi_0 = \frac{\varpi}{4}\sqrt{\frac{2}{\pi}} \tag{30}$$

and

$$\gamma_0 = \frac{\sqrt{\pi}}{1+\beta\sqrt{\pi}}. \tag{31}$$

In a similar manner we substitute Eq. (5a) into Eq. (21) to find

$$\Delta(\xi) = \begin{cases} \Delta_0, & \xi \in [0, \gamma_0], \\ \Delta_0 \mathrm{erfc}[m(\xi)], & \xi \in [\gamma_0, \gamma), \end{cases} \tag{32}$$

with

$$\Delta_0 = \frac{\varpi}{2}. \tag{33}$$

In regard to the constant L, we find we can use Eqs. (19) to find

$$L = \frac{1 - \varpi + \varpi\rho[1 - W(\beta\sqrt{\pi})]}{1 - \varpi W(\beta\sqrt{\pi})} \tag{34}$$

with

$$W(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\exp(-2x^2)}{\exp(-x^2) + z} \mathrm{d}x. \tag{35}$$

Considering now the case of the Lorentz profile as defined by Eq. (5b), we find that the function $\Psi(\xi)$ can be expressed as

$$\Psi(\xi) = \begin{cases} \Psi_0, & \xi \in [0, \gamma_0], \\ \Psi_0 \left\{ 1 - \dfrac{2}{\pi} \left[\dfrac{m(\xi)}{1 + m^2(\xi)} + \tan^{-1} m(\xi) \right] \right\}, & \xi \in [\gamma_0, \gamma), \end{cases} \tag{36}$$

where

$$m(\xi) = \sqrt{\frac{\xi(1 + \beta\pi) - \pi}{\pi(1 - \beta\xi)}}, \qquad \xi \in [\gamma_0, \gamma). \tag{37}$$

In addition we now have

$$\Psi_0 = \frac{\varpi}{4\pi} \tag{38}$$

and

$$\gamma_0 = \frac{\pi}{1 + \beta\pi}. \tag{39}$$

For the function $\Delta(\xi)$ we find

$$\Delta(\xi) = \begin{cases} \Delta_0, & \xi \in [0, \gamma_0], \\ \Delta_0 \left[1 - \dfrac{2}{\pi} \tan^{-1} m(\xi) \right], & \xi \in [\gamma_0, \gamma), \end{cases} \tag{40}$$

where again,

$$\Delta_0 = \frac{\varpi}{2}. \tag{41}$$

To complete this case we express the constant L as

$$L = \frac{1 - \varpi + \rho\varpi f}{1 - \varpi + \varpi f} \tag{42}$$

with

$$f = \sqrt{\frac{\beta\pi}{1 + \beta\pi}}. \tag{43}$$

It is clear that the first thing we must do in order to evaluate the established results, *viz.* Eqs. (24)–(27) of the previous section, is to compute the H function. Since we wish simply to solve Eq. (17) by iteration our first decision concerns the type of quadrature scheme we use to represent the integral term. We have done two things: first of all for "large" values of the constant β, say $\beta \geq 0.01$, we used a Gauss–Legendre scheme in each of the two intervals $[0, \gamma_0]$ and $[\gamma_0, \gamma]$ and encountered no difficulties. On the other hand, for "small" values of β, say $\beta < 0.01$, we did encounter a loss of accuracy that became more and more significant as $\beta \to 0$ in using our first quadrature scheme for these cases. After some experimentation with a Gauss–Laguerre quadrature scheme, we found good results by mapping the variable $z \in [\gamma_0,\gamma]$ into the variable $u \in [0,1]$ by way of the transformation

$$u(z) = \mathrm{e}^{-\alpha m(z)}, \qquad z \in [\gamma_0, \gamma), \tag{44}$$

with $u(\gamma) = 0$, and then using a Gauss–Legendre scheme on the original first interval $[0, \gamma_0]$ and on the second interval transformed to [0,1]. After some (casual) experimentation with the parameter α in Eq. (44) we found that while some value of $\alpha \in [1,2]$ worked well for the Doppler case, a much smaller value, say $\alpha \in [10^{-3},10^{-2}]$, was better for the Lorentz case.

In regard to a listing of some of our numerical results, we note first of all that we have not found any H-function or other results related to our Tables 1 and 2 for either the Doppler or the Lorentz profile when $\beta > 0$. And so naturally (since no expert code writers were involved in this work) we hope our confidence in these results will prove justified. For the case $\beta = 0$ we found that Ivanov and Nagirner[6] reported quite a few years ago an excellent computation and tabulation of the H function for the case of a Doppler profile. To test our quadrature schemes we repeated and confirmed the computations of Ref. 6 for selected (difficult) cases. For the case of the Lorentz profile, we used Warming's[7] H-function calculations, again for the case $\beta = 0$, to establish some confidence in our H-function results. It has to be said that these early works of Ivanov,[4] Ivanov and Nagirner,[6] and Warming[7] are very good in that much care was taken with the computations and significant asymptotic analysis was incorporated into the numerical work. In addition to extending these calculations reported in Refs. 4, 6 and 7 to the interesting cases of Doppler and Lorentz profiles with $\beta > 0$, our goal here has been to get good numerical results without having to do any asymptotic analysis, so that when we take on a version of this problem that includes polarization effects we will not (hopefully) have to face a great deal of asymptotic analysis in order to obtain good results.

