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ON SOLUTIONS OF AN EQUATION OF TRANSFER FOR A PLANETARY ATMOSPHERE

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ABSTRACT

A set of general solutions to the homogeneous vector equation of transfer for the two azimuth-independent components of a partially plane-polarized radiation field is obtained. In addition to describing the scattering of light by anisotropic particles, the model considered here is appropriate for the quantum theory of resonance line scattering. The analysis, which is based on Case's method of normal modes, yields two discrete eigenvectors and two linearly independent, degenerate, singular, continuum eigenvectors. A full-range completeness theorem for the aforementioned eigenvectors is proved by solving two coupled singular integral equations. Further, a full-range orthogonality theorem is proved, the necessary normalization integrals are determined, and the full-range adjoint vectors are constructed.

I. INTRODUCTION

In his classic book on radiative transfer (hereafter referred to as R.T.) Chandrasekhar (1950) developed the vector equation of transfer for the scattering of light by anisotropic particles (molecules). If $I(\tau,\mu)$ is a two-component vector whose elements $I_l(\tau,\mu)$ and $I_r(\tau,\mu)$ denote the intensities of the two states of polarization, then Chandrasekhar's equation of transfer for the azimuth-independent quantities takes the form (see R.T., § 74)

$$\mu\frac{\partial}{\partial\tau}I(\tau,\mu)+I(\tau,\mu)=\tfrac{1}{2}c_1\int_{-1}^{1}K(\mu,\mu')I(\tau,\mu')d\mu'+\tfrac{1}{2}c_2E\int_{-1}^{1}I(\tau,\mu')d\mu'\,,\tag{1}$$

where

$$K(\mu,\mu')=\tfrac{3}{4}\begin{vmatrix}2(1-\mu^2)(1-\mu'^2)+\mu'^2\mu^2 & \mu^2\\ \mu'^2 & 1\end{vmatrix}\,,\quad\text{and}\quad E=\tfrac{1}{2}\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}\,.\tag{2}$$

Here μ is the direction cosine (as measured from the *inward* normal to the free surface) of the directed radiation, and τ is the optical variable.

The scattering kernel for this model thus comprises two parts: the first, with weight factor c_1, is of the Rayleigh-scattering type, and the second, weighted by c_2, is isotropic. Also, since the medium considered is conservative, the two constants c_1 and c_2 must sum to unity. Chandrasekhar (1950) noted that, in addition to describing the scattering of light by anisotropic particles, equations (1) and (2) form an appropriate mathematical model for representing the quantum theory of resonance line scattering (see R.T., §§ 18 and 19).

Equation (1) in the Rayleigh-scattering limit ($c_2 = 0$, and hence $c_1 = 1$) has been investigated very thoroughly. This case also was formulated initially by Chandrasekhar (1946). In a subsequent paper (Chandrasekhar 1947) he obtained, by passing to an infinite limit in a discrete-ordinates procedure, the exact solutions for the laws of darkening in the Milne problem. More recently, Mullikin (1966) has made an exhaustive study of finite slab problems and has suggested new techniques for numerical calculations.

Taking a different tack, Siewert and Fraley (1967) found a set of general solutions to equation (1) for the case c_2 equal to zero. Their method was based on the singular eigenfunction expansion technique introduced by Case (1960) in a classic paper in neutron transport theory. In addition to finding the elementary solutions to equation (1) for the Rayleigh-scattering approximation, Siewert and Fraley (1967) proved the necessary half-range completeness and orthogonality theorems; they thus were able to solve any of the standard half-space problems. Smith and Siewert (1967) constructed the half-space Green's function, in order to illustrate the utility of the method. By utilizing both the Case method (Case and Zweifel 1967) and the S-matrix developed by Chandrasekhar (1950), Shieh and Siewert (1969) have obtained tractable results for finite slabs.

There has been a noticeable lack of results regarding the solution of equation (1) for arbitrary c_1. The difficulty, as was suggested by Chandrasekhar (see § 74 of R.T.), is derived from the fact that the characteristic function for this problem does not factorize. Although this does not increase prohibitively the difficulty of finding solutions to equation (1) or, for that matter, of proving that the resulting eigenvectors are complete on the full range, severe complications are encountered when half-space applications are considered.

