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A COMPUTATION OF THE X AND Y FUNCTIONS FOR A NON-GREY MODEL WITH COMPLETE FREQUENCY REDISTRIBUTION

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Abstract—The F_N method is used to compute the classical X and Y functions for use in the solution of a class of non-grey radiative-transfer problems defined for finite layers. The model considered allows for scattering with complete frequency redistribution (completely non-coherent scattering) and continuum absorption. Some test problems based on Doppler and Lorentz profiles of the line-scattering coefficient are discussed, and numerical results (thought to be correct to five significant figures) are given for selected cases. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

We consider here radiative-transfer problems based on the equation of transfer written, after Hummer,[1] as

$$\mu \frac{\partial}{\partial \tau} I_x(\tau, \mu) + [\phi(x) + \beta]\, I_x(\tau, \mu) = [\phi(x) + \beta]\, S_x(\tau), \tag{1}$$

where $S_x(\tau)$ is the source function,

$$[\phi(x) + \beta]S_x(\tau) = \frac{1}{2}\, \varpi\phi(x) \int_{-\infty}^{\infty} \phi(x') \int_{-1}^{1} I_{x'}(\tau, \mu')\, \mathrm{d}\mu'\, \mathrm{d}x' + [\rho\beta + (1 - \varpi)\phi(x)]B(\tau), \tag{2}$$

and $B(\tau)$ is the Planck function. Here $\tau \in [0, \tau_0]$ is the optical variable, τ_0 is the optical thickness of the considered layer and $\mu \in [-1, 1]$ is the *cosine* of the polar angle (as measured from the *positive* τ axis) that describes the direction of propagation of the radiation. In addition, $\varpi \in [0, 1]$ is the albedo for single scattering, $\beta \geq 0$ is the ratio of the continuum absorption coefficient to the average line coefficient, ρ is the ratio of the continuum source function to the Planck function and $\phi(x)$ is the line-scattering profile. In a recent work[2] on non-grey radiative-transfer problems based on Eq. (1) for semi-infinite media, we used the non-linear H equation and the F_N method to compute the distribution of radiation exiting the medium and the source function within the medium for a linearly varying Planck function. Here, as a first extension of Ref. 2 to the case of a finite medium, we compute the X and Y functions that have been much discussed in the classical radiative-transfer literature[3–5] and in terms of which solutions to some problems in finite media can be expressed.

In order to keep our formulation and our calculation concise, we in this work explicitly exclude the special (conservative) case of $\varpi = 1$ with $\beta = 0$, since the defining equations (either the non-linear equations or the linear equations) for the X and Y functions in this conservative case may require the addition of two linear constraints to define unique solutions.[6–8] We can mention here that using the second constraint[6–8] for either the Lorentz or the Doppler line-scattering profile appears somewhat problematical since the first moments of the X and Y functions do not exist for these cases.[6]

We note that by letting $\beta \to 0$ with $\varpi = 1$ and by letting $\varpi \to 1$ with $\beta = 0$ (see our Tables 3 and 4), we believe we have computed, to at least five significant figures, the X and Y functions also for the conservative case.

2. THE DEFINING EQUATIONS FOR THE X AND Y FUNCTIONS

In addition to Refs. 3–5 that provide much of the background and history of the use of the X and Y functions in the theory of radiative transfer, we make use here of Ivanov's[6] text that discusses the use of the X and Y functions in the context of the class of non-grey problems we are considering here. In Ref. 6 the non-linear X and Y equations are written (using the notation of Ref. 2) as

$$X(z) = 1 + z\int_0^{\gamma} \Psi(z')[X(z)X(z') - Y(z)Y(z')]\,\frac{dz'}{z'+z} \tag{3a}$$

and

$$Y(z) = e^{-\tau_0/z} + z\int_0^{\gamma} \Psi(z')[X(z)Y(z') - X(z')Y(z)]\,\frac{dz'}{z'-z} \tag{3b}$$

for $z \in [0, \gamma]$. Here $\gamma = 1/\beta$ and $\Psi(z)$ is the characteristic function. Quoting from Ref. 2, we note that the characteristic function for the case of a Doppler profile can be written as

$$\Psi(\xi) = \begin{cases} \Psi_0, & \xi \in [0, \gamma_0], \\ \Psi_0\,\mathrm{erfc}[\sqrt{2}m(\xi)], & \xi \in [\gamma_0, \gamma), \end{cases} \tag{4}$$

where $\mathrm{erfc}(z)$ is the complementary error function and

$$m(\xi) = \sqrt{\ln\left[\frac{\xi}{\sqrt{\pi}(1-\beta\xi)}\right]}, \quad \xi \in [\gamma_0, \gamma). \tag{5}$$