We have found that Eqs. (25)–(27) become difficult to evaluate accurately (even having already good results for the H function) as β tends to zero. Of course this is not surprising

Table 1. Basic functions for a Doppler profile with $1 - \varpi = 10^{-6}$ and $\beta = 10^{-4}$

z	$\Upsilon_0(\sqrt{\pi}z)$	$\Upsilon_1(\sqrt{\pi}z)$	$\Xi_0(\sqrt{\pi}z)$	$\Xi_1(\sqrt{\pi}z)$	$H(\sqrt{\pi}z)$
0.0	9.7581(−1)	1.2153(1)	2.6409(1)	2.8791(4)	1.0000
0.1	9.7203(−1)	1.3880(1)	3.0051(1)	3.3283(4)	1.1562
0.2	9.6919(−1)	1.5132(1)	3.2745(1)	3.6649(4)	1.2733
0.5	9.6215(−1)	1.8159(1)	3.9334(1)	4.5003(4)	1.5643
1.0	9.5253(−1)	2.2157(1)	4.8130(1)	5.6404(4)	1.9621
2.0	9.3684(−1)	2.8405(1)	6.1996(1)	7.4938(4)	2.6104
5.0	9.0163(−1)	4.1423(1)	9.1244(1)	1.1625(5)	4.0658
1.0(1)	8.5907(−1)	5.5568(1)	1.2353(2)	1.6552(5)	5.8251
2.0(1)	7.9799(−1)	7.3190(1)	1.6457(2)	2.3457(5)	8.3498
5.0(1)	6.8316(−1)	9.8620(1)	2.2619(2)	3.5701(5)	1.3096(1)
1.0(2)	5.7108(−1)	1.1425(2)	2.6726(2)	4.6307(5)	1.7729(1)
2.0(2)	4.4539(−1)	1.2072(2)	2.8948(2)	5.5739(5)	2.2924(1)
5.0(2)	2.8305(−1)	1.0930(2)	2.7303(2)	6.1123(5)	2.9634(1)
1.0(3)	1.8221(−1)	8.7851(1)	2.2746(2)	5.7004(5)	3.3802(1)
2.0(3)	1.0868(−1)	6.2471(1)	1.6786(2)	4.6566(5)	3.6841(1)
5.0(3)	5.0040(−2)	3.3735(1)	9.4680(1)	2.9141(5)	3.9265(1)
$\gamma/\sqrt{\pi}$	4.4905(−2)	3.0739(1)	8.6692(1)	2.6969(5)	3.9478(1)

Table 2. Basic functions for a Lorentz profile with $1-\varpi = 10^{-6}$ and $\beta = 10^{-4}$

z	$\Upsilon_0(\pi z)$	$\Upsilon_1(\pi z)$	$\Xi_0(\pi z)$	$\Xi_1(\pi z)$	$H(\pi z)$
0.0	8.6687(−1)	1.1056(2)	4.7533(2)	1.4403(6)	1.0000
0.1	8.5413(−1)	1.2087(2)	5.1980(2)	1.5780(6)	1.0957
0.2	8.4545(−1)	1.2782(2)	5.4992(2)	1.6717(6)	1.1609
0.5	8.2607(−1)	1.4315(2)	6.1669(2)	1.8808(6)	1.3065
1.0	8.0319(−1)	1.6093(2)	6.9461(2)	2.1271(6)	1.4784
2.0	7.7171(−1)	1.8475(2)	7.9983(2)	2.4649(6)	1.7149
5.0	7.1589(−1)	2.2471(2)	9.7911(2)	3.0585(6)	2.1342
1.0(1)	6.6275(−1)	2.5926(2)	1.1383(3)	3.6130(6)	2.5333
2.0(1)	6.0022(−1)	2.9430(2)	1.3065(3)	4.2432(6)	3.0030
5.0(1)	5.0365(−1)	3.3310(2)	1.5120(3)	5.1326(6)	3.7284
1.0(2)	4.2166(−1)	3.4783(2)	1.6179(3)	5.7488(6)	4.3442
2.0(2)	3.3538(−1)	3.4124(2)	1.6394(3)	6.1605(6)	4.9924
5.0(2)	2.2418(−1)	2.9217(2)	1.4827(3)	6.0646(6)	5.8277
1.0(3)	1.5119(−1)	2.2993(2)	1.2236(3)	5.3393(6)	6.3759
2.0(3)	9.4084(−2)	1.6121(2)	8.9886(2)	4.1534(6)	6.8049
γ/π	6.5655(−2)	1.1941(2)	6.8452(2)	3.2635(6)	7.0185

since, in fact, the integrals in those equations do not exist for a Doppler or Lorentz line-scattering profile with $\beta = 0$. In order to report some numerical results we choose a meaningful (but not impossibly difficult) case, *viz*., $1-\varpi = 10^{-6}$ and $\beta = 10^{-4}$, and so we list in Tables 1 and 2 our results for the Υ and Ξ functions, along with our results for the H function, that we believe accurate to plus or minus one unit in the last digit given.

4. THE F_N METHOD FOR SURFACE QUANTITIES

While solutions to half-space problems based on isotropic-scattering models can usually be solved concisely in terms of H functions, similar analysis generally becomes considerably more difficult for problems in finite media or for problems with anisotropic scattering phase functions. Anticipating using the F_N method[8,9] to solve more general problems in the area of noncoherent scattering in finite media (including polarization effects), we wish to test the method in the current setting. We seek again, as in Section 2, a solution that vanishes at infinity of

$$\xi \frac{\partial}{\partial \tau} G(\tau,\xi) + G(\tau,\xi) = \int_{-\gamma}^{\gamma} \Psi(\xi')G(\tau,\xi')\mathrm{d}\xi' \tag{45}$$

such that

$$\Psi(\xi)G(0,\xi) = \Gamma(\xi), \qquad \xi \in (0,\gamma). \tag{46}$$

To start we follow Ref. 8, change ξ to $-\xi$ and rewrite Eq. (45) as

$$-\xi \frac{\partial}{\partial \tau} G(\tau,-\xi) + G(\tau,-\xi) = G(\tau) \tag{47}$$

where

$$G(\tau) = \int_{-\gamma}^{\gamma} \Psi(\xi)G(\tau,\xi)\mathrm{d}\xi. \tag{48}$$