In § II, the elementary solutions to equation (1) are derived; in § III, we prove that the resulting eigenvectors form a complete set for the expansion of arbitrary two-component vectors defined on the full range, $\mu \in (-1,1)$. Section IV is devoted to the proof of a full-range orthogonality theorem, to the evaluation of the necessary normalization integrals, and to the construction of the full-range adjoint vectors.

Since the medium considered is conservative, there is no *immediate* application of the full-range completeness and orthogonality theorems. However, the above theorems would be required for the construction of the pseudo–infinite-medium Green's function that is often used to develop the half-space Green's function (McCormick and Kuščer 1965; Siewert and Zweifel 1966; Smith and Siewert 1967). Clearly if an *exact* analytical solution for the half-space Green's function is desired, then additional half-range theorems, yet to be proved, would be required.

We believe the procedure here developed has merit because (*a*) it yields exact solutions to the homogeneous equation (1), (*b*) it permits the exact solution for the pseudo–infinite-medium Green's function, (*c*) it should serve as a basis for further analysis of half-space problems, and (*d*) if exact solutions for half-space problems remain elusive, then solutions that meet approximately the appropriate half-range boundary conditions could be constructed easily from our results.

II. EIGENVALUES AND EIGENVECTORS

Noting that equation (1) possesses translational invariance, we seek solutions of the form

$$I_\eta(\tau,\mu) = e^{-\tau/\eta}\mathbf{\Phi}(\eta,\mu) , \tag{3}$$

where the allowable eigenvalues η and the corresponding eigenvectors $\mathbf{\Phi}(\eta,\mu)$ are to be determined. Substitution of the above *Ansatz* into equation (1) yields

$$(\eta - \mu)\mathbf{\Phi}(\eta,\mu) = \frac{\eta}{2}\int_{-1}^{1}[c_1\mathbf{K}(\mu,\mu') + c_2\mathbf{E}]\mathbf{\Phi}(\eta,\mu')d\mu' . \tag{4}$$

Equation (4) can be reduced to a more tractable form by following the procedure used by Siewert and Fraley (1967) for the case c_2 equal to zero. Although the algebra involved in this reduction is tedious, it is straightforward; for the sake of brevity, we state

$$(\eta - \mu)\mathbf{\Phi}(\eta,\mu) = \frac{\eta}{2}[4 - 3c_1\eta^2]^{-1}[\mathbf{Q}_0(\eta) + P_2(\mu)\mathbf{Q}_1(\eta)]\int_{-1}^{1}\mathbf{\Phi}(\eta,\mu')d\mu' , \tag{5}$$

where $P_2(\mu)$ is the Legendre polynomial of order 2,

$$Q_0(\eta) = \begin{vmatrix} c_1(4 - 3\eta^2) + 2c_2 & 1 + c_2 \\ 2c_2 & 3c_1(1 - \eta^2) + 2c_2 \end{vmatrix} , \tag{6a}$$

and

$$Q_1(\eta) = \begin{vmatrix} -c_1(4 - 3\eta^2) & c_1(2 - 3\eta^2) \\ 0 & 0 \end{vmatrix} . \tag{6b}$$

We consider first the discrete spectrum, i.e., $\eta \notin (-1, 1)$. For this case, equation (5) yields

$$\Phi(\eta,\mu) = \frac{\eta}{2(\eta - \mu)} [4 - 3c_1\eta^2]^{-1}[Q_0(\eta) + P_2(\mu)Q_1(\eta)] \int_{-1}^{1} \Phi(\eta,\mu')d\mu' . \tag{7}$$

