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On the Equivalence Between the Discrete Ordinates and the Spherical Harmonics Methods in Radiative Transfer

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Abstract—*In this work concerning steady-state radiative-transfer calculations in plane-parallel media, the equivalence between the discrete ordinates method and the spherical harmonics method is proved. More specifically, it is shown that for standard radiative-transfer problems without the imposed restriction of azimuthal symmetry the two methods yield identical results for the radiation intensity when the quadrature scheme for the discrete ordinates method is defined by the zeros of the associated Legendre functions and when generalized Mark boundary conditions are used to define the spherical harmonics solution. It is also shown that, with these choices for a quadrature scheme and for the boundary conditions, the two methods can be formulated so as to require the same computational effort. Finally a justification for using the generalized Mark boundary conditions in the spherical harmonics solution is given.*

I. INTRODUCTION

It is well known[1] that the problem of computing the radiation intensity $I(\tau,\mu,\phi)$ in a plane-parallel medium can be decomposed into a collection of ϕ-independent problems for the components of $I(\tau,\mu,\phi)$ in a finite Fourier decomposition. We use m as our Fourier-component index, but to avoid too much heavy notation we suppress the m dependence on the intensity $I(\tau,\mu)$ and the inhomogeneous source term $Q(\tau,\mu)$, and so we start our work here by considering the Fourier-component problems defined for $m = 0,1,\dots,L$ by

$$\mu \frac{\partial}{\partial\tau} I(\tau,\mu) + I(\tau,\mu) = \frac{\varpi}{2}\sum_{l=m}^{L}\beta_l P_l^m(\mu)\int_{-1}^{1} P_l^m(\mu')I(\tau,\mu')\,d\mu' + Q(\tau,\mu) \tag{1}$$

and the boundary conditions

$$I(0,\mu) = F_1(\mu) + \int_0^1 R_1(\mu',\mu)I(0,-\mu')\mu'\,d\mu' \tag{2a}$$

and

$$I(\tau_0,-\mu) = F_2(\mu) + \int_0^1 R_2(\mu',\mu)I(\tau_0,\mu')\mu'\,d\mu' \tag{2b}$$

for $\mu \in (0,1]$. Here $\beta_0 = 1$ and the $|\beta_l| < 2l+1$ for $0 < l \le L$ are the coefficients in an L'th order Legendre polynomial expansion of the phase function, and $\varpi \in [0,1]$ is the single-scattering albedo. Also $\tau \in [0,\tau_0]$ is the optical variable and $\mu \in [-1,1]$ is the cosine of the polar angle (as measured from the *positive* τ axis). The inhomogeneous source term $Q(\tau,\mu)$, the boundary source terms $F_1(\mu)$ and $F_2(\mu)$, and the boundary reflection functions $R_1(\mu',\mu)$ and $R_2(\mu',\mu)$ are considered specified. To include the case of specular reflection, we allow that $\mu' R_1(\mu',\mu)$ and $\mu' R_2(\mu',\mu)$ can have components of the forms $\rho_1^s \delta(\mu'-\mu)$ and $\rho_2^s \delta(\mu'-\mu)$ where ρ_1^s and ρ_2^s are specular-reflection coefficients. Finally, we note that in this work we are using the *normalized* associated Legendre functions defined by

$$P_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_l(\mu) \ . \quad (3)$$

II. A SPHERICAL HARMONICS SOLUTION

To solve the collection of Fourier-component problems defined by Eqs. (1) and (2), we use a form of the solution to the moments of the homogeneous version of Eq. (1) that was reported in Ref. 2 and a particular solution that was worked out by Siewert and McCormick in a work[3] on polarization that contains the scalar case considered here as the first component in a Stokes-vector formulation. In view of Refs. 2 and 3, our presentation here is brief. First of all, we note that, for N odd,

$$I(\tau,\mu) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu) \times \sum_{j=1}^{J} [A_j e^{-\tau/\xi_j} + (-1)^{l-m} B_j e^{-(\tau_0-\tau)/\xi_j}] g_l^m(\xi_j) + I_p(\tau,\mu) \quad (4)$$

