PERGAMON

Journal of Quantitative Spectroscopy & Radiative Transfer 62 (1999) 665–675

Journal of Quantitative Spectroscopy & Radiative Transfer

A discrete-ordinates solution for a non-grey model with complete frequency redistribution

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Received 17 October 1997

Abstract

The discrete-ordinates method is used to develop a solution to a class of non-grey problems in the theory of radiative transfer. The model considered allows for scattering with complete frequency redistribution (completely non-coherent scattering) and continuum absorption. In addition to a general formulation for semi-infinite and finite plane-parallel media, specific computations, for both the Doppler and the Lorentz profiles of the line-scattering coefficient, are discussed in regard to a half-space application concerning a linearly varying Planck function and also in regard to a basic problem from which, except for the conservative case, the classical X and Y functions can be extracted. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

In two recent papers [1,2] we used exact analysis and the F_N method to develop and evaluate solutions to some basic problems specific to a class of non-grey problems in radiative transfer that are based on the equation of transfer written, after Hummer [3], as

$$\mu \frac{\partial}{\partial \tau} I_x(\tau, \mu) + [\phi(x) + \beta] I_x(\tau, \mu) = [\phi(x) + \beta] S_x(\tau) \tag{1}$$

where $S_x(\tau)$ is the source function

$$[\phi(x) + \beta] S_x(\tau) = \frac{1}{2} \varpi \phi(x) \int_{-\infty}^{\infty} \phi(x') \int_{-1}^{1} I_{x'}(\tau, \mu') \mathrm{d}\mu' \, \mathrm{d}x' + [\rho\beta + (1 - \varpi)\phi(x)] B(\tau), \tag{2}$$

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PII: S0022-4073(98)00096-X

with $B(\tau)$ being the Planck function at the center of the line. Continuing, we note that x is the normalized frequency variable measured (in dimensionless units) from the line center, $\tau \in [0, \tau_0]$ is the optical variable, τ_0 is the optical thickness of the plane-parallel medium and $\mu \in [-1, 1]$ is the cosine of the polar angle (as measured from the *positive* τ-axis) that describes the direction of propagation of the radiation. In addition, $\varpi \in [0, 1)$ is the albedo for single scattering, $\beta \geq 0$ is the ratio of the continuum absorption coefficient to the average line coefficient, ρ is the ratio of the continuum source function to the Planck function and $\phi(x)$ is the line-scattering profile.

For a specified Planck function $B(\tau)$ we seek a solution of Eq. (1) subject to boundary conditions of the form

$$I_x(0, \mu) = I_{x,1}(\mu) \tag{3a}$$

and

$$I_x(\tau_0, -\mu) = I_{x,2}(\mu) \tag{3b}$$

for $\mu \in (0, 1]$. Here we consider that the functions $I_{x,1}(\mu)$ and $I_{x,2}(\mu)$ that describe any radiation incident on the layer are specified.

2. A reduction to a simpler problem

In Ref. [4] some transformations were used, for the case of a semi-infinite medium with no radiation incident on the surface, that made it possible to construct the solution to Eq. (1) from the solution of what is considered a much simpler problem. We therefore proceed to use similar transformations to obtain a simpler problem for the case considered in this work, viz. the case of a finite layer with prescribed radiation incident on both surfaces. First of all, a change of the angular variable by $\xi = \mu\gamma_x$ with

$$\gamma_x = [\phi(x) + \beta]^{-1} \tag{4}$$

allows us, after the changes in notation

$$I_x(\tau, \xi/\gamma_x) \to \gamma_x I_x(\tau, \xi), \qquad I_{x,1}(\xi/\gamma_x) \to \gamma_x I_{x,1}(\xi) \quad \text{and} \quad I_{x,2}(\xi/\gamma_x) \to \gamma_x I_{x,2}(\xi),$$

to rewrite Eqs. (1) and (3) as

$$\xi \frac{\partial}{\partial \tau} I_x(\tau, \xi) + I_x(\tau, \xi) = \frac{1}{2} \varpi \phi(x) \int_{-\gamma}^{\gamma} \int_{M_{\xi'}} \phi(x') I_{x'}(\tau, \xi') \, \mathrm{d}x' \, \mathrm{d}\xi' + Q_x(\tau), \tag{5}$$