Next we multiply Eq. (47) by $\exp(-\tau/s)$, $\mathscr{R}s > 0$ and $s \notin [0,\gamma]$, and integrate over τ from zero to infinity to find (after an integration by parts)

$$\int_0^\infty e^{-\tau/s} G(\tau, -\xi) d\tau = \frac{s}{s-\xi} G_*(s) + \frac{s\xi}{s-\xi} G(0, -\xi) \tag{49}$$

where

$$G_*(s) = \int_0^\infty e^{-\tau/s} G(\tau) d\tau. \tag{50}$$

We now multiply Eq. (49) by $\Psi(\xi)$ and integrate the resulting equation over ξ from $-\gamma$ to γ to obtain

$$\Lambda(s) G_*(s) + s \int_0^\gamma \xi \Psi(\xi) G(0, -\xi) \frac{d\xi}{\xi - s} = s \int_0^\gamma \xi \Gamma(\xi) \frac{d\xi}{\xi + s} \tag{51}$$

after we have used Eq. (11) and where

$$\Lambda(s) = 1 + s \int_{-\gamma}^{\gamma} \Psi(\xi) \frac{d\xi}{\xi - s}. \tag{52}$$

If in Eq. (51) we let $s \to \eta \in (0, \gamma)$ from above (+) and below (–) the real axis, we can use the Plemelj formulas[10] to find

$$\lambda(\eta) G_*(\eta) + \eta \fint_0^\gamma \xi \Psi(\xi) G(0, -\xi) \frac{d\xi}{\xi - \eta} \pm \pi i \eta \Psi(\eta) \left[G_*(\eta) + \eta G(0, -\eta) \right] = \eta \int_0^\gamma \xi \Gamma(\xi) \frac{d\xi}{\xi + \eta} \tag{53}$$

where we use the symbol $\fint$ to denote that integrals are to be evaluated in the Cauchy principal-value sense, and where

$$\lambda(\eta) = 1 + \eta \fint_{-\gamma}^{\gamma} \Psi(\xi) \frac{d\xi}{\xi - \eta}. \tag{54}$$

Finally upon eliminating $G_*(\eta)$ between the two versions of Eq. (53), we find we can write

$$\lambda(\eta) G(0, -\eta) - \fint_0^\gamma \xi \Psi(\xi) G(0, -\xi) \frac{d\xi}{\xi - \eta} = \int_0^\gamma \xi \Gamma(\xi) \frac{d\xi}{\xi + \eta}, \quad \eta \in (0, \gamma). \tag{55}$$

It is apparent that Eq. (55) is a singular-integral equation for the unknown function $G(0, -\xi)$, $\xi \in (0, \gamma)$.

In regard to possible zeros of $\Lambda(s)$, we have confirmed a result reported by Ivanov,[4] *viz*., that $\lambda(\gamma) > 1 - \varpi$, and so we can use the argument principle and the fact that $\Psi(\xi) \geq 0$ to confirm (since $\beta > 0$) another result quoted by Ivanov: the function $\Lambda(s)$ has no zeros in the complex plane cut from $-\gamma$ to γ along the real axis.

It is clear that a solution to Eq. (55) will yield the desired solution to our problem, and so here we do not actually require the H function. However, in order to provide an alternative way to express the solution to the G problem, and since H satisfies a singular-integral equation of the same form as Eq. (55), *viz*.,

$$\lambda(\eta) H(\eta) - \fint_0^\gamma \xi \Psi(\xi) H(\xi) \frac{d\xi}{\xi - \eta} = \sqrt{\Lambda(\infty)}, \qquad \eta \in (0, \gamma), \tag{56}$$

we also use the F_N method to compute the H function. We note in regard to Eq. (56) that

$$\Lambda(\infty) = 1 - \varpi W(\beta \sqrt{\pi}) \tag{57a}$$

for the Doppler profile and that

$$\Lambda(\infty) = 1 - \varpi (1 - f\ ) \tag{57b}$$

for the Lorentz profile. We recall that $W(z)$ is defined by Eq. (35) and f by Eq. (43).

Noting the definition of $\Gamma(\xi)$ as given by Eq. (22), we conclude that everything we require here can be expressed in terms of the five problems ($k = 1, 2, \ldots, 5$)

$$\lambda(\eta)N_k(\eta) - \fint_0^{\gamma}\xi\Psi(\xi)N_k(\xi)\frac{\mathrm{d}\xi}{\xi-\eta} = R_k(\eta), \qquad \eta \in (0,\gamma), \tag{58}$$

with

$$R_k(\eta) = \int_0^{\gamma}\xi^k\Psi(\xi)\frac{\mathrm{d}\xi}{\xi+\eta}, \qquad k = 1,2, \tag{59}$$

$$R_k(\eta) = \int_0^{\gamma}\xi^{k-2}\Delta(\xi)\frac{\mathrm{d}\xi}{\xi+\eta}, \qquad k = 3,4, \tag{60}$$

and $R_5(\eta)=\sqrt{\Lambda(\infty)}$. With these definitions it is clear that the results listed in Tables 1 and 2 follow from the identifications $\Upsilon_{k-1}(\xi) = N_k(\xi)$, $k = 1,2$, $\Xi_{k-3}(\xi) = N_k(\xi)$, $k = 3,4$, and $H(\xi) = N_5(\xi)$. To avoid too much notation, we suppress the subscripts in Eq. (58) and consider the general form

$$\lambda(\eta)N(\eta) - \fint_0^{\gamma}\xi\Psi(\xi)N(\xi)\frac{\mathrm{d}\xi}{\xi-\eta} = R(\eta), \qquad \eta \in (0,\gamma). \tag{61}$$

It should be noted that if we wished only to compute the H function we could use the fact that $H(0) = 1$ to rewrite Eq. (56) as

$$\lambda(\eta)H(\eta) - \eta\fint_0^{\gamma}\Psi(\xi)H(\xi)\frac{d\xi}{\xi-\eta} = 1, \qquad \eta \in (0,\gamma), \tag{62}$$

which could be a better form, especially as $\Lambda(\infty) \to 0$.