where

$$I_p(\tau,\mu) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu) \times \sum_{j=1}^{J} \frac{C_j}{\xi_j} [\mathcal{A}_j(\tau) + (-1)^{l-m} \mathcal{B}_j(\tau)] g_l^m(\xi_j) \quad (5)$$

is a particular solution, which satisfies the first $N+1$ associated Legendre moments of Eq. (1). Here we use $M = N + m$, $J = (N+1)/2$, and we use the *normalized* Chandrasekhar polynomials,[1,4] with the starting value

$$g_m^m(\xi) = (2m-1)!![(2m)!]^{-1/2} \ , \quad (6)$$

that satisfy, for $l \geq m$,

$$h_l \xi g_l^m(\xi) = a_{l+1}^m g_{l+1}^m(\xi) + a_l^m g_{l-1}^m(\xi) \quad (7)$$

where

$$h_l = 2l + 1 - \varpi \beta_l \ , \quad \text{for } 0 \leq l \leq L \ , \quad (8a)$$

and

$$h_l = 2l + 1 \ , \quad \text{for } l > L \ , \quad (8b)$$

and where

$$a_l^m = (l^2 - m^2)^{1/2} \ . \quad (9)$$

In addition, the eigenvalues $\{\xi_j\}$ are the positive zeros of $g_{M+1}^m(\xi)$, and the constants $\{A_j\}$ and $\{B_j\}$ are to be determined from the boundary conditions. In regard to the particular solution given by Eq. (5), we note that the constants $\{C_j\}$ are given by

$$C_j = 2\left(\sum_{l=m}^{M} h_l [g_l^m(\xi_j)]^2\right)^{-1} \quad (10)$$

and that

$$\mathcal{A}_j(\tau) = \int_0^\tau \left\{ \int_{-1}^1 X_j(\mu) Q(x,\mu)\, d\mu \right\} e^{-(\tau-x)/\xi_j}\, dx \quad (11a)$$

and

$$\mathcal{B}_j(\tau) = \int_\tau^{\tau_0} \left\{ \int_{-1}^1 Y_j(\mu) Q(x,\mu)\, d\mu \right\} e^{-(x-\tau)/\xi_j}\, dx \quad (11b)$$

where

$$X_j(\mu) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu) g_l^m(\xi_j) \quad (12a)$$

and

$$Y_j(\mu) = \sum_{l=m}^{M} \frac{2l+1}{2} (-1)^{l-m} P_l^m(\mu) g_l^m(\xi_j) \ . \quad (12b)$$

To complete the spherical harmonics solution, we note (again) that the required eigenvalues $\{\xi_j\}$ are the zeros of the polynomial $g_{M+1}^m(\xi)$, and in Ref. 2 it was shown that the squares ξ_j^2 can be computed very efficiently as the eigenvalues of a tridiagonal matrix of half-order (J) size. In addition, the Chandrasekhar polynomials $g_l^m(\xi_j)$ can be effectively computed from the eigenvectors of the mentioned tridiagonal matrix, or the recursion formula given by Eq. (7) can be used, if sufficient care is given to using the recursive approach.[4] Finally, we must find the arbitrary constants $\{A_j\}$ and $\{B_j\}$ that appear in Eq. (4). To define a system of linear algebraic equations for the $N+1$ unknowns $\{A_j\}$ and $\{B_j\}$, we use what we are calling generalized Mark boundary conditions, i.e., we substitute Eq. (4) into Eqs. (2) and evaluate the resulting equations at the J positive zeros of the associated Legendre function

$P_{M+1}^m(\xi)$. We note that these zeros can also be found from the eigenvalues of a tridiagonal matrix. In fact, putting $\varpi = 0$ in the formulation for computing the zeros of the Chandrasekhar polynomial $g_{M+1}^m(\xi)$ will yield the zeros of $P_{M+1}^m(\xi)$.