for $\xi \in (-\gamma, \gamma)$ and $\tau \in (0, \tau_0)$, and

$$I_x(0, \xi) = I_{x,1}(\xi) \tag{6a}$$

and

$$I_x(\tau_0, -\xi) = I_{x,2}(\xi) \tag{6b}$$

for $\xi \in (0, \gamma)$. Here

$$Q_x(\tau) = [\rho\beta + (1 - \varpi)\phi(x)]B(\tau) \tag{7}$$

is an inhomogeneous source term and $\gamma = \sup \gamma_x$. We consider profiles $\phi(x)$ that vanish at infinity, and so it is clear that $\gamma = 1/\beta$. In addition, the set M_ξ is defined such that $x \in M_\xi$ if and only if $[\phi(x) + \beta]|\xi| \le 1$.

We now write

$$I_x(\tau, \xi) = I_x^{\mathrm{p}}(\tau, \xi) + \phi(x)G(\tau, \xi) - [I_x^{\mathrm{p}}(0, \xi) - I_{x,1}(\xi) + \phi(x)G(0, \xi)]\mathrm{e}^{-\tau/\xi} \tag{8a}$$

and

$$I_x(\tau, -\xi) = I_x^{\mathrm{p}}(\tau, -\xi) + \phi(x)G(\tau, -\xi) - [I_x^{\mathrm{p}}(\tau_0, -\xi) - I_{x,2}(\xi) + \phi(x)G(\tau_0, -\xi)]\mathrm{e}^{-(\tau_0-\tau)/\xi} \tag{8b}$$

for $\xi \in (0, \gamma)$. Here $I_x^{\mathrm{p}}(\tau, \xi)$ is a particular solution of Eq. (5) corresponding to the specified inhomogeneous source term $Q_x(\tau)$ and the function $G(\tau, \xi)$ is to be defined. Substituting Eqs. (8) into Eq. (5) and using Eqs. (6), we find that Eqs. (8) are correct if $G(\tau, \xi)$ satisfies

$$\xi \frac{\partial}{\partial \tau} G(\tau, \xi) + G(\tau, \xi) = \int_{-\gamma}^{\gamma} \Psi(\xi')G(\tau, \xi')\,\mathrm{d}\xi', \tag{9}$$

for $\xi \in (-\gamma, \gamma)$ and $\tau \in (0, \tau_0)$, and the boundary conditions

$$\Psi(\xi)G(0, \xi) = G_1(\xi) \tag{10a}$$

and

$$\Psi(\xi)G(\tau_0, -\xi) = G_2(\xi) \tag{10b}$$

for $\xi \in (0, \gamma)$. Here

$$\Psi(\xi) = \frac{\varpi}{2} \int_{M_\xi} \phi^2(x)\,\mathrm{d}x, \tag{11}$$

$$G_1(\xi) = \frac{\varpi}{2} \int_{M_\xi} \phi(x)[I_{x,1}(\xi) - I_x^{\mathrm{p}}(0, \xi)]\,\mathrm{d}x \tag{12a}$$

and

$$G_2(\xi) = \frac{\varpi}{2} \int_{M_\xi} \phi(x)[I_{x,2}(\xi) - I_x^{\mathrm{p}}(\tau_0, -\xi)]\,\mathrm{d}x. \tag{12b}$$

Since in Ref. [4] a general procedure was given for constructing a particular solution $I_x^{\mathrm{p}}(\tau, \xi)$, we have only to solve Eq. (9) subject to the boundary conditions given by Eqs. (10) to have the complete solution we seek, $I_x(\tau, \xi)$. We therefore proceed to define our discrete-ordinates solution of the "G problem."

3. A discrete-ordinates solution

We note first of all that the characteristic function $\Psi(\xi)$ as defined by Eq. (11) is an even function, and so we write our discrete-ordinates equations as

$$\xi_i \frac{\mathrm{d}}{\mathrm{d}\tau} G(\tau, \xi_i) + G(\tau, \xi_i) = \sum_{k=1}^{N} w_k \Psi(\xi_k)[G(\tau, \xi_k) + G(\tau, -\xi_k)] \tag{13a}$$

and

$$-\xi_i \frac{\mathrm{d}}{\mathrm{d}\tau} G(\tau, -\xi_i) + G(\tau, -\xi_i) = \sum_{k=1}^{N} w_k \Psi(\xi_k)[G(\tau, \xi_k) + G(\tau, -\xi_k)] \tag{13b}$$