At this point, rather than seek an analytical solution of Eq. (61), we wish to use the F_N method[8,9] to develop an approximate (but sufficiently accurate) solution; we therefore propose the approximation

$$N(\xi) = \sum_{\alpha=0}^{N} a_\alpha\Phi_\alpha(\xi), \qquad \xi \in (0,\gamma), \tag{63}$$

where $\{\Phi_\alpha(\xi)\}$ is a set of approximating functions (to be specified later) and $\{a_\alpha\}$ is a set of constants to be determined. If we substitute Eq. (63) into Eq. (61) and consider the resulting equation at a set of distinct points $\{\eta_i\}$ we obtain a system of linear algebraic equations to determine the required constants, *viz*.,

$$\sum_{\alpha=0}^{N} a_\alpha B_\alpha(\eta_i) = R(\eta_i), \qquad i = 1,2,\ldots,N+1. \tag{64}$$

Here we have defined

$$B_\alpha(\eta) = \lambda(\eta)\Phi_\alpha(\eta) - \fint_0^{\gamma}\xi\Psi(\xi)\Phi_\alpha(\xi)\frac{\mathrm{d}\xi}{\xi-\eta}, \qquad \eta \in (0,\gamma). \tag{65}$$

Once we have specified a set of functions $\{\Phi_\alpha(\xi)\}$ and determined the constants $\{a_\alpha\}$ we can, of course, obtain our first approximate solution from Eq. (63). On the other hand, we can also go back to Eq. (61) to obtain a "post-processed" result. To see this we first rewrite Eq. (61) as

$$N(\eta) = R(\eta) + \fint_0^{\gamma}\xi\Psi(\xi)N(\xi)\frac{\mathrm{d}\xi}{\xi-\eta} + [1-\lambda(\eta)]N(\eta), \qquad \eta \in (0,\gamma). \tag{66}$$

We now substitute Eq. (63) into the r.h.s. of Eq. (66) and thus express our "post-processed" result as

$$N(\xi) = R(\xi) + \sum_{\alpha=0}^{N} a_\alpha C_\alpha(\xi) \tag{67}$$

where

$$C_\alpha(\xi) = \int_0^\gamma \Psi(x)\left[\frac{x\Phi_\alpha(x) - \xi\Phi_\alpha(\xi)}{x-\xi} + \frac{\xi\Phi_\alpha(\xi)}{x+\xi}\right]\mathrm{d}x. \tag{68}$$

To complete this formulation we note that, for this first work on using the F_N method for the considered class of problems, we intend to evaluate the functions $B_\alpha(\eta)$ and $C_\alpha(\xi)$ by numerical integration. While the form of Eq. (68) is considered convenient for numerical integration, we prefer, for this purpose, to rewrite Eq. (65) as

$$B_\alpha(\eta) = \Phi_\alpha(\eta) - \int_0^\gamma \Psi(x)\left[\frac{x\Phi_\alpha(x) - \eta\Phi_\alpha(\eta)}{x-\eta} + \frac{\eta\Phi_\alpha(\eta)}{x+\eta}\right]\mathrm{d}x. \tag{69}$$

In order to proceed with our F_N solution we must now choose the functions $\{\Phi_\alpha(\xi)\}$ and a collocation scheme. In the past[9] good results have been obtained for various radiative-transfer problems by using a version of the F_N method based on the use of shifted Legendre polynomials as expansion functions and a collocation scheme defined by the zeros of the Chebycheff polynomials.

We thus first propose to use

$$\Phi_\alpha(\xi) = P_\alpha(2\beta\xi - 1) \tag{70}$$

and

$$\eta_i = \frac{\gamma}{2}\left\{1 + \cos\left[\frac{\pi}{2}\left(\frac{2i-1}{N+1}\right)\right]\right\}, \qquad i = 1, 2, \ldots, N+1, \tag{71}$$

to define the linear system given by Eq. (64). Having chosen the functions defined by Eq. (70), we have to admit that the solution is going to fail in principle for the excluded case of $\beta = 0$, and it is also going to fail numerically for some sufficiently small β. It is clear that we must decide how well we can do this calculation using these approximating functions as the parameter β becomes small.

We started this numerical study by considering $\beta = 0.5$, and we found with $N = 99$ results for all of the functions listed in Tables 1 and 2 that agreed to five significant figures with the results obtained by evaluating the exact expressions. Continuing, we considered smaller values of β and found that we could still get results good to five significant figures for β as small as 10^{-3}. For smaller values of β our results began to deteriorate, and so we started a search for another set of expansion functions and a collocation scheme that could improve the F_N method for small β.

Having had some success using the transformations

$$u(z) = z/\gamma_0, \qquad z \in [0, \gamma_0], \tag{72a}$$

and

$$u(z) = \mathrm{e}^{-\alpha m(z)}, \qquad z \in [\gamma_0, \gamma), \tag{72b}$$

with $u(\gamma) = 0$, to define a quadrature scheme for solving the nonlinear H equation and evaluating Eqs. (24)–(27), we now introduce a set of (discontinuous) functions defined as

$$\Phi_\alpha(z) = \begin{cases} P_\alpha(2z/\gamma_0 - 1), & z \in [0, \gamma_0], \\ 0, & z \in (\gamma_0, \gamma], \end{cases} \tag{73a}$$

for $\alpha = 0,1,2,\ldots, N_1$ and

$$\Phi_\alpha(z) = \begin{cases} 0, & z \in [0, \gamma_0), \\ P_{\alpha-N_1-1}(2\mathrm{e}^{-a[m(z)]^b} - 1), & z \in [\gamma_0, \gamma), \end{cases} \tag{73b}$$

with $\Phi_\alpha(\gamma) = P_{\alpha-N_1-1}(-1)$, for $\alpha = N_1+1,\ N_1+2, \ldots, N$. Of course the integer N_1 in Eqs. (73)

must be specified; we have used $N_1 = [m(N-1)/10]$ with $m = 1$, 2 or 3, typically. We note that in Eq. (73b) we have two "scaling factors" a and b.

To have a collocation scheme we can use for all values of β we have used the zeros of the Chebycheff polynomials and transformations, similar to those given by Eqs. (72), on the variable η in Eq. (61) so as to obtain

$$\eta_i = \frac{\gamma_0}{2}\left\{1 + \cos\left[\frac{\pi}{2}\left(\frac{2i-1}{N_1+1}\right)\right]\right\} \tag{74a}$$

for $i = 1, 2, \ldots, N_1 + 1$ and

$$\eta_{N_1+1+i} = m^{-1}\left[\left(-\ln\left\{\frac{1}{2}+\frac{1}{2}\cos\left[\frac{\pi}{2}\left(\frac{2i-1}{N-N_1}\right)\right]\right\}/a\right)^{1/b}\right] \tag{74b}$$

for $i = 1,2,\ldots, N-N_1$. At this point we note that although the notation used in Eqs. (73) and Eqs. (74) may appear unnecessarily complicated, that notation allows us to use our general formulation of the F_N method without introducing any additional equations or concepts.