In the papers of Karp[5] and Karp and Petrack,[6] the intensity as computed from Eq. (4) is shown to oscillate about the correct result, and so in order to remove these oscillations and thus to define our best result for the intensity, we substitute Eq. (4) into the right side of Eq. (1) and then solve analytically the resulting equation to find $I(\tau,\mu)$. While Eq. (4) can be used with confidence for computing either full- or half-range moments of the intensity, we consider that this "postprocessing" procedure[7] is a very important element of our spherical harmonics solution for the intensity.

III. A DISCRETE ORDINATES SOLUTION

To start we consider a discrete ordinates representation of Eq. (1) written as

$$\mu_i \frac{d}{d\tau} I(\tau,\mu_i) + I(\tau,\mu_i) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu_i) \sum_{\alpha=1}^{N+1} w_\alpha P_l^m(\mu_\alpha) I(\tau,\mu_\alpha) + Q(\tau,\mu_i) \tag{13}$$

for $i = 1,2,\ldots,N+1$. Here, to use the same N as in the spherical harmonics solution, we consider N to be odd and define the quadrature scheme by the same zeros of $P_{M+1}^m(\xi)$. We must emphasize here that this quadrature scheme is not the one used by Chandrasekhar[1] for the cases $m > 0$, but as we will see, it is a better one. We note that Karp and Petrack[6] have pointed out that a quadrature scheme based on the zeros of the associated Legendre function $P_{M+1}^m(\xi)$ is computationally better than a scheme based on the usual Gauss points, i.e., the zeros of the Legendre polynomial $P_{N+1}(\xi)$.

Following Chandrasekhar,[1] we first seek solutions of the homogeneous equation, and so we substitute

$$I(\tau,\mu_i) = \Phi(\nu,\mu_i) e^{-\tau/\nu} \tag{14}$$

into the homogeneous version of Eq. (13) to obtain

$$\Phi(\nu,\mu_i) = \frac{\varpi\nu}{2} \frac{1}{\nu - \mu_i} \sum_{l=m}^{L} \beta_l P_l^m(\mu_i) g_l^m(\nu) , \quad \nu \neq \mu_i . \tag{15}$$

Here we have defined

$$g_l^m(\nu) = \sum_{\alpha=1}^{N+1} w_\alpha P_l^m(\mu_\alpha) \Phi(\nu,\mu_\alpha) . \tag{16}$$

Noting the recursion relation

$$(2l+1)\mu P_l^m(\mu) = a_{l+1}^m P_{l+1}^m(\mu) + a_l^m P_{l-1}^m(\mu) , \quad l \geq m , \tag{17}$$

where the a_l^m are defined by Eq. (9), we can multiply Eq. (16) by $(2l+1)\nu$ and use Eqs. (15) and (17) to obtain

$$h_l \nu g_l^m(\nu) = a_{l+1}^m g_{l+1}^m(\nu) + a_l^m g_{l-1}^m(\nu) , \quad l \geq m , \tag{18}$$

where the h_l are defined by Eqs. (8). In obtaining Eq. (18), we have used

$$\sum_{\alpha=1}^{N+1} w_\alpha P_l^m(\mu_\alpha) P_{l'}^m(\mu_\alpha) = \frac{2}{2l+1} \delta_{l,l'} , \quad l + l' \leq 2M + 1 . \tag{19}$$

We note that the Gauss quadrature scheme based on the zeros of $P_{M+1}^m(\xi)$ can integrate exactly polynomials $p(\mu)$ of up to and including order $2N+1$ against the weight function $(1-\mu^2)^m$. In other words,

$$\int_{-1}^{1} (1-\mu^2)^m \mu^\beta \, d\mu = \sum_{\alpha=1}^{N+1} w_\alpha \mu_\alpha^\beta , \quad \beta = 0,1,\ldots,2N+1 . \tag{20}$$

We also note that in the process of deriving Eq. (18), the maximum value of l' for which we used Eq. (19) was $l' = L$. Now, since we wish to use Eq. (18) for $l = m, m+1, \ldots, M$, we conclude that we have the condition $M + 1 \geq L$ that we must respect in our discrete ordinates solution. It is clear that if, instead of using a Gauss quadrature scheme based on the zeros of $P_{M+1}^m(\xi)$, we had used the more common Gauss-Legendre scheme, then Eq. (20) would have been exact only for $\beta = 0,1,\ldots,2(N-m)+1$.