The factor $[4 - 3c_1\eta^2]^{-1}$ in equations (5) and (7) is to be noted, since for $\eta \notin (-1,1)$ a singularity appears to be present. As will be seen, however, $\eta = \pm\infty$ are the only discrete eigenvalues. If equation (7) is now integrated over μ from -1 to 1, there results

$$[4 - 3c_1\eta^2]^{-1} D(\eta) \int_{-1}^{1} \Phi(\eta,\mu')d\mu' = 0 ; \tag{8}$$

the discrete eigenvalues are thus the zeros of the dispersion function

$$\Lambda(z) \triangleq \det D(z) . \tag{9}$$

We prefer not to write explicitly the form of the matrix $D(z)$; it is found, however, that $\Lambda(z)$ can ultimately be written as

$$\Lambda(z) = c_1\Lambda_1(z)\Lambda_2(z) + 8c_2\Lambda_0(z) , \tag{10}$$

where

$$\Lambda_0(z) = 1 + \frac{z}{2} \int_{-1}^{1} \frac{d\mu}{\mu - z} \tag{11a}$$

and

$$\Lambda_\alpha(z) = (-1)^\alpha + 3(1 - z^2)\Lambda_0(z) , \qquad \alpha = 1 \text{ or } 2 . \tag{11b}$$

We note that equation (10) agrees perfectly with Chandrasekhar's result (i.e., the first of equations [246] from § 74 of R.T.) in the infinite limit of his discrete-ordinates procedure. In the Appendix we show, by utilizing the argument principle (Churchill 1960), that $\Lambda(z)$ has no zeros in the finite complex plane cut from -1 to 1 along the real axis. It follows immediately from inspection of equation (10) that $\Lambda(z)$ has a double zero at infinity. The discrete eigenvectors are thus degenerate; again, following the work of Siewert and Fraley (1967), we find

$$\Phi_+ \triangleq \Phi_+(\eta,\mu) = I_+(\tau,\mu) = \begin{vmatrix} 1 \\ 1 \end{vmatrix} \tag{12a}$$

and

$$I_-(\tau,\mu) = (\tau - \mu) \begin{vmatrix} 1 \\ 1 \end{vmatrix} , \tag{12b}$$

which can be easily shown to be solutions of equation (1). It should be noted that $I_+(\tau,\mu)$ satisfies equations (1) and (4), whereas $I_-(\tau,\mu)$ is a solution of equation (1) *only*.

In developing the dispersion relation given by equation (10), we have tacitly neglected the factor $[4 - 3c_1\eta^2]^{-1}$ appearing in equation (8). Since this factor itself has a double zero at infinity, one might suspect that there are other solutions for $\eta = \infty$, which correspond to higher-order representations of the translation group. As suggested by Case and Zweifel (1967), we propose additional solutions to equation (1) of the form

$$I(\tau,\mu) = \tau^n A_n(\mu) + \tau^{n-1} A_{n-1}(\mu) + \ldots + A_0(\mu) , \tag{13}$$

where the $A_n(\mu)$ are vectors to be determined. Substituting equation (13) into equation (1), we find that it can be a solution only for $n = 1$; this, in fact, leads to the two solutions given by equations (12). Thus we are justified in neglecting the factor $[4 - 3c_1\eta^2]^{-1}$ in the formulation of the dispersion function $\Lambda(z)$.

We proceed now to find the elementary solutions to equation (1) for $\eta \in (-1,1)$. These are the so-called continuum eigenvectors; they are, in fact, generalized functions. We write

$$\begin{aligned}\mathbf{\Phi}(\eta,\mu) = \frac{\eta}{2}\left[\frac{P}{\eta-\mu} + \lambda(\eta)\delta(\eta-\mu)\right][4 - 3c_1\eta^2]^{-1}[Q_0(\eta) \\ + P_2(\mu)Q_1(\eta)]\int_{-1}^{1}\mathbf{\Phi}(\eta,\mu')d\mu' ,\end{aligned} \tag{14}$$

where P is a mnemonic symbol used to denote the Cauchy principal-value function, and $\delta(x)$ is the Dirac delta-function. Again an integration over μ from -1 to 1 is used to reduce equation (14) to a homogeneous equation. This time, however, instead of obtaining the dispersion relation, we determine two linearly independent expressions for the arbitrary function $\lambda(\eta)$; these, in turn, yield two linearly independent solutions for $\mathbf{\Phi}(\eta,\mu)$, $\eta \in (-1,1)$. Thus we find a twofold degeneracy for the continuum eigenvectors.