We have carried out many numerical experiments with the expansion functions and collocation scheme defined by Eqs. (73) and Eqs. (74) and have generally found excellent results. For example, with $1-\varpi = 10^{-6}$ we found results for the functions listed in Tables 1 and 2 that agreed with results obtained from the exact formulation to five significant figures for β as small as 10^{-7} for both the Doppler and the Lorentz profiles.

In regard to the scaling factors $\{a, b\}$ used in Eq. (73b) and Eq. (74b), we note that we have typically used $a \in [1,1.5]$ with $b \in [0.8,1.0]$ for the Doppler case and $a \in [10^{-3},10^{-2}]$ with $b \in [0.6,1.0]$ for the Lorentz case. Again we have done only some casual experimentation with these two scaling factors, but we have seen that the results can vary greatly on the choice of these quantities. We have also seen that the choice required for $\{a, b\}$, in order to obtain good results, depends mostly on the physical parameter β.

Since we wished here to be able to investigate easily various sets of expansion functions $\{\Phi_\alpha(\xi)\}$ we have used only numerical integration to evaluate the required functions $B_\alpha(\eta)$ and $C_\alpha(\xi)$, and so, perhaps by developing and using recursion formulas to evaluate these quantities we can still improve the numerical results.

5. THE F_N METHOD FOR THE SOURCE FUNCTION

It is clear from Eq. (2) and Eq. (7) that the source function $S_x(\tau)$ can be computed from

$$S_x(\tau) = \gamma_x \lim_{\xi\to 0} I_x(\tau, -\xi), \tag{75}$$

or, after we use Eq. (9b),

$$S_x(\tau) = \gamma_x \lim_{\xi\to 0}\left[I_x^p(\tau,-\xi) + \phi(x)G(\tau,-\xi)\right] \tag{76}$$

or, after we use Eq. (18),

$$S_x(\tau) = \gamma_x(B_0 + B_1\tau)[L\phi(x) + \rho\beta] + \gamma_x\phi(x)\lim_{\xi\to 0} G(\tau,-\xi). \tag{77}$$

So to compute the source function, and thus to complete this work, we now wish to use the F_N method to compute

$$S(\tau) = \lim_{\xi\to 0} G(\tau,-\xi), \qquad \tau > 0. \tag{78}$$

A set of singular-integral equations for $G(\tau,-\xi)$ and $G(\tau,\xi)$ for $\xi \in (0,\gamma)$ can be derived[8] from Eq. (47) in much the same way as Eq. (55) was derived. This time we carry out two separate integral transformations: in one case we multiply Eq. (47) by $\exp(-\tau/s)$ and integrate over τ from τ to infinity, and in the other case we multiply by the same factor and integrate over τ

from zero to τ. As the development here follows closely what we did in the previous section of this paper (see also Ref. 8), we omit some steps and write

$$\lambda(\eta)G(\tau,-\eta) - \int_0^{\gamma}\xi\Psi(\xi)G(\tau,-\xi)\frac{\mathrm{d}\xi}{\xi-\eta} - \int_0^{\gamma}\xi\Psi(\xi)G(\tau,\xi)\frac{\mathrm{d}\xi}{\xi+\eta} = 0 \tag{79a}$$

and

$$\lambda(\eta)G(\tau,\eta) - \int_0^{\gamma}\xi\Psi(\xi)G(\tau,\xi)\frac{\mathrm{d}\xi}{\xi-\eta} - \int_0^{\gamma}\xi\Psi(\xi)G(\tau,-\xi)\frac{\mathrm{d}\xi}{\xi+\eta} = \mathrm{e}^{-\tau/\eta}K(\eta) \tag{79b}$$

for $\eta \in (0, \gamma)$ and $\tau > 0$. Here

$$K(\eta) = \lambda(\eta)G(0,\eta) - \int_0^{\gamma}\xi\Psi(\xi)G(0,\xi)\frac{\mathrm{d}\xi}{\xi-\eta} - \int_0^{\gamma}\xi\Psi(\xi)G(0,-\xi)\frac{\mathrm{d}\xi}{\xi+\eta}. \tag{80}$$

We choose to decompose $G(\tau, \eta)$ into scattered and unscattered components, and so we substitute

$$G(\tau,\xi) = G(0,\xi)\mathrm{e}^{-\tau/\xi} + G^*(\tau,\xi), \tag{81}$$

for $\xi \in (0, \gamma)$, into Eqs. (79) to find

$$\lambda(\eta)G(\tau,-\eta) - \int_0^{\gamma}\xi\Psi(\xi)G(\tau,-\xi)\frac{\mathrm{d}\xi}{\xi-\eta} - \int_0^{\gamma}\xi\Psi(\xi)G^*(\tau,\xi)\frac{\mathrm{d}\xi}{\xi+\eta} = T_1(\tau,\eta) \tag{82a}$$

and

$$\lambda(\eta)G^*(\tau,\eta) - \int_0^{\gamma}\xi\Psi(\xi)G^*(\tau,\xi)\frac{\mathrm{d}\xi}{\xi-\eta} - \int_0^{\gamma}\xi\Psi(\xi)G(\tau,-\xi)\frac{\mathrm{d}\xi}{\xi+\eta} = T_2(\tau,\eta) \tag{82b}$$

for $\eta \in (0, \gamma)$ and $\tau > 0$. Here

$$T_1(\tau,\eta) = \int_0^{\gamma}\xi\Gamma(\xi)\mathrm{e}^{-\tau/\xi}\frac{\mathrm{d}\xi}{\xi+\eta} \tag{83a}$$

and

$$T_2(\tau,\eta) = \int_0^{\gamma}\xi\Gamma(\xi)C(\tau{:}\xi,\eta)\mathrm{d}\xi - \mathrm{e}^{-\tau/\eta}\int_0^{\gamma}\xi\Psi(\xi)G(0,-\xi)\frac{\mathrm{d}\xi}{\xi+\eta} \tag{83b}$$

where

$$C(\tau{:}\xi,\eta) = \frac{\mathrm{e}^{-\tau/\xi} - \mathrm{e}^{-\tau/\eta}}{\xi-\eta}. \tag{84}$$