In addition,

$$\Psi_0 = \frac{\varpi}{4}\sqrt{\frac{2}{\pi}} \tag{6}$$

and

$$\gamma_0 = \frac{\sqrt{\pi}}{1+\beta\sqrt{\pi}}. \tag{7}$$

Quoting again from Ref. 2, we write the characteristic function for the case of a Lorentz profile as

$$\Psi(\xi) = \begin{cases} \Psi_0, & \xi \in [0, \gamma_0], \\ \Psi_0\,\dfrac{2}{\pi}\left\{\cot^{-1}[m(\xi)] - \dfrac{m(\xi)}{1+m^2(\xi)}\right\}, & \xi \in [\gamma_0, \gamma), \end{cases} \tag{8}$$

where

$$m(\xi) = \sqrt{\frac{\xi(1+\beta\pi)-\pi}{\pi(1-\beta\xi)}}, \quad \xi \in [\gamma_0, \gamma). \tag{9}$$

Here we now have

$$\Psi_0 = \frac{\varpi}{4\pi} \tag{10}$$

and

$$\gamma_0 = \frac{\pi}{1+\beta\pi}. \tag{11}$$

In addition to the non-linear X and Y equations, Ivanov[6] also lists the linear, singular-integral equations

$$\lambda(\eta)X(\eta) - \eta \,⨍_0^{\gamma} \Psi(\xi)X(\xi)\,\frac{d\xi}{\xi-\eta} + \eta e^{-\tau_0/\eta}\int_0^{\gamma}\Psi(\xi)Y(\xi)\,\frac{d\xi}{\xi+\eta} = 1 \tag{12a}$$

and

$$\lambda(\eta)Y(\eta) - \eta \,⨍_0^{\gamma} \Psi(\xi)Y(\xi)\,\frac{d\xi}{\xi-\eta} + \eta e^{-\tau_0/\eta}\int_0^{\gamma}\Psi(\xi)X(\xi)\,\frac{d\xi}{\xi+\eta} = e^{-\tau_0/\eta} \tag{12b}$$

for $\eta \in (0, \gamma)$. We note that we use the symbol $⨍$ to denote that integrals are to be evaluated in the Cauchy principal-value sense. In addition,

$$\lambda(\eta) = 1 + \eta ⨍_{-\gamma}^{\gamma} \Psi(\xi)\,\frac{d\xi}{\xi - \eta}. \tag{13}$$

For our work here, which excludes the conservative case, we consider that the X and Y functions are defined by Eqs. (12a) and (12b), and so these are the equations we wish to solve in an approximate but accurate and concise way.

3. THE F_N SOLUTIONS FOR THE X AND Y FUNCTIONS

To establish some numerical results for the X and Y functions, we choose to use the F_N method[8–10] to develop our approximate solutions to Eqs. (12a) and (12b), and since our development here has many common features with the solution technique reported in Ref. 2 for the semi-infinite case, our presentation in this work is brief.