for $i = 1, 2, \ldots, N$. In writing Eqs. (13) as we have, we clearly are considering that the N quadrature points $\{\xi_k\}$ and the N weights $\{w_k\}$ are defined for use on the integration interval $[0, \gamma]$. Of course, we are free to use a single quadrature scheme on the interval $[0, \gamma]$, or, as we will in fact do, we can use a composite quadrature defined over sub-intervals of $[0, \gamma]$. Now seeking exponential solutions of Eqs. (13), we substitute

$$G(\tau, \pm \xi_i) = \phi(\nu, \pm \xi_i) \mathrm{e}^{-\tau/\nu} \tag{14}$$

into Eqs. (13) to find

$$\frac{1}{\nu} \Xi \Phi_+ = (\mathbf{I} - \mathbf{W})\Phi_+ - \mathbf{W}\Phi_- \tag{15a}$$

and

$$-\frac{1}{\nu} \Xi \Phi_- = (\mathbf{I} - \mathbf{W})\Phi_- - \mathbf{W}\Phi_+ \tag{15b}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix,

$$\Phi_\pm = [\phi(\nu, \pm \xi_1), \phi(\nu, \pm \xi_2), \ldots, \phi(\nu, \pm \xi_N)]^{\mathrm{T}}, \tag{16}$$

$$(\mathbf{W})_{i,j} = w_j \Psi(\xi_j) \tag{17}$$

and

$$\Xi = \mathrm{diag}\{\xi_1, \xi_2, \ldots, \xi_N\}. \tag{18}$$

If we now let

$$\mathbf{U} = \Phi_+ + \Phi_- \tag{19}$$

and

$$\mathbf{V} = \Phi_+ - \Phi_- \tag{20}$$

then we can eliminate $\mathbf{V}$ between the sum and the difference of Eqs. (15) to find

$$(\Xi^{-2} - 2\Xi^{-1}\mathbf{W}\Xi^{-1})\Xi\mathbf{U} = \frac{1}{\nu^2} \Xi\mathbf{U} \tag{21}$$

where to have Ξ^{-1} exist we cannot allow any of the quadrature points to be zero. Multiplying Eq. (21) by the diagonal matrix $\mathbf{T}$ with the diagonal elements given by

$$T_i = [w_i \Psi(\xi_i)]^{1/2} \tag{22}$$

we can make $\mathbf{TWT}^{-1}$ symmetric so we can rewrite Eq. (21) as

$$(\mathbf{D} - 2\mathbf{z}\mathbf{z}^{\mathrm{T}})\mathbf{X} = \lambda \mathbf{X} \tag{23}$$

where

$$\mathbf{D} = \mathrm{diag}\,\{\xi_1^{-2}, \xi_2^{-2}, \ldots, \xi_N^{-2}\}, \tag{24}$$

$$\mathbf{X} = \mathbf{T}\Xi\mathbf{U}, \tag{25}$$

$$\mathbf{z} = \left[\frac{\sqrt{w_1\Psi(\xi_1)}}{\xi_1}, \frac{\sqrt{w_2\Psi(\xi_2)}}{\xi_2}, \ldots, \frac{\sqrt{w_N\Psi(\xi_N)}}{\xi_N}\right]^{\mathrm{T}} \tag{26}$$

and $\lambda = 1/\nu^2$. We note that the eigenvalue problem defined by Eq. (23) is of a form that is encountered when the so-called "divide and conquer" method [5] is used to find the eigenvalues of tridiagonal matrices.

Considering that we have found the required eigenvalues from Eq. (23), we impose the normalization condition

$$\sum_{k=1}^{N} w_k\Psi(\xi_k)[\phi(\nu, \xi_k) + \phi(\nu, -\xi_k)] = 1 \tag{27}$$

so that we can write our discrete-ordinates solution as

$$G(\tau, \pm\xi_i) = \sum_{j=1}^{N}\left[A_j\frac{\nu_j}{\nu_j \mp \xi_i}\mathrm{e}^{-\tau/\nu_j} + B_j\frac{\nu_j}{\nu_j \pm \xi_i}\mathrm{e}^{-(\tau_0-\tau)/\nu_j}\right] \tag{28}$$

where the arbitrary constants $\{A_j\}$ and $\{B_j\}$ are to be determined from the boundary conditions and the separation constants $\{\nu_j\}$ are the reciprocals of the positive square roots of the eigenvalues defined by Eq. (23).