Continuing, we multiply Eq. (17) by $g_l^m(\nu)$, we multiply Eq. (18) by $P_l^m(\mu)$, and we then subtract the two equations, one from the other. We next sum the resulting equation from $l = m$ to $l = M$ to find

$$(\mu - \nu) \sum_{l=m}^{M} (2l+1) P_l^m(\mu) g_l^m(\nu) + \varpi\nu \sum_{l=m}^{M} \beta_l P_l^m(\mu) g_l^m(\nu) = a_{M+1}^m [P_{M+1}^m(\mu) g_M^m(\nu) - P_M^m(\mu) g_{M+1}^m(\nu)] . \tag{21}$$

If we now change our previously imposed condition $M + 1 \geq L$ to our final condition $M \geq L$, we can use Eq. (21) to rewrite Eq. (15), for $\nu \neq \mu_i$, as

$$\Phi(\nu,\mu_i) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu_i) g_l^m(\nu) - \frac{1}{2} a_{M+1}^m \frac{1}{\nu - \mu_i} P_M^m(\mu_i) g_{M+1}^m(\nu) , \tag{22}$$

and if we multiply Eq. (22) by $P_\alpha^m(\mu_i)$, for $\alpha = m, m+1, \ldots, M$, and integrate (discretely) we find, after we use Eq. (16),

$$g_{M+1}^m(\nu) F_\alpha^m(\nu) = 0 \tag{23}$$

where, in general,

$$F_\alpha^m(\nu) = \sum_{i=1}^{N+1} w_i P_\alpha^m(\mu_i) P_M^m(\mu_i) \frac{1}{\nu - \mu_i} \ . \tag{24}$$

We can now multiply Eq. (24) by $(2\alpha + 1)\nu$ and use Eq. (17) to find the recursion relation

$$(2\alpha+1)\nu F_\alpha^m(\nu) = a_{\alpha+1}^m F_{\alpha+1}^m(\nu) + a_\alpha^m F_{\alpha-1}^m(\nu) + 2\delta_{\alpha,M} \tag{25}$$

for $\alpha = m, m+1, \ldots, M$. We see from Eq. (24) that $F_{M+1}^m(\nu) = 0$, and so it is clear from Eq. (25) that $F_\alpha^m(\nu)$ cannot be zero for all $\alpha = m, m+1, \ldots, M$. We therefore conclude from Eq. (23) that the acceptable values of ν are the $N+1$ zeros of $g_{M+1}^m(\xi)$, the same as the eigenvalues for the spherical harmonics method. It follows that we can write our discrete ordinates solution of the homogeneous version of Eq. (13) as

$$I_h(\tau, \mu_i) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu_i) \times \sum_{j=1}^{J} [A_j e^{-\tau/\xi_j} + (-1)^{l-m} B_j e^{-(\tau_0 - \tau)/\xi_j}] \times g_l^m(\xi_j) \ . \tag{26}$$

It is clear that we would not be able to rewrite Eq. (15) as we have in Eq. (22) without using in Eq. (21) the fact that the quadrature points $\{\mu_i\}$ are the zeros of $P_{M+1}^m(\mu)$.