The choice of the arbitrary normalization for the two continuum eigenvectors is extremely critical if a simple final form is to be obtained. We prefer to write (after many algebraic manipulations) these vectors as

$$\mathbf{\Phi}_1(\eta,\mu) = \left|\begin{matrix}\frac{3\eta}{2}(1-\mu^2)(1-\eta^2)\frac{P}{\eta-\mu} + [(1-\eta^2)\lambda_1(\eta) + C]\delta(\eta-\mu) \\ -C\delta(\eta-\mu)\end{matrix}\right| \tag{15a}$$

and

$$\mathbf{\Phi}_2(\eta,\mu) = \left|\begin{matrix}\frac{3\eta}{2}(1-\eta^2)\frac{P}{\eta-\mu} + \lambda_1(\eta)\delta(\eta-\mu) \\ \frac{3\eta}{2}(1-\eta^2)\frac{P}{\eta-\mu} + \lambda_2(\eta)\delta(\eta-\mu)\end{matrix}\right| . \tag{15b}$$

Here $C = 4c_2/3c_1$, and

$$\lambda_a(\eta) = (-1)^a + 3(1-\eta^2)[1 - \eta\tanh^{-1}(\eta)] , \quad a = 1 \text{ or } 2 . \tag{16}$$

The eigenvectors are determined; thus we write our general solution to equation (1) as

$$\begin{aligned}I(\tau,\mu) = A_+\mathbf{\Phi}_+ + A_-I_-(\tau,\mu) + \int_{-1}^{1} A_1(\eta)\mathbf{\Phi}_1(\eta,\mu)e^{-\tau/\eta}d\eta \\ + \int_{-1}^{1} A_2(\eta)\mathbf{\Phi}_2(\eta,\mu)e^{-\tau/\eta}d\eta ,\end{aligned} \tag{17}$$

where A_+, A_-, $A_1(\eta)$, and $A_2(\eta)$ are arbitrary expansion coefficients which are to be determined from the boundary conditions of a given physical problem. Before we turn to the next section where the full-range completeness theorem is proved, two observations should be made: first, the vector $\mathbf{\Phi}_1(\eta,\mu)$ should be multiplied by c_1 before the limiting case c_1 equal to zero is investigated; second, we note that three of the four eigenvectors we have found are independent of the parameters c_1 and c_2. This last fact can definitely be used to advantage when half-space problems are considered.

III. FULL-RANGE COMPLETENESS

THEOREM I: *The eigenvectors* $\mathbf{\Phi}_+$, $I_-(0,\mu)$, $\mathbf{\Phi}_1(\eta,\mu)$, *and* $\mathbf{\Phi}_2(\eta,\mu)$ *are complete on the full range,* $\mu \in (-1,1)$, *in the sense that an arbitrary two-component vector* $\mathbf{\Psi}(\mu)$ *defined for* $\mu \in (-1,1)$ *can be expanded in the form*

$$\mathbf{\Psi}(\mu) = A_+\mathbf{\Phi}_+ + A_-I_-(0,\mu) + \int_{-1}^{1} A_1(\eta)\mathbf{\Phi}_1(\eta,\mu)d\eta + \int_{-1}^{1} A_2(\eta)\mathbf{\Phi}_2(\eta,\mu)d\eta \quad (18)$$

To prove the theorem, we show that a solution to the above coupled singular integral equations exists. This is done by using the methods of Muskhelishvili (1953) to solve equation (18) for the unknown expansion coefficients. We begin by attempting to expand an arbitrary function $\mathbf{\Psi}'(\mu)$ in terms of the continuum modes alone. It is found that this expansion is valid only if two constraints are placed on $\mathbf{\Psi}'(\mu)$. These restrictions are later removed by introducing the discrete modes. We thus consider

$$\mathbf{\Psi}'(\mu) = \int_{-1}^{1} A_1(\eta)\mathbf{\Phi}_1(\eta,\mu)d\eta + \int_{-1}^{1} A_2(\eta)\mathbf{\Phi}_2(\eta,\mu)d\eta \,. \quad (19)$$