We introduce the approximations

$$X(\xi) = \sum_{\alpha=0}^{N} a_\alpha \Phi_\alpha(\xi) \tag{14a}$$

and

$$Y(\xi) = \sum_{\alpha=0}^{N} b_\alpha \Phi_\alpha(\xi), \tag{14b}$$

for $\xi \in (0, \gamma)$, into Eqs. (12a) and (12b) and consider the resulting equations at selected collocation points $\{\eta_i\}$ to find

$$\sum_{\alpha=0}^{N} [a_\alpha B_\alpha(\eta_i) + e^{-\tau_0/\eta_i} b_\alpha A_\alpha(\eta_i)] = 1 \tag{15a}$$

and

$$\sum_{\alpha=0}^{N} [b_\alpha B_\alpha(\eta_i) + e^{-\tau_0/\eta_i} a_\alpha A_\alpha(\eta_i)] = e^{-\tau_0/\eta_i} \tag{15b}$$

for $i = 1, 2, \ldots, N+1$. Here, $\{\Phi_\alpha(\xi)\}$ denotes a set of expansion functions to be specified and the functions $A_\alpha(\eta)$ and $B_\alpha(\eta)$ are defined (slightly differently than in Ref. 2) by

$$A_\alpha(\eta) = \eta \int_0^{\gamma} \Psi(x)\,\Phi_\alpha(x)\,\frac{dx}{x + \eta} \tag{16a}$$

and

$$B_\alpha(\eta) = \Phi_\alpha(\eta) - \eta \int_0^{\gamma} \Psi(x) \left[\frac{\Phi_\alpha(x) - \Phi_\alpha(\eta)}{x - \eta} + \frac{\Phi_\alpha(\eta)}{x + \eta}\right] dx. \tag{16b}$$

Our procedure now is to choose a set of expansion functions $\{\Phi_\alpha(\xi)\}$ and a collocation scheme, evaluate the A and B functions required to define the system of linear algebraic equations given by Eqs. (15a) and (15b) and then solve those equations to find the expansion coefficients $\{a_\alpha\}$ and $\{b_\alpha\}$. In this way we complete the first forms of our solutions, viz. Eqs. (14a) and (14b). We can also obtain our "post-processed" results by first rewriting Eqs. (12a) and (12b) as

$$X(\eta) = 1 + \eta ⨍_0^{\gamma} \Psi(\xi)\,X(\xi)\,\frac{d\xi}{\xi - \eta} - \eta e^{-\tau_0/\eta} \int_0^{\gamma} \Psi(\xi)\,Y(\xi)\,\frac{d\xi}{\xi + \eta} + [1 - \lambda(\eta)]X(\eta) \tag{17a}$$

and

$$Y(\eta) = e^{-\tau_0/\eta} + \eta ⨍_0^{\gamma} \Psi(\xi) Y(\xi)\,\frac{d\xi}{\xi - \eta} - \eta e^{-\tau_0/\eta} \int_0^{\gamma} \Psi(\xi)\,X(\xi)\,\frac{d\xi}{\xi + \eta} + [1 - \lambda(\eta)]\,Y(\eta), \tag{17b}$$

for $\eta \in (0, \gamma)$, and then using Eqs. (14a) and (14b) on the right-hand sides of these equations. In this way, we find

$$X(\xi) = 1 + \sum_{\alpha=0}^{N} a_\alpha C_\alpha(\xi) - e^{-\tau_0/\xi} \sum_{\alpha=0}^{N} b_\alpha A_\alpha(\xi) \tag{18a}$$

and

$$Y(\xi) = e^{-\tau_0/\xi} + \sum_{\alpha=0}^{N} b_\alpha C_\alpha(\xi) - e^{-\tau_0/\xi} \sum_{\alpha=0}^{N} a_\alpha A_\alpha(\xi) \tag{18b}$$

for $\xi \in (0, \gamma)$. Here we have used the definition

$$C_\alpha(\xi) = \xi \int_0^\gamma \Psi(x) \left[\frac{\Phi_\alpha(x) - \Phi_\alpha(\xi)}{x - \xi} + \frac{\Phi_\alpha(\xi)}{x + \xi} \right] dx. \tag{19}$$