To define the constants $\{A_j\}$ and $\{B_j\}$ we substitute Eq. (28) into Eqs. (10) evaluated at the quadrature points $\{\xi_i\}$ to find the system of linear algebraic equations

$$\Psi(\xi_i)\sum_{j=1}^{N}\left[A_j\frac{\nu_j}{\nu_j - \xi_i} + B_j\frac{\nu_j}{\nu_j + \xi_i}\mathrm{e}^{-\tau_0/\nu_j}\right] = G_1(\xi_i) \tag{29a}$$

and

$$\Psi(\xi_i)\sum_{j=1}^{N}\left[B_j\frac{\nu_j}{\nu_j - \xi_i} + A_j\frac{\nu_j}{\nu_j + \xi_i}\mathrm{e}^{-\tau_0/\nu_j}\right] = G_2(\xi_i) \tag{29b}$$

for $i = 1, 2, \ldots, N$. Once we have solved Eqs. (29) to find the required constants $\{A_j\}$ and $\{B_j\}$, we substitute Eq. (28) into the right-hand sides of Eqs. (13). We next replace ξ_i with ξ in the resulting equations to find

$$\xi\frac{\mathrm{d}}{\mathrm{d}\tau}G(\tau, \xi) + G(\tau, \xi) = \sum_{j=1}^{N}[A_j\mathrm{e}^{-\tau/\nu_j} + B_j\mathrm{e}^{-(\tau_0-\tau)/\nu_j}] \tag{30a}$$

and

$$-\xi\frac{\mathrm{d}}{\mathrm{d}\tau}G(\tau, -\xi) + G(\tau, -\xi) = \sum_{j=1}^{N}[A_j\mathrm{e}^{-\tau/\nu_j} + B_j\mathrm{e}^{-(\tau_0-\tau)/\nu_j}]. \tag{30b}$$

We can now solve Eqs. (30) to get our "post-processed" results, viz.

$$G(\tau,\xi)=\Psi^{-1}(\xi)G_1(\xi)\mathrm{e}^{-\tau/\xi}+\sum_{j=1}^{N}\nu_j[A_jC(\tau:\nu_j,\xi)+B_j\mathrm{e}^{-(\tau_0-\tau)/\nu_j}S(\tau:\nu_j,\xi)] \tag{31a}$$

and

$$G(\tau,-\xi)=\Psi^{-1}(\xi)G_2(\xi)\,\mathrm{e}^{-(\tau_0-\tau)/\xi}+\sum_{j=1}^{N}\nu_j[A_j\mathrm{e}^{-\tau/\nu_j}S(\tau_0-\tau:\nu_j,\xi)+B_jC(\tau_0-\tau:\nu_j,\xi)] \tag{31b}$$

for $\xi\in(0,\gamma)$. Here

$$S(\tau:x,y)=\frac{1-\mathrm{e}^{-\tau/x}\mathrm{e}^{-\tau/y}}{x+y} \tag{32a}$$

and

$$C(\tau:x,y)=\frac{\mathrm{e}^{-\tau/x}-\mathrm{e}^{-\tau/y}}{x-y}. \tag{32b}$$

Having developed our discrete-ordinates solution to a general problem, we are ready to report on some numerical aspects of our solution and to consider some specific problems.

4. Some numerical aspects of our discrete-ordinates solution

The first thing we must do is to define the quadrature scheme to be used in our discrete-ordinates solution. To be specific we first list the characteristic functions for the Doppler and the Lorentz line-scattering profiles. Quoting from Ref. [1], we note that the profile for the Doppler case is

$$\phi(x)=\frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-x^2} \tag{33}$$

and that the characteristic function is

$$\Psi(\xi)=\begin{cases}\Psi_0, & \xi\in[0,\gamma_0],\\ \Psi_0\mathrm{erfc}[\sqrt{2}m(\xi)], & \xi\in[\gamma_0,\gamma),\end{cases} \tag{34}$$

where erfc(z) is the complementary error function, and where

$$m(\xi)=\sqrt{\ln\left[\frac{\xi}{\sqrt{\pi}(1-\beta\xi)}\right]},\qquad \xi\in[\gamma_0,\gamma). \tag{35}$$