Let us introduce a vector $N(z)$, defined as

$$N(z) = \frac{1}{2\pi i}\int_{-1}^{1} \eta(1-\eta^2)A(\eta)\frac{d\eta}{\eta - z}\,, \quad (20)$$

where $A(\eta)$ is a vector whose two components are respectively $A_1(\eta)$ and $A_2(\eta)$. We note that $N(z)$ has the following properties: (*a*) it is analytic in the complex plane cut from -1 to 1; (*b*) it vanishes as $1/z$ as z tends to infinity. In addition, the boundary values of $N(z)$ as z approaches the branch cut from above (+) and below (−) are given by the Plemelj formulae (Muskhelishvili 1953):

$$N^{\pm}(\mu) = \frac{1}{2\pi i}\int_{-1}^{1} \eta(1-\eta^2)A(\eta)\frac{P}{\eta-\mu}d\eta \pm \tfrac{1}{2}\mu(1-\mu^2)A(\mu)\,. \quad (21)$$

It follows that

$$\pi i[N^+(\mu) + N^-(\mu)] = \int_{-1}^{1} \eta(1-\eta^2)A(\eta)\frac{P}{\eta-\mu}d\eta\,, \quad (22a)$$

and

$$N^+(\mu) - N^-(\mu) = \mu(1-\mu^2)A(\mu)\,. \quad (22b)$$

The functions $\Lambda_\alpha(z)$, $\alpha = 1$ or 2, also are analytic in the complex plane cut from -1 to 1 on the real axis. Further, their boundary values satisfy

$$\Lambda_\alpha^+(\mu) + \Lambda_\alpha^-(\mu) = 2\lambda_\alpha(\mu)\,, \qquad \alpha = 1 \text{ or } 2\,, \quad (23a)$$

and

$$\Lambda_\alpha^+(\mu) - \Lambda_\alpha^-(\mu) = 3\pi i\mu(1-\mu^2)\,, \qquad \alpha = 1 \text{ or } 2\,. \quad (23b)$$

If we now enter into equation (19) the explicit forms of $\boldsymbol{\Phi}_\alpha(\eta,\mu)$, $\alpha = 1$ and 2, and make use of equations (22) and (23), we find that the coupled singular integral equations can be cast in the form

$$\mu(1-\mu^2)\boldsymbol{\Psi}'(\mu) = \boldsymbol{\Lambda}^+(\mu)N^+(\mu) - \boldsymbol{\Lambda}^-(\mu)N^-(\mu) \,, \tag{24}$$

where $\boldsymbol{\Lambda}(z)$ is defined as

$$\boldsymbol{\Lambda}(z) \triangleq \begin{vmatrix} (1-z^2)\Lambda_1(z) + C & \Lambda_1(z) \\ -C & \Lambda_2(z) \end{vmatrix} \,. \tag{25}$$

The completeness proof thus reduces to solving the inhomogeneous Hilbert problem, given by equation (24). Following the method of Muskhelishvili (1953), we write the solution for $N(z)$ (with the understood proviso that the analytic properties of $N(z)$ are to be made consistent with the original definition given by equation [20]) as

$$N(z) = \frac{1}{2\pi i}\boldsymbol{\Lambda}^{-1}(z)\int_{-1}^{1}\mu(1-\mu^2)\boldsymbol{\Psi}'(\mu)\frac{d\mu}{\mu - z} \,. \tag{26}$$

We note here that

$$\det \boldsymbol{\Lambda}(z) = (1-z^2)\Lambda(z)/c_1 \,. \tag{27}$$

In order to investigate the behavior of $N(z)$ as z tends to infinity, we observe that, for large z,

$$\Lambda_0(z) \sim -1/3z^2 \,, \quad \Lambda_1(z) \sim -2/5z^2 \,, \quad \Lambda_2(z) \sim 2 \,, \tag{28a}$$

and thus

$$\Lambda(z) \sim -8(1 - \tfrac{7}{10}c_1)/3z^2 \,. \tag{28b}$$

Since c_1 is at most unity, $\Lambda(z)$ cannot vanish faster than $1/z^2$. We note finally that

$$\lim_{z\to\infty}\boldsymbol{\Lambda}^{-1}(z) = \frac{1}{4(1 - 7c_1/10)}\begin{vmatrix} 3c_1 & 0 \\ 2c_2 & 2(1 - 7c_1/10) \end{vmatrix} \,. \tag{29}$$

It follows now that $N(z)$, as given by equation (26), does in fact vanish as $1/z$ for large z. This then is consistent with the definition, equation (20).