Following the procedure we used in the previous section to decompose $\Gamma(\xi)$ and $G(0, -\xi)$ into four basic components, we do the same thing here for $G(\tau, -\xi)$ and $G^*(\tau, \xi)$, and so we consider the set of equations

$$\lambda(\eta)N_k(\tau,-\eta) - \int_0^{\gamma}\xi\Psi(\xi)N_k(\tau,-\xi)\frac{\mathrm{d}\xi}{\xi-\eta} - \int_0^{\gamma}\xi\Psi(\xi)N_k^*(\tau,\xi)\frac{\mathrm{d}\xi}{\xi+\eta} = R_{1,k}(\tau,\eta) \tag{85a}$$

and

$$\lambda(\eta)N_k^*(\tau,\eta) - \int_0^{\gamma}\xi\Psi(\xi)N_k^*(\tau,\xi)\frac{\mathrm{d}\xi}{\xi-\eta} - \int_0^{\gamma}\xi\Psi(\xi)N_k(\tau,-\xi)\frac{\mathrm{d}\xi}{\xi+\eta} = R_{2,k}(\tau,\eta) \tag{85b}$$

where

$$R_{1,k}(\tau,\eta) = \int_0^{\gamma}\xi^k\Psi(\xi)\mathrm{e}^{-\tau/\xi}\frac{\mathrm{d}\xi}{\xi+\eta}, \qquad k = 1, 2, \tag{86a}$$

$$R_{1,k}(\tau,\eta) = \int_0^{\gamma} \xi^{k-2}\Delta(\xi)\mathrm{e}^{-\tau/\xi}\frac{\mathrm{d}\xi}{\xi+\eta}, \qquad k = 3, 4, \tag{86b}$$

$$R_{2,k}(\tau,\eta) = \int_0^{\gamma} \xi^{k}\Psi(\xi)C(\tau{:}\xi,\eta)\mathrm{d}\xi - \mathrm{e}^{-\tau/\eta}\int_0^{\gamma}\xi\Psi(\xi)N_k(\xi)\frac{\mathrm{d}\xi}{\xi+\eta}, \tag{86c}$$

for $k = 1, 2$, and

$$R_{2,k}(\tau,\eta) = \int_0^{\gamma} \xi^{k}\Delta(\xi)C(\tau{:}\xi,\eta)\mathrm{d}\xi - \mathrm{e}^{-\tau/\eta}\int_0^{\gamma}\xi\Psi(\xi)N_k(\xi)\frac{\mathrm{d}\xi}{\xi+\eta}, \tag{86d}$$

for $k = 3, 4$. We note that the functions $N_k(\xi)$, for $k = 1, 2, 3$ and 4, are the surface quantities established in Section 4, *viz*., solutions to Eq. (61).

Here again, we suppress the explicit notation and consider

$$\lambda(\eta)N(\tau,-\eta) - \fint_0^{\gamma}\xi\Psi(\xi)N(\tau,-\xi)\frac{\mathrm{d}\xi}{\xi-\eta} - \int_0^{\gamma}\xi\Psi(\xi)N^*(\tau,\xi)\frac{\mathrm{d}\xi}{\xi+\eta} = R_1(\tau,\eta) \tag{87a}$$

and

$$\lambda(\eta)N^*(\tau,\eta) - \fint_0^{\gamma}\xi\Psi(\xi)N^*(\tau,\xi)\frac{\mathrm{d}\xi}{\xi-\eta} - \int_0^{\gamma}\xi\Psi(\xi)N(\tau,-\xi)\frac{\mathrm{d}\xi}{\xi+\eta} = R_2(\tau,\eta) \tag{87b}$$

for $\eta \in (0,\gamma)$ and $\tau > 0$.

To establish our F_N solution, we substitute the approximations

$$N(\tau,-\xi) = \sum_{\alpha=0}^{N} c_\alpha(\tau)\Phi_\alpha(\xi), \qquad \xi \in (0,\gamma), \tag{88a}$$

and

$$N^*(\tau,\xi) = \sum_{\alpha=0}^{N} d_\alpha(\tau)\Phi_\alpha(\xi), \qquad \xi \in (0,\gamma), \tag{88b}$$

into Eqs. (87) and note Eq. (65); we then consider the resulting equations at the collocation points $\{\eta_i\}$ to obtain

$$\sum_{\alpha=0}^{N}[c_\alpha(\tau)B_\alpha(\eta_i) - d_\alpha(\tau)A_\alpha(\eta_i)] = R_1(\tau,\eta_i) \tag{89a}$$

and

$$\sum_{\alpha=0}^{N}[d_\alpha(\tau)B_\alpha(\eta_i) - c_\alpha(\tau)A_\alpha(\eta_i)] = R_2(\tau,\eta_i) \tag{89b}$$

for $i = 1, 2, \ldots, N+1$. Here we have defined

$$A_\alpha(z) = \int_0^{\gamma}\xi\Psi(\xi)\Phi_\alpha(\xi)\frac{\mathrm{d}\xi}{\xi+z}, \qquad z \in [0,\gamma]. \tag{90}$$

We note that since the coefficient matrix in the linear system defined by Eqs. (89) is independent of τ, only one LU factorization of the coefficient matrix is required to have the desired solution for any value of τ.