In addition

$$\Psi_0=\frac{\varpi}{4}\sqrt{\frac{2}{\pi}} \tag{36}$$

and

$$\gamma_0 = \frac{\sqrt{\pi}}{1 + \beta\sqrt{\pi}}. \tag{37}$$

Quoting again from Ref. [1], we note that the profile for the Lorentz case is

$$\phi(x) = \frac{1}{\pi(1 + x^2)} \tag{38}$$

and that the characteristic function can be written as

$$\Psi(\xi) = \begin{cases} \Psi_0, & \xi \in [0, \gamma_0], \\ \Psi_0 \dfrac{2}{\pi} \left\{ \cot^{-1}[m(\xi)] - \dfrac{m(\xi)}{1 + m^2(\xi)} \right\}, & \xi \in [\gamma_0, \gamma), \end{cases} \tag{39}$$

where

$$m(\xi) = \sqrt{\frac{\xi(1 + \beta\pi) - \pi}{\pi(1 - \beta\xi)}}, \quad \xi \in [\gamma_0, \gamma). \tag{40}$$

Here

$$\Psi_0 = \frac{\varpi}{4\pi} \tag{41}$$

and

$$\gamma_0 = \frac{\pi}{1 + \beta\pi}. \tag{42}$$

Now to be explicit we note that the quadrature scheme we employed is defined by using a Gauss–Legendre scheme on the interval [0, 1] after using a linear transformation to map the first part of the integration interval, viz. $[0, \gamma_0]$, onto [0, 1]. For the second part of the integration interval we used either the transformation [6]

$$u(z) = \frac{1}{1 + m(z)}, \tag{43}$$

with $u(\gamma) = 0$, for the Lorentz case or the transformation

$$u(z) = e^{-am(z)}, \tag{44}$$

with $u(\gamma) = 0$, for the Doppler case to map the interval $[\gamma_0, \gamma]$ onto [0, 1], and again we used a Gauss–Legendre scheme on the interval [0, 1]. In Eq. (44) we generally used $a = 1$. So if we put N_1 Gauss points in the interval $[0, \gamma_0]$ and N_2 points in the interval $[\gamma_0, \gamma]$ then we clearly have $N = N_1 + N_2$ quadrature points in the interval $[0, \gamma]$.

Having defined our quadrature scheme, we found the required separation constants $\{v_j\}$ by solving the eigenvalue problem given by Eq. (23). Initially, in computing these eigenvalues we did not use any specialized software that takes into account the special form of Eq. (23). We simply used the driver program RG from the EISPACK collection [7] to find the eigenvalues; however, since doing our first calculation, we have been able to use a recent work of Siewert and Wright [8] to improve this aspect of our solution.

Finally, but importantly, we note that since the characteristic function $\Psi(\xi)$ used in this work can, from a computational point-of-view, be zero, we can have some, say a total of N_0, of the separation constants $\{\nu_j\}$ equal to some of the quadrature points $\{\xi_j\}$. Of course, this is not allowed in Eqs. (28), and so, since the quadrature points where $\Psi(\xi)$ is effectively zero make no contribution to the right-hand side of Eqs. (13), we have simply omitted from our calculation these quadrature points. Omitting these N_0 quadrature points changes N to $N - N_0$ in our final solution.

5. First application of the discrete-ordinates solution

In Ref. [1] we used exact analysis (the H function) and the F_N method to compute $G(0, -\xi)$, and thus $I_x(0, -\xi)$, for $\xi \in (0, \gamma)$, for the case of a linearly varying Planck function $B(\tau)$ in a semi-infinite medium. Since $S_x(\tau)$ can be expressed as

$$S_x(\tau) = \gamma_x \lim_{\xi \to 0} I_x(\tau, -\xi), \tag{45}$$

we also reported in Ref. [1] a computation of the source function for the same linearly varying Planck function. In that work [1] numerical results were reported for both the Doppler and the Lorentz profiles of the line-scattering coefficient. Here we have used our discrete-ordinates solution to confirm all of the numerical results given in Ref. 1. We can say that the discrete-ordinates solution was easily implemented and that we obtained results in perfect agreement with those previous calculations.