4. NUMERICAL RESULTS

In regard to defining a set of expansion functions $\{\Phi_\alpha(\xi)\}$ and a collocation scheme, we note that these matters were first mentioned in the context of the current application of the F_N method in Ref. 2, and so here we add to that discussion and report the way these quantities are defined in this work. In general, we can say that the two important choices one must make in using the F_N method for solving radiative-transfer problems are those concerning the expansion functions and the collocation scheme. For the considered class of radiative-transfer problems based on Eqs. (1) and (2), the expansion functions and the collocation scheme we use here are somewhat different from those typically made for an application of the F_N method to monochromatic problems.[10] We can say that these differences are derived from two observations: (i) since the constant β can be arbitrarily small, the support for the expansion functions can be arbitrarily large, and (ii) since the spectrum over which the F_N equations are defined can also be unbounded, sampling the spectrum (the collocation scheme) can be very delicate, and a poor choice can lead to systems of linear algebraic equations that are, from a numerical point of view, only marginally linearly independent. We are of the opinion that the choices for the expansion functions and the collocation scheme we have made in this work are good ones.

So here we use the (discontinuous) expansion functions defined by

$$\Phi_\alpha(\xi) = \begin{cases} P_\alpha(2\xi/\gamma_0 - 1), & \xi \in [0, \gamma_0], \\ 0, & \xi \in (\gamma_0, \gamma], \end{cases} \tag{20a}$$

for $\alpha = 0, 1, 2, \ldots, N_1$ and

$$\Phi_\alpha(\xi) = \begin{cases} 0, & \xi \in [0, \gamma_0), \\ P_{\alpha - N_1 - 1}(2e^{-a[m(\xi)]^b} - 1), & \xi \in [\gamma_0, \gamma), \end{cases} \tag{20b}$$

with $\Phi_\alpha(\gamma) = P_{\alpha - N_1 - 1}(-1)$, for $\alpha = N_1 + 1, N_1 + 2, \ldots, N$. Of course, the integer N_1 in Eqs. (20a) and (20b) must be specified; we have used $N_1 = [m(N-1)/10]$ with $m = 1, 2$ or 3, typically. We note that in Eq. (20b) we have two "scaling factors" $a > 0$ and $b > 0$ that we will define.

To have a collocation scheme for all values of β we have used the zeros of the Chebycheff polynomials of the second kind and transformations, similar to those that lead to Eqs. (20a) and (20b), on the variable η so as to obtain

$$\eta_\alpha = \frac{\gamma_0}{2} \left[1 + \cos\left(\frac{\alpha\pi}{N_1 + 2} \right) \right] \tag{21a}$$

for $\alpha = 1, 2, \ldots, N_1 + 1$ and

$$\eta_{N_1+1+\alpha} = m^{-1} \left(\left\{ -\frac{1}{a} \ln \frac{1}{2} \left[1 + \cos\left(\frac{\alpha\pi}{N - N_1 + 1} \right) \right] \right\}^{1/b} \right) \tag{21b}$$

for $\alpha = 1, 2, \ldots, N - N_1$.

In regard to the scaling factors a and b that appear in Eqs. (20a), (20b), (21a) and (21b), we have carried out numerous numerical studies to see the impact of these two factors on our calculation of the X and Y functions. Not surprisingly, we found that these two factors can affect greatly the numerical results obtained. However, we have defined a scheme that at least for the considered cases of ϖ and β can be used with some confidence. For the case of the Doppler line-scattering profile we use $a = 0.6$ with $b = 2$, and for the Lorentz case we use $a = 0.001$ with $b = 1$. For emphasis, we can say here that these choices of a and b have no theoretical basis, but to date we have found our best results using these choices.

It is generally known[4] that in the limit of infinite optical thickness τ_0 the Y function approaches zero and the X function becomes the classical H function. It is also known[6] that for the non-grey model considered here the function $H(\xi)$ diverges as ξ tends to infinity for the conservative case $\varpi = 1$ with $\beta = 0$. And so for the nearly conservative case we consider, viz. $1 - \varpi = 10^{-11}$ with $\beta = 0$, we have found it convenient when τ_0 is sufficiently large, say $\tau_0 > 10^{10}$, to include with our expansion functions $\{\Phi_\alpha(\xi)\}$ an asymptotic factor to account for this (nearly) unbounded behavior. Therefore, we have multiplied the right-hand side of Eq. (14a) by either the asymptotic factor $\sqrt{(\xi\sqrt{\ln \xi})}$ for the Doppler case[6] or the factor $\xi^{1/4}$ for the case of the Lorentz line-scattering profile[6] to improve our calculation of the H function for nearly conservative cases.