In regard to the required particular solution, we omit some details of the calculation and note simply that the solution we seek is, like the solution to the homogeneous equation, very similar to the equivalent expression for the spherical harmonics method. We find we can write

$$I_p(\tau, \mu_i) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu_i) \times \sum_{j=1}^{J} \frac{C_j}{\xi_j} [\mathcal{A}_j(\tau) + (-1)^{l-m} \mathcal{B}_j(\tau)] \times g_l^m(\xi_j) \tag{27}$$

where the $\{C_j\}$ are again given by Eq. (10). In addition

$$\mathcal{A}_j(\tau) = \int_0^\tau \left\{ \sum_{\alpha=1}^{N+1} w_\alpha X_j(\mu_\alpha) Q(x, \mu_\alpha) \right\} e^{-(\tau - x)/\xi_j} \, dx \tag{28a}$$

and

$$\mathcal{B}_j(\tau) = \int_\tau^{\tau_0} \left\{ \sum_{\alpha=1}^{N+1} w_\alpha Y_j(\mu_\alpha) Q(x, \mu_\alpha) \right\} e^{-(x - \tau)/\xi_j} \, dx \tag{28b}$$

where

$$X_j(\mu_\alpha) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu_\alpha) g_l^m(\xi_j) \tag{29a}$$

and

$$Y_j(\mu_\alpha) = \sum_{l=m}^{M} \frac{2l+1}{2} (-1)^{l-m} P_l^m(\mu_\alpha) g_l^m(\xi_j) \ . \tag{29b}$$

Having developed our discrete ordinates solution to the homogeneous equation and our particular solution, we are ready to substitute

$$I(\tau, \mu_i) = \sum_{l=m}^{M} \frac{2l+1}{2} P_l^m(\mu_i) \times \sum_{j=1}^{J} [A_j e^{-\tau/\xi_j} + (-1)^{l-m} B_j e^{-(\tau_0 - \tau)/\xi_j}] \times g_l^m(\xi_j) + I_p(\tau, \mu_i) \tag{30}$$

into the boundary conditions

$$I(0, \mu_i) = F_1(\mu_i) + \int_0^1 R_1(\mu', \mu_i) I(0, -\mu') \mu' \, d\mu' \tag{31a}$$

and

$$I(\tau_0, -\mu_i) = F_2(\mu_i) + \int_0^1 R_2(\mu', \mu_i) I(\tau_0, \mu') \mu' \, d\mu' \ , \tag{31b}$$

for $i = 1, 2, \ldots, J$, in order to find the required arbitrary coefficients $\{A_j\}$ and $\{B_j\}$. But, we see a problem: to evaluate the reflection terms in Eqs. (31), we must have the solution defined on a set of quadrature points appropriate to the integration interval $[0,1]$, and at this point we do not have that. Of course if $\mu' R_1(\mu', \mu)$ and $\mu' R_2(\mu', \mu)$ contain only components for specular reflection, then there is no problem; however, we wish to consider cases more general than that.

In his computational implementation of an accurate and efficient discrete ordinates solution to the radiative-transfer problem defined by Eqs. (1) and (2), Chalhoub[8] used a special "half-range" quadrature scheme developed especially by Chalhoub and Garcia[9,10] in order to better take into account discontinuities that can occur on the surfaces of the considered finite layer. As a way of postprocessing his solution in order to compute the intensity at values of μ different from the quadrature points, Chalhoub[8] introduced additional points into the half-range quadrature scheme, assigned weights equal to zero to these additional points and solved a larger (than $N + 1$) system of linear algebraic equations, to find the intensity at all quadrature points and all introduced points. While this procedure of introducing additional points into the quadrature scheme could allow us to evaluate the reflection terms in Eqs. (31), we clearly would have to solve a much larger system of linear algebraic equations in order to find the arbitrary coefficients $\{A_j\}$ and $\{B_j\}$, and so this scheme, though it should work, is not considered a very efficient way to deal with the difficulties associated with reflection functions that require integration. Perhaps a better way of dealing with these types of boundary conditions is to extend the solution given by Eq. (30) to continuous values of μ, just by replacing μ_i with μ, and doing the integration either analytically or with any appropriate half-range quadrature scheme.