6. A second application of the discrete-ordinates solution

In his classic work on radiative transfer, Ivanov [9] expresses the solution to the "albedo" problem for a finite layer and the solution to the Schuster problem in terms of the Chandrasekhar–Ambartzumian [10,11] X and Y functions generalized to the non-grey model we are considering in this work. We note that the solutions to both of these problems can, by way of the reduction given in Section 2 of this work, be expressed in terms of the solution to the G problem defined by the equation of transfer

$$\xi \frac{\partial}{\partial \tau} G(\tau, \xi : \xi_0) + G(\tau, \xi : \xi_0) = \int_{-\gamma}^{\gamma} \Psi(\xi') G(\tau, \xi' : \xi_0)\, d\xi' \tag{46}$$

for $\xi \in (-\gamma, \gamma)$ and $\tau \in (0, \tau_0)$ and the boundary conditions

$$\Psi(\xi) G(0, \xi : \xi_0) = \delta(\xi - \xi_0) \tag{47a}$$

and

$$\Psi(\xi) G(\tau_0, -\xi : \xi_0) = 0 \tag{47b}$$

for $\xi, \xi_0 \in (0, \gamma)$. Of course, we could convert this G problem with a delta "function" on the boundary to a problem for the diffuse field

$$G_*(\tau, \xi : \xi_0) = G(\tau, \xi : \xi_0) - \Psi^{-1}(\xi)\delta(\xi - \xi_0)\,\mathrm{e}^{-\tau/\xi}, \tag{48}$$

for $\xi \in (-\gamma, \gamma)$, and then use our discrete-ordinates solution, with the addition of a particular solution to the inhomogeneous equation, to compute the diffuse field $G_*(\tau, \xi : \xi_0)$. However, since Ivanov [9] has expressed the solutions to the two mentioned problems in terms of X and Y functions, and since we wish to use our discrete-ordinates solution to confirm the results [2] of our recently reported F_N computation of the X and Y functions, we proceed here to use our discrete-ordinates solution to compute the X and Y functions.

If we multiply Eqs. (46) and (47) by $\Psi(\xi_0)$, integrate over ξ_0 and define

$$G(\tau, \xi) = \int_0^{\gamma} \Psi(\xi_0) G(\tau, \xi : \xi_0)\,\mathrm{d}\xi_0 \tag{49}$$

then we find a new G problem, viz.

$$\xi \frac{\partial}{\partial \tau} G(\tau, \xi) + G(\tau, \xi) = \int_{-\gamma}^{\gamma} \Psi(\xi') G(\tau, \xi')\,\mathrm{d}\xi', \tag{50}$$

for $\xi \in (-\gamma, \gamma)$ and $\tau \in (0, \tau_0)$, with the boundary conditions

$$G(0, \xi) = 1 \tag{51a}$$

and

$$G(\tau_0, -\xi) = 0 \tag{51b}$$

for $\xi \in (0, \gamma)$. For this problem we find we can use the basic definitions of the X and Y functions [9,10] to express the "exiting intensities" as

$$G(0, -\xi) = \int_0^{\gamma} \Psi(\xi_0)[X(\xi)X(\xi_0) - Y(\xi)Y(\xi_0)]\,\frac{\xi_0}{\xi_0 + \xi}\,\mathrm{d}\xi_0 \tag{52a}$$

and

$$G(\tau_0, \xi) = \mathrm{e}^{-\tau_0/\xi} + \int_0^{\gamma} \Psi(\xi_0)[X(\xi)Y(\xi_0) - Y(\xi)X(\xi_0)]\,\frac{\xi_0}{\xi_0 - \xi}\,\mathrm{d}\xi_0 \tag{52b}$$

for $\xi \in (0, \gamma)$. Here the X and Y functions satisfy the non-linear equations [9,10]

$$X(z) = 1 + z\int_0^{\gamma} \Psi(z')[X(z)X(z') - Y(z)Y(z')]\,\frac{\mathrm{d}z'}{z' + z} \tag{53a}$$

and

$$Y(z) = \mathrm{e}^{-\tau_0/z} + z\int_0^{\gamma} \Psi(z')[X(z)Y(z') - Y(z)X(z')]\,\frac{\mathrm{d}z'}{z' - z} \tag{53b}$$