It is clear that one of the important aspects of our F_N calculation, once the expansion functions and a collocation scheme have been specified, is the evaluation of the basic functions defined by Eqs. (16) and (19). In this work, we have used numerical integration to evaluate these quantities. The numerical quadrature scheme we employed is defined by using a Gauss–Legendre scheme on the interval [0, 1] after using a linear transformation to map the first part of the integration interval, viz. $[0, \gamma_0]$, onto [0, 1]. For the second part of the integration interval we used a suggestion forwarded to us by Rutily,[11] viz. we used the transformation

$$u(z) = \frac{1}{1 + m(z)}, \tag{22}$$

with $u(\gamma) = 0$, to map the interval $[\gamma_0, \gamma]$ onto [0, 1], and then we again employed a Gauss–Legendre scheme on the interval [0, 1].

For our first calculation we consider the case of $1 - \varpi = 10^{-6}$ with $\beta = 10^{-4}$, and so in Tables 1 and 2 we list results for the Doppler and Lorentz line-scattering profiles. We note that for this calculation our results for the cases of $\tau_0 = 1$, $\tau_0 = 10$ and $\tau_0 = 100$ were first obtained with $N = 99$ in the F_N method and then refined with $N = 199$ and $N = 309$. In addition, since the version of the F_N method we used for these two cases (for $\tau_0 = 1$, $\tau_0 = 10$ and $\tau_0 = 100$, as well as for $\tau_0 \to \infty$) lead to very stable systems of linear algebraic equations, we were also able to see that our results remained valid for N as large as 499.

The second set of calculations we wish to report is for the (somewhat challenging) essentially conservative case defined by $1 - \varpi = 10^{-11}$ with $\beta = 0$. Our results for this case, again for $\tau_0 = 1$, $\tau_0 = 10$ and $\tau_0 = 100$, are listed in Tables 3 and 4. Here too the results summarized in Tables 3 and 4 were first obtained with $N = 99$ in the F_N method and then refined with $N = 199$ and

Table 1. The X and Y functions for a Doppler profile with $1 - \varpi = 10^{-6}$ and $\beta = 10^{-4}$

	$\tau_0 = 1$		$\tau_0 = 10$		$\tau_0 = 100$	
z	$X(\sqrt{\pi}z)$	$Y(\sqrt{\pi}z)$	$X(\sqrt{\pi}z)$	$Y(\sqrt{\pi}z)$	$X(\sqrt{\pi}z)$	$Y(\sqrt{\pi}z)$
0.0	1.0000	0.0	1.0000	0.0	1.0000	0.0
0.1	1.1264	5.7222 (− 2)	1.1502	1.0853 (− 2)	1.1558	1.1034 (− 3)
0.2	1.2019	1.7743 (− 1)	1.2600	2.4171 (− 2)	1.2724	2.4341 (− 3)
0.5	1.3123	5.7035 (− 1)	1.5227	7.7054 (− 2)	1.5614	7.5113 (− 3)
1.0	1.3759	9.0239 (− 1)	1.8533	2.1040 (− 1)	1.9548	1.8995 (− 2)
2.0	1.4161	1.1460	2.2979	6.0766 (− 1)	2.5909	5.1403 (− 2)
5.0	1.4435	1.3263	2.8917	1.6294	3.9868	2.1205 (− 1)
1.0 (1)	1.4533	1.3930	3.2145	2.4049	5.5810	6.8720 (− 1)
2.0 (1)	1.4583	1.4277	3.4136	2.9513	7.5535	2.1698
5.0 (1)	1.4614	1.4490	3.5479	3.3471	1.0015 (1)	5.8888
1.0 (2)	1.4624	1.4562	3.5954	3.4922	1.1301 (1)	8.6442
2.0 (2)	1.4629	1.4598	3.6197	3.5673	1.2081 (1)	1.0563 (1)
5.0 (2)	1.4632	1.4620	3.6344	3.6133	1.2603 (1)	1.1943 (1)
1.0 (3)	1.4633	1.4627	3.6394	3.6288	1.2787 (1)	1.2447 (1)
2.0 (3)	1.4634	1.4630	3.6419	3.6366	1.2880 (1)	1.2708 (1)
5.0 (3)	1.4634	1.4633	3.6434	3.6412	1.2937 (1)	1.2868 (1)
$\gamma/\sqrt{\pi}$	1.4634	1.4633	3.6435	3.6416	1.2942 (1)	1.2880 (1)