Of course once we have solved the linear system to find the coefficients $c_\alpha(\tau)$ and $d_\alpha(\tau)$ we can compute our first results from Eqs. (88). We can also rewrite Eqs. (87) as

$$N(\tau,-\eta) = R_1(\tau,\eta) + \fint_0^{\gamma}\xi\Psi(\xi)N(\tau,-\xi)\frac{\mathrm{d}\xi}{\xi-\eta} + \int_0^{\gamma}\xi\Psi(\xi)N^*(\tau,\xi)\frac{\mathrm{d}\xi}{\xi+\eta} + [1-\lambda(\eta)]N(\tau,-\eta) \tag{91a}$$

and

$$N^*(\tau,\eta) = R_2(\tau,\eta) + \fint_0^\gamma \xi\Psi(\xi)N^*(\tau,\xi)\frac{d\xi}{\xi-\eta} + \int_0^\gamma \xi\Psi(\xi)N(\tau,-\xi)\frac{d\xi}{\xi+\eta} + [1-\lambda(\eta)]N^*(\tau,\eta) \quad (91b)$$

and substitute Eqs. (88) into the r.h.s. of these equations to find, after noting Eq. (68) and Eq. (90), the "post-processed" results, *viz.*,

$$N(\tau,-\xi) = R_1(\tau,\xi) + \sum_{\alpha=0}^{N}\left[c_\alpha(\tau)C_\alpha(\xi) - d_\alpha(\tau)A_\alpha(\xi)\right] \quad (92a)$$

and

$$N^*(\tau,\xi) = R_2(\tau,\xi) + \sum_{\alpha=0}^{N}\left[d_\alpha(\tau)C_\alpha(\xi) - c_\alpha(\tau)A_\alpha(\xi)\right] \quad (92b)$$

for $\xi \in (0, \gamma)$ and $\tau > 0$.

It is clear that Eqs. (88) or Eqs. (92) define $G(\tau, \xi)$ and $G(\tau, -\xi)$ for $\xi \in (0,\gamma)$ and $\tau > 0$, and so the complete solution for $I_x(\tau, \mu)$ can be considered as established. We have now only to evaluate our solution numerically and to check out the achieved accuracy. For numerical purposes, we choose to evaluate the source function as given by Eq. (77), and since everything in that equation is known except for $S(\tau)$, as defined by Eq. (78), we need compute only $S(\tau)$. If we write

$$S(\tau) = B_1\left[LU_1(\tau) + \rho\beta X_1(\tau)\right] - B_0\left[LU_0(\tau) + \rho\beta X_0(\tau)\right] \quad (93)$$

then we can make the identifications

$$U_{k-1}(\tau) = N_k(\tau,0), \qquad k = 1,2, \quad (94a)$$

and

$$X_{k-3}(\tau) = N_k(\tau,0), \qquad k = 3,4, \quad (94b)$$

where, in general,

$$N(\tau,0) = R_1(\tau,0) + \sum_{\alpha=0}^{N}\left[c_\alpha(\tau)C_\alpha(0) - d_\alpha(\tau)A_\alpha(0)\right]. \quad (95)$$

For a first test of our source-function calculation we return to Ref. 3 and note that in that work Hummer expressed the total source function in the form

$$S_x(\tau) = \phi(x)\gamma_x S_L(\tau) + \rho\beta\gamma_x B(\tau) \quad (96)$$

where $S_L(\tau)$ is the line-source function and $B(\tau)$ is the Planck function. Thus for the considered case of a linear Planck function we can write

$$S_L(\tau) = (B_0 + B_1\tau)L + \lim_{\xi\to 0} G(\tau,-\xi). \quad (97)$$

To compare with some of Hummer's[3] results, we considered the case of Doppler profile with $B_0 = 1$ and $B_1 = 0$, and we have computed

$$p_0(\tau) = S_L(\tau)/S_L(0), \qquad \rho = 0, \quad (98a)$$

and

$$p_1(\tau) = S_L(\tau)/S_L(0), \qquad \rho = 1. \quad (98b)$$

Of course we can express $p_0(\tau)$ and $p_1(\tau)$ in terms of the basic functions of this section. We find

$$p_0(\tau) = \left[1 - U_0(\tau)\right]/\left[1 - U_0(0)\right] \quad (99a)$$

Table 3. Results for a Doppler profile with $1-\varpi = 10^{-6}$ and $\beta = 10^{-4}$

τ	$U_0(\tau)$	$U_1(\tau)$	$X_0(\tau)$	$X_1(\tau)$
0.0	9.7581(–1)	1.2153(1)	2.6409(1)	2.8791(4)
1.0	9.5949(–1)	1.9384(1)	4.1941(1)	4.8176(4)
2.0	9.4780(–1)	2.4302(1)	5.2751(1)	6.2037(4)
5.0	9.2005(–1)	3.5442(1)	7.7496(1)	9.4863(4)
1.0(1)	8.8469(–1)	4.8696(1)	1.0726(2)	1.3647(5)
2.0(1)	8.3156(–1)	6.6819(1)	1.4852(2)	1.9841(5)
5.0(1)	7.2455(–1)	9.7359(1)	2.1990(2)	3.2011(5)
1.0(2)	6.1014(–1)	1.2167(2)	2.7952(2)	4.4360(5)
2.0(2)	4.6839(–1)	1.3957(2)	3.2830(2)	5.8144(5)
5.0(2)	2.6449(–1)	1.3683(2)	3.3755(2)	7.2189(5)
1.0(3)	1.3416(–1)	1.0801(2)	2.8191(2)	7.2149(5)
2.0(3)	5.1455(–2)	6.5774(1)	1.8685(2)	5.8930(5)
5.0(3)	8.7937(–3)	2.0433(1)	6.9223(1)	2.9412(5)
1.0(4)	1.4477(–3)	4.9719	2.0514(1)	1.0814(5)
2.0(4)	1.2497(–4)	5.8897(–1)	3.1690	2.0186(4)
5.0(4)	7.2650(–7)	4.6553(–3)	4.0291(–2)	3.1184(2)
1.0(5)	8.1572(–10)	6.1134(–6)	8.3808(–5)	7.1836(–1)

and

$$p_1(\tau) = \{L[1 - U_0(\tau)] - \beta X_0(\tau)\}/\{L[1 - U_0(0)] - \beta X_0(0)\}. \tag{99b}$$

We have used the approximating functions and collocation scheme defined by Eqs. (73) and Eqs. (74) to recompute $p_0(\tau)$ and $p_1(\tau)$ for the various cases, *viz.*, $1 - \varpi = 10^{-6}$ with $\beta \in [0, 10^{-3}]$, considered by Hummer.[3] We note that Hummer's results[3] are listed with three significant figures and that we found agreement with those results.