for $z \in [0, \gamma]$. We can use Eqs. (53) to rewrite Eqs. (52) as

$$G(0, -\xi) - 1 = (x_0 - 1)X(\xi) - y_0 Y(\xi) \tag{54a}$$

and

$$G(\tau_0, \xi) = y_0 X(\xi) + (1 - x_0) Y(\xi) \tag{54b}$$

for $\xi \in (0, \gamma)$. Here

$$x_0 = \int_0^\gamma \Psi(\xi) X(\xi)\,\mathrm{d}\xi \tag{55a}$$

and

$$y_0 = \int_0^\gamma \Psi(\xi) Y(\xi)\,\mathrm{d}\xi. \tag{55b}$$

Normally, of course, we would think of using Eqs. (54) to compute the exiting intensities $G(0, -\xi)$ and $G(\tau_0, \xi)$ for $\xi \in (0, \gamma)$; however, here we proceed differently. We consider that we have used our discrete-ordinates solution to compute $G(0, -\xi)$ and $G(\tau_0, \xi)$ for $\xi \in (0, \gamma)$, and so we can solve Eqs. (54) to find the X and Y functions. Multiplying Eqs. (54) by $\Psi(\xi)$, integrating over ξ and using [9,10]

$$(1 - x_0)^2 - y_0^2 = 1 - 2\int_0^\gamma \Psi(\xi)\,\mathrm{d}\xi, \tag{56}$$

we find

$$x_0 = \int_0^\gamma \Psi(\xi)[G(0, -\xi) + 1]\,\mathrm{d}\xi \tag{57a}$$

and

$$y_0 = \int_0^\gamma \Psi(\xi) G(\tau_0, \xi)\,\mathrm{d}\xi. \tag{57b}$$

Since we now know x_0 and y_0 we can solve Eqs. (54) to find

$$X(\xi) = \frac{1}{\Lambda(\infty)}\{(x_0 - 1)[G(0, -\xi) - 1] - y_0 G(\tau_0, \xi)\} \tag{58a}$$

and

$$Y(\xi) = \frac{1}{\Lambda(\infty)}\{(1 - x_0) G(\tau_0, \xi) + y_0[G(0, -\xi) - 1]\} \tag{58b}$$

where

$$\Lambda(\infty) = 1 - 2\int_0^\gamma \Psi(\xi)\,\mathrm{d}\xi. \tag{59}$$

Equations (58) are our final expressions for the X and Y functions we seek. Of course, for the special case of $\varpi = 1$ with $\beta = 0$, we cannot use Eqs. (58) since $\Lambda(\infty) = 0$ for this, the conservative, case.

In regard to computing the X and Y functions as discussed here, we note that we first implemented our discrete-ordinates solution and confirmed, to all five significant figures given, the numerical results reported in Ref. [2] for both the Doppler and the Lorentz profiles for the cases of $\tau_0 = 1$, 10 and 100 with $1 - \varpi = 10^{-6}$ and $\beta = 10^{-4}$. In regard to the results given in Ref. [2] for the nearly conservative case of $1 - \varpi = 10^{-11}$ and $\beta = 0$, we did have some numerical difficulties using Eqs. (55) because $\Lambda(\infty)$ is very nearly zero for this case. However, by using $1 - \varpi = 10^{-m}$, with $m = 6$, 7 and 8 we did confirm all of the significant figures reported in Ref. [2] for the nearly conservative case.

7. Concluding remarks

Having used exact analysis [1], the F_N method [1, 2] and now the discrete-ordinates approximation to solve several basic non-grey problems in semi-infinite and in finite media, we can say that we were able to obtain the same accuracy from all three of the techniques. Of course, having done already a lot of basic work on the considered non-grey model of scattering with complete frequency redistribution, we were well prepared by the time we considered the discrete-ordinates method. But, in the end of the day, we are of the opinion that, for the considered application, the discrete-ordinates method is easier to implement than the F_N method. Of course both the F_N method and the discrete-ordinates method, as we have implemented them, can be used with confidence for finite-media applications, while the exact analysis is considered limited to half-space problems.

Acknowledgements

The authors wish to thank R. D. M. Garcia, A. H. Karp, N. J. McCormick and B. Rutily for some helpful discussions concerning this (and other) work. Also, one of the authors (CES) would like to express his thanks to the Universidade Federal do Rio Grande do Sul (CPGMAp/PROMEC/PROPESQ) and to FAPERGS for their generous financial support during a recent visit to Porto Alegre and for the kind hospitality extended throughout the period during which most of this work was done. Finally, it is noted that the work of LBB was supported in part by CNPq of Brazil.

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