Table 2. The X and Y functions for a Lorentz profile with $1-\varpi=10^{-6}$ and $\beta=10^{-4}$

	$\tau_0=1$		$\tau_0=10$		$\tau_0=100$	
z	$X(\pi z)$	$Y(\pi z)$	$X(\pi z)$	$Y(\pi z)$	$X(\pi z)$	$Y(\pi z)$
0.0	1.0000	0.0	1.0000	0.0	1.0000	0.0
0.1	1.0810	8.9419 (− 2)	1.0934	9.2503 (− 3)	1.0955	8.5257 (− 4)
0.2	1.1191	2.9271 (− 1)	1.1559	2.0226 (− 2)	1.1605	1.8124 (− 3)
0.5	1.1614	6.7230 (− 1)	1.2915	6.5620 (− 2)	1.3054	5.1489 (− 3)
1.0	1.1814	8.9819 (− 1)	1.4381	2.0619 (− 1)	1.4760	1.1852 (− 2)
2.0	1.1929	1.0400	1.5969	5.5322 (− 1)	1.7092	2.8536 (− 2)
5.0	1.2003	1.1362	1.7611	1.1399	2.1149	1.0643 (− 1)
1.0 (1)	1.2029	1.1703	1.8355	1.4755	2.4763	3.6735 (− 1)
2.0 (1)	1.2042	1.1878	1.8776	1.6833	2.8256	1.0085
5.0 (1)	1.2050	1.1984	1.9047	1.8232	3.1629	2.0754
1.0 (2)	1.2053	1.2020	1.9141	1.8727	3.3108	2.6801
2.0 (2)	1.2054	1.2037	1.9188	1.8979	3.3935	3.0530
5.0 (2)	1.2055	1.2048	1.9216	1.9133	3.4463	3.3035
1.0 (3)	1.2055	1.2052	1.9226	1.9184	3.4644	3.3919
2.0 (3)	1.2055	1.2053	1.9231	1.9210	3.4736	3.4371
γ/π	1.2055	1.2054	1.9233	1.9219	3.4771	3.4540

Table 3. The X and Y functions for a Doppler profile with $1-\varpi=10^{-11}$ and $\beta=0$

	$\tau_0=1$		$\tau_0=10$		$\tau_0=100$	
z	$X(\sqrt{\pi}z)$	$Y(\sqrt{\pi}z)$	$X(\sqrt{\pi}z)$	$Y(\sqrt{\pi}z)$	$X(\sqrt{\pi}z)$	$Y(\sqrt{\pi}z)$
0.0	1.0000	0.0	1.0000	0.0	1.0000	0.0
0.1	1.1264	5.7238 (− 2)	1.1503	1.0889 (− 2)	1.1560	1.1721 (− 3)
0.2	1.2019	1.7746 (− 1)	1.2602	2.4250 (− 2)	1.2728	2.5859 (− 3)
0.5	1.3124	5.7041 (− 1)	1.5231	7.7296 (− 2)	1.5627	7.9806 (− 3)
1.0	1.3760	9.0246 (− 1)	1.8542	2.1099 (− 1)	1.9580	2.0184 (− 2)
2.0	1.4161	1.1461	2.2995	6.0898 (− 1)	2.5989	5.4622 (− 2)
5.0	1.4436	1.3264	2.8946	1.6320	4.0140	2.2505 (− 1)
1.0 (1)	1.4534	1.3931	3.2182	2.4084	5.6446	7.2476 (− 1)
2.0 (1)	1.4584	1.4278	3.4177	2.9554	7.6794	2.2623
5.0 (1)	1.4615	1.4491	3.5523	3.3515	1.0236 (1)	6.0830
1.0 (2)	1.4625	1.4563	3.5999	3.4967	1.1577 (1)	8.9026
2.0 (2)	1.4630	1.4599	3.6243	3.5719	1.2392 (1)	1.0863 (1)
5.0 (2)	1.4633	1.4621	3.6391	3.6180	1.2937 (1)	1.2273 (1)
1.0 (3)	1.4634	1.4628	3.6441	3.6335	1.3129 (1)	1.2787 (1)
2.0 (3)	1.4635	1.4631	3.6465	3.6412	1.3227 (1)	1.3054 (1)
5.0 (3)	1.4635	1.4634	3.6480	3.6459	1.3286 (1)	1.3216 (1)
1.0 (4)	1.4635	1.4634	3.6485	3.6475	1.3306 (1)	1.3271 (1)