IV. WHEN EQUIVALENT AND WHEN NOT

In regard to reviewing previous works concerning the equivalence between the discrete ordinates method and the spherical harmonics method, we note, first of all, that Gast, in a very early work,[11] Karp,[15] and Sanchez and McCormick[12] have all discussed this issue for a class of problems based on the $m = 0$ case. In addition, and for the general case of $m \geq 0$, Karp and Petrack[6] have reported an equivalence between the two solutions of the classical albedo problem.[1]

It is clear that if in the boundary conditions given by Eqs. (2) we have no reflection terms that must be evaluated by (nontrivial) integration, if we use the generalized Mark boundary conditions to define the spherical harmonics solution, and if the inhomogeneous source term $Q(\tau,\mu)$ can be written as

$$Q(\tau,\mu) = (1-\mu^2)^{m/2}\sum_{\alpha=0}^{N+1} q_\alpha(\tau)\mu^\alpha \ , \qquad (32)$$

then the two methods, as defined in this work, will yield identical results at the quadrature points and will require the same computational effort. In addition, if the mentioned conditions are satisfied, the two methods will also yield identical results, and require the same computational effort, at all values of τ and μ once the "postprocessing" procedure[7] has been implemented. We note that the form of $Q(\tau,\mu)$ given by Eq. (32) is the one that is encountered when a problem with a beam, described by a delta "function," incident on the boundary is reformulated as a problem with an internal source and without the incident beam.[1,2]

If some special quadrature scheme is used with the discrete ordinates method, or if, for example, Marshak boundary conditions are used with the spherical harmonics method, or if the inhomogeneous source term is not of the form given by Eq. (32), or if reflective boundary conditions that require some numerical integration are applicable, then the two methods are not equivalent.

It is our opinion that some good justification would have to be found to use a "full-range" quadrature scheme different from the one used for the discrete ordinates method in this work. It is also our opinion, since the two methods are essentially always equivalent and since the spherical harmonics method requires no special consideration for dealing with general reflective boundary conditions or with a source more general than that given by Eq. (32), that there is little, if any, reason to use the classical discrete ordinates method for solving the class of problems considered in this work.

V. A JUSTIFICATION FOR THE GENERALIZED MARK BOUNDARY CONDITIONS

In Ref. 13, a work devoted to polarization, a justification was given for defining a new class of boundary conditions that proved to be very important for that polarization study. Here, we repeat a component of that earlier work in order to justify the use of the zeros of $P_{M+1}^m(\xi)$ to define what we are calling the generalized Mark boundary conditions. Since the demonstration we wish to make applies to a finite slab embedded in a vacuum or a totally absorbing medium ($\varpi = 0$), we consider that we have no reflection at the surfaces and that we have reformulated the problem so as to have the effects of $F_1(\mu)$ and $F_2(\mu)$ included in the inhomogeneous source term. Therefore, we consider a finite slab with a defined internal source and with no radiation incident on the two surfaces. We will consider this a three-region problem with the two added regions, one with $\tau \in (-\infty,0]$ and the other with $\tau \in [\tau_0,\infty)$, having $\varpi = 0$. Because the eigenvalues for the case of $\varpi = 0$ are just the zeros of $P_{M+1}^m(\xi)$, we write the solutions in the two outside regions as

$$I(\tau,\mu) = \sum_{l=m}^{M}\frac{2l+1}{2}(-1)^{l-m}P_l^m(\mu)\sum_{j=1}^{J}B_j e^{\tau/\mu_j}P_l^m(\mu_j) \ , \quad \tau \leq 0 \ , \qquad (33a)$$

and

$$I(\tau,\mu) = \sum_{l=m}^{M}\frac{2l+1}{2}P_l^m(\mu)\sum_{j=1}^{J}A_j e^{-(\tau-\tau_0)/\mu_j}P_l^m(\mu_j) \ , \quad \tau \geq \tau_0 \ . \qquad (33b)$$