Table 4. Results for a Lorentz profile with $1-\varpi = 10^{-6}$ and $\beta = 10^{-4}$

τ	$U_0(\tau)$	$U_1(\tau)$	$X_0(\tau)$	$X_1(\tau)$
0.0	8.6687(–1)	1.1056(2)	4.7533(2)	1.4403(6)
1.0	8.3333(–1)	1.3757(2)	5.9211(2)	1.8027(6)
2.0	8.1221(–1)	1.5429(2)	6.6496(2)	2.0305(6)
5.0	7.6881(–1)	1.8796(2)	8.1269(2)	2.4978(6)
1.0(1)	7.2333(–1)	2.2201(2)	9.6367(2)	2.9855(6)
2.0(1)	6.6815(–1)	2.6090(2)	1.1391(3)	3.5718(6)
5.0(1)	5.8131(–1)	3.1444(2)	1.3899(3)	4.4729(6)
1.0(2)	5.0457(–1)	3.5120(2)	1.5758(3)	5.2284(6)
2.0(2)	4.1851(–1)	3.7694(2)	1.7296(3)	5.9929(6)
5.0(2)	2.9433(–1)	3.7577(2)	1.8069(3)	6.7993(6)
1.0(3)	1.9981(–1)	3.3364(2)	1.6949(3)	6.9421(6)
2.0(3)	1.1539(–1)	2.5230(2)	1.3883(3)	6.3151(6)
5.0(3)	3.7540(–2)	1.1665(2)	7.5507(2)	4.0516(6)
1.0(4)	9.8343(–3)	3.9390(1)	3.0689(2)	1.8872(6)
2.0(4)	1.2657(–3)	6.3958	6.4577(1)	4.5390(5)
5.0(4)	1.1519(–5)	7.4901(–2)	1.2247	1.0000(4)
1.0(5)	1.7765(–8)	1.3241(–4)	3.4779(–3)	3.0807(1)

To report some general numerical results we list in Tables 3 and 4 our basic functions for the case $1 - \varpi = 10^{-6}$ with $\beta = 10^{-4}$ for both the Doppler and the Lorentz profiles. These results (obtained with $N = 299$) are thought to be correct to plus or minus 1 unit in the last digits given.

6. CONCLUDING REMARKS

We note first of all that using the nonlinear H equation, we found no difficulty in accurately evaluating the functions $H(\xi)$ and $\Upsilon_0(\xi)$ for any of the considered values of β; however, we did find a loss of accuracy in computing the functions $\Upsilon_1(\xi)$, $\Xi_0(\xi)$ and $\Xi_1(\xi)$ as β became very small —not a surprising observation since those functions do not even exist for the Doppler or Lorentz line-scattering profiles for the case of $\beta = 0$.

Having completed this, our first work on using the F_N method for investigating the problem of spectral-line formation by completely noncoherent scattering, we are of the opinion that the method can be used to solve well this class of problems. As is typical in the use of the F_N method, we found here the choice of a set of expansion functions $\{\Phi_\alpha(\xi)\}$ and an accompanying collocation scheme to be somewhat of an "art form", but finally we believe the two schemes we have used, one for "large" values of β and another for "small" values of β are good ones for this application. We have found $\beta = 10^{-3}$ to be a good transitional value since we were able to obtain excellent results for this value of β from each of the two schemes.

It is expected that in future work we will seek ways to evaluate the basic functions of the F_N method, *viz*., $B_\alpha(\xi)$, $C_\alpha(\xi)$ and $A_\alpha(\xi)$, that are faster and more accurate than the numerical integration methods we have used here. More attention will also be given to deciding how to define the scaling factors $\{a, b\}$ used in Eq. (73b) and Eq. (74b). We also intend to extend this current calculation to the case of a finite medium, and it is anticipated that we will go on to spectral-line problems that also include polarization effects.[11,12]

Acknowledgements—The authors would like to thank J. Bergeat, V. V. Ivanov and B. Rutily for numerous helpful communications concerning this (and related) work. In addition, one of the authors (LBB) would like to express her thanks to the Mathematics Department of North Carolina State University for providing some partial financial support and for the kind hospitality extended throughout a period during which a part of this work was done. Finally, it is noted that the work of LBB was supported in part by CNPq of Brazil and that some computer time was made available to CES by the North Carolina Supercomputing Center and to LBB by the National Supercomputer Center of the Universidade Federal do Rio Grande do Sul.

REFERENCES

1. McCormick, N. J. and Siewert, C. E., *Astrophys. J.*, 1970, **162,** 633.
2. Case, K. M., *Ann. Phys.*, 1960, **9,** 1.
3. Hummer, D. G., *Mon. Not. R. Astron. Soc.*, 1968, **138,** 73.
4. Ivanov, V. V., *Transfer of Radiation in Spectral Lines*. National Bureau of Standards Publication 385, Washington, DC, 1973.
5. Chandrasekhar, S., *Radiative Transfer*. Oxford University Press, London, 1950.
6. Ivanov, V. V. and Nagirner, D. I., *Astrofiz*., 1965, **1,** 143.
7. Warming, R. F., *Astrophys. J.*, 1970, **159,** 593.
8. Siewert, C. E., *Z. Angew. Math. Phys.*, 1984, **35,** 144.
9. Garcia, R. D. M., *Transport Theory Stat. Phys.*, 1985, **14,** 391.
10. Muskhelishvili, N. I., *Singular Integral Equations*. Noordhoff, Groningen, 1953.
11. Ivanov, V. V., Grachev, S. I. and Loskutov, V. M., *Astron. Astrophys.*, 1997, **318,** 315.
12. Ivanov, V. V., Grachev, S. I. and Loskutov, V. M., *Astron. Astrophys.*, 1997, **321,** 968.