$N=309$, and again since the version of the F_N method we used for these two cases (for $\tau_0=1$, $\tau_0=10$ and $\tau_0=100$, as well as for $\tau_0\to\infty$) lead to very stable systems of linear algebraic equations, we were also able to see that our results remained valid for N as large as 499.

In regard to the literature concerning calculations of X and Y functions for the non-grey model of radiative transfer considered here, we first looked at the work of Fuller and Hyett,[12] who considered the case of a Doppler line-scattering profile with $\beta=0$ and $\varpi\le 0.9$. In regard to Ref. 12 we first of all note that the $X(z)$ and $Y(z)$ functions of Fuller and Hyett correspond to our $X(\sqrt{\pi}z)$ and $Y(\sqrt{\pi}z)$ and that the definition of the optical thickness τ_0 used by Fuller and Hyett must be multiplied by the factor $\sqrt{\pi}$ to yield our τ_0. These differences in definitions are due to differing definitions of the optical variable τ used in, say, Eq. (1) and, of course, are unimportant except when comparing results. We have redone several (for $\varpi=0.1$ and $\varpi=0.9$) of the calculations summarized in Ref. 12, and we have to say that we found some significant differences between our results and those of Fuller and Hyett. Looking at the cases of $\tau_0=1$ and $\tau_0=100$, in the notation of Ref. 12, we found some agreement (3 or 4 and sometimes 5 figures of agreement) for the X function, and we also found some similar agreement for the Y function; but we also found some entries where we did not agree on any of the

Table 4. The X and Y functions for a Lorentz profile with $1-\varpi=10^{-11}$ and $\beta=0$

	$\tau_0=1$		$\tau_0=10$		$\tau_0=100$	
z	$X(\pi z)$	$Y(\pi z)$	$X(\pi z)$	$Y(\pi z)$	$X(\pi z)$	$Y(\pi z)$
0.0	1.0000	0.0	1.0000	0.0	1.0000	0.0
0.1	1.0810	8.9439 (− 2)	1.0934	9.2802 (− 3)	1.0956	8.8252 (− 4)
0.2	1.1191	2.9275 (− 1)	1.1560	2.0290 (− 2)	1.1607	1.8760 (− 3)
0.5	1.1615	6.7236 (− 1)	1.2918	6.5801 (− 2)	1.3059	5.3290 (− 3)
1.0	1.1815	8.9825 (− 1)	1.4386	2.0658 (− 1)	1.4770	1.2263 (− 2)
2.0	1.1929	1.0401	1.5977	5.5392 (− 1)	1.7113	2.9508 (− 2)
5.0	1.2004	1.1363	1.7622	1.1409	2.1206	1.0957 (− 1)
1.0 (1)	1.2030	1.1704	1.8368	1.4767	2.4868	3.7462 (− 1)
2.0 (1)	1.2043	1.1879	1.8790	1.6846	2.8422	1.0221
5.0 (1)	1.2051	1.1985	1.9061	1.8247	3.1863	2.0970
1.0 (2)	1.2053	1.2020	1.9155	1.8741	3.3374	2.7057
2.0 (2)	1.2055	1.2038	1.9203	1.8994	3.4220	3.0808
5.0 (2)	1.2055	1.2049	1.9231	1.9147	3.4759	3.3329
1.0 (3)	1.2056	1.2052	1.9241	1.9199	3.4945	3.4219
2.0 (3)	1.2056	1.2054	1.9246	1.9225	3.5039	3.4673
5.0 (3)	1.2056	1.2055	1.9248	1.9240	3.5096	3.4949
1.0 (4)	1.2056	1.2056	1.9249	1.9245	3.5115	3.5041

digits for the Y function. We must say, however, that the entries where we found no agreement were for cases when the Y function is very small. Of course, we have no proof that our results are correct, but we believe that at least some of the results given in Ref. 12 should be considered somewhat suspect.