Considering Eq. (21) for the special case of $\varpi = 0$, $\nu = \mu_j$, and μ changed to $-\mu$, we find we can write

$$\sum_{l=m}^{M}\frac{2l+1}{2}(-1)^{l-m}P_l^m(\mu)P_l^m(\mu_j) = \frac{1}{2}a_{M+1}^m\frac{(-1)^N}{\mu+\mu_j}P_{M+1}^m(\mu)P_M^m(\mu_j) \ . \qquad (34)$$

If we now use Eq. (34) in Eq. (33a) with $\mu > 0$ and in Eq. (33b) with $\mu < 0$, we can write

$$I(0,\mu) = \frac{1}{2} a^m_{M+1}(-1)^N P^m_{M+1}(\mu) \sum_{j=1}^{J} B_j P^m_M(\mu_j) \frac{1}{\mu + \mu_j} , \quad \mu > 0 , \tag{35a}$$

and

$$I(\tau_0,-\mu) = \frac{1}{2} a^m_{M+1}(-1)^N P^m_{M+1}(\mu) \sum_{j=1}^{J} A_j P^m_M(\mu_j) \frac{1}{\mu + \mu_j} , \quad \mu > 0 . \tag{35b}$$

Finally, because both of Eqs. (35) are zero for values of μ equal to the quadrature points, we consider it some justification for defining the radiation incident on our finite slab to be zero at these same points.

VI. A FINAL COMMENT

We note that the spherical harmonics solution and the discrete ordinates solution that we have reported in this work are not valid for the special, conservative case, i.e., $m = 0$ with $\varpi = 1$. The required modifications for the spherical harmonics solution are given in Ref. 14. We are of the opinion that the required modifications for the discrete ordinates solution are sufficiently evident from the spherical harmonics solution for this special case that we do not list them here.

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REFERENCES

1. S. CHANDRASEKHAR, *Radiative Transfer*, Oxford University Press, London, United Kingdom (1950).

2. M. BENASSI, R. D. M. GARCIA, A. H. KARP, and C. E. SIEWERT, *Astrophys. J.*, **280**, 853 (1984).

3. C. E. SIEWERT and N. J. McCORMICK, *J. Quant. Spectrosc. Radiat. Transfer*, **50**, 531 (1993).

4. R. D. M. GARCIA and C. E. SIEWERT, *J. Quant. Spectrosc. Radiat. Transfer*, **43**, 201 (1990).

5. A. H. KARP, *J. Quant. Spectrosc. Radiat. Transfer*, **25**, 403 (1981).

6. A. H. KARP and S. PETRACK, *J. Quant. Spectrosc. Radiat. Transfer*, **30**, 351 (1983).

7. V. KOURGANOFF, *Basic Methods in Transfer Problems*, Clarendon Press, Oxford, United Kingdom (1952).

8. E. S. CHALHOUB, "The Discrete Ordinates Method in the Solution of the Azimuthally Dependent Transport Equation in Plane Geometry," ScD Dissertation, Instituto de Pesquisas Energéticas e Nucleares/Universidade de São Paulo, São Paulo, Brazil (1997) (in Portuguese).

9. E. S. CHALHOUB and R. D. M. GARCIA, *Ann. Nucl. Energy*, **24**, 1069 (1997).

10. E. S. CHALHOUB and R. D. M. GARCIA, "A New Quadrature Scheme for Solving Azimuthally Dependent Transport Problems," *Transport Theory Stat. Phys.* (in press).

11. R. GAST, "On the Equivalence of the Spherical Harmonics Method and the Discrete Ordinate Method Using Gauss Quadrature for the Boltzmann Equation," WAPD-TM-118, Westinghouse Electric Corporation (1958).

12. R. SANCHEZ and N. J. McCORMICK, *Nucl. Sci. Eng.*, **80**, 481 (1982).

13. R. D. M. GARCIA and C. E. SIEWERT, *J. Quant. Spectrosc. Radiat. Transfer*, **36**, 401 (1986).

14. L. B. BARICHELLO, R. D. M. GARCIA, and C. E. SIEWERT, *J. Quant. Spectrosc. Radiat. Transfer*, **60**, 247 (1998).