We also note that some discussion of a calculation of the X and Y functions for the case of a Lorentz line-scattering profile with $\beta=0$ is given in a paper by Gabrielyan et al.[13] However, looking at the numerical results for the X and Y functions reported in Ref. 13 for the two considered cases, viz. $\varpi=0.65$ and $\varpi=0.99$, we found little agreement with our calculations for the case of $\varpi=0.65$ and even less agreement for the case $\varpi=0.99$.

Needless to say, we would like to see the issue of these differences between our calculations and those of Ref. 12 and, in particular, those of Ref. 13 resolved by other, independent calculations.

In regard to the accuracy of the results we are reporting we would like to note that we have been unable to find any published computations of the X and Y functions for the general cases we are considering here, and so it has proved impossible to have some independent checks of our calculations. Therefore, we have to admit that, although we have taken various steps to establish some confidence in the accuracy of our results, there certainly exists the possibility that some numbers may be less accurate than what we are reporting here. So without proof, but with some confidence, we can say that we believe the results in Tables 1–4 to be correct to plus or minus one unit in the final digit given.

5. CONCLUDING REMARKS

It seems clear that from an analytical point of view, a study of the X and Y for this model of non-coherent scattering is really very simple and, in fact, differs very little from the classical case discussed, for example, by Chandrasekhar;[4] however, because of the rapid variation of the characteristic function $\Psi(\xi)$, and because the domain of definition for X and Y can be arbitrarily large, the problem is not without some interest when it comes to computing the desired functions.

We consider that this work demonstrates the F_N method can be used for this (in our opinion, difficult) radiative-transfer calculation, and we intend to continue with this work in ways that should make the calculation faster, more accurate and perhaps better defined in regard to the choice of the scaling factors a and b used here.

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REFERENCES

1. Hummer, D. G., *Monthly Notices of the Royal Astronomical Society*, 1968, **138,** 73.
2. Barichello, L. B. and Siewert, C. E., *Journal of Quantitative Spectroscopy & Radiative Transfer*, 1998, **60,** 261–276.
3. Ambartsumian, V. A., *Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS*, 1943, **38,** 229.
4. Chandrasekhar, S., *Radiative Transfer*. Oxford University Press, London, 1950.
5. Busbridge, I. W., *Mathematics of Radiative Transfer*. Cambridge University Press, Cambridge, 1960.
6. Ivanov, V. V., *Transfer of Radiation in Spectral Lines*. National Bureau of Standards Publication 385, Washington, DC, 1973.
7. Heaslet, M. A. and Warming, R. F., *Journal of Quantitative Spectroscopy & Radiative Transfer*, 1968, **8,** 1101.
8. Dunn, W. L., Garcia, R. D. M. and Siewert, C. E., *The Astrophysical Journal*, 1982, **260,** 849.
9. Siewert, C. E., *Zeitschrift für Angewandte Mathematik und Physik*, 1984, **35,** 144.
10. Garcia, R. D. M., *Transport Theory and Statistical Physics*, 1985, **14,** 391.
11. Rutily, B., personal communication, Observatoire de Lyon, France, 1997.
12. Fuller, F. B. and Hyett, B. J., *Calculations of Chandrasekhar's X- and Y-Functions and Their Analogs for Noncoherent Isotropic Scattering*, NASA Technical Note D-4712, 1968.
13. Gabrielyan, R. G., Mkrtchyan, A. R., Mnatsakanyan, M. A. and Kotandzhyan, Kh. V., *Astrophysics*, 1988, **28,** 262.