Since we know that the dispersion function $\Lambda(z)$ has no zeros in the finite cut plane (see the Appendix), $\boldsymbol{\Lambda}^{-1}(z)$ is clearly regular everywhere in the cut plane. We observe, however, that $\boldsymbol{\Lambda}(z)$ is singular at the branch points $z = \pm 1$, i.e., $\det \boldsymbol{\Lambda}(\pm 1) = 0$, and thus special care must be taken to insure that $N(z)$ has the correct end-point behavior (Muskhelishvili 1953). In order to illustrate the end-point problem, we write equation (26) for the limits $z = \pm 1$:

$$\lim_{z\to\pm 1} N(z) \sim \{2\pi i(1-z^2)[-1 + 6C\Lambda_0(z)]\}^{-1}\begin{vmatrix} 1 & 1 \\ C & C \end{vmatrix}\int_{-1}^{1}\mu(1-\mu^2)\boldsymbol{\Psi}'(\mu)\frac{d\mu}{\mu - z} \,. \tag{30}$$

The points $z = \pm 1$ are, in the terminology of Muskhelishvili (1953), special end points; the behavior of $N(z)$ thus is not correct because of the singularities introduced by the factor $(1-z^2)^{-1}$ in equation (30). These singularities, however, are easily removed by imposing on $\boldsymbol{\Psi}'(\mu)$ the two constraints

$$\int_{-1}^{1}\mu(1-\mu^2)\tilde{\boldsymbol{\Phi}}_+\boldsymbol{\Psi}'(\mu)\frac{d\mu}{\mu \pm 1} = 0 \,, \tag{31a}$$

or, alternatively,

$$\int_{-1}^{1} \mu^{\alpha} \tilde{\mathbf{\Phi}}_{+} \mathbf{\Psi}'(\mu) d\mu = 0 \,, \qquad \alpha = 1 \text{ and } 2 \,. \quad (31b)$$

Here the superscript tilde denotes the transpose operation.

If we recall now that

$$\mathbf{\Psi}'(\mu) = \mathbf{\Psi}(\mu) - (A_{+} - \mu A_{-}) \mathbf{\Phi}_{+} \,, \quad (32)$$

where $\mathbf{\Psi}(\mu)$ is the arbitrary function we wish to expand, then the conditions given by equations (31) are clearly satisfied if

$$A_{+} = \tfrac{3}{4} \int_{-1}^{1} \mu^{2} \tilde{\mathbf{\Phi}}_{+} \mathbf{\Psi}(\mu) d\mu \quad (33a)$$

and

$$A_{-} = -\tfrac{3}{4} \int_{-1}^{1} \mu \tilde{\mathbf{\Phi}}_{+} \mathbf{\Psi}(\mu) d\mu \,. \quad (33b)$$

If we were to substitute equation (32) into equation (26) and use equations (33) for the discrete coefficients A_{+} and A_{-}, the resulting expression for $N(z)$ would have the correct analytic properties. Thus the completeness proof is established.

Although it is straightforward to obtain explicit expressions for $A_1(\eta)$ and $A_2(\eta)$ from equation (22b), we prefer to use the orthogonality relations developed in the next section.

IV. ORTHOGONALITY, NORMALIZATION INTEGRALS, AND ADJOINT FUNCTIONS

THEOREM II: *The eigenvectors* $\mathbf{\Phi}_{+}$, $\mathbf{\Phi}_{1}(\eta,\mu)$, *and* $\mathbf{\Phi}_{2}(\eta,\mu)$ *are orthogonal on the full range, with respect to weight function* μ, i.e.,

$$\int_{-1}^{1} \mu \tilde{\mathbf{\Phi}}_{i}(\eta,\mu) \mathbf{\Phi}_{j}(\eta',\mu) d\mu = 0 \,, \quad \eta \neq \eta' , \qquad i, j = +, 1, \text{ or } 2 \,. \quad (34)$$

To prove the theorem, equation (4) is multiplied by $\tilde{\mathbf{\Phi}}(\eta',\mu)$ from the left; equation (4) for $\eta = \eta'$ is then transposed and multiplied from the right by $\mathbf{\Phi}(\eta,\mu)$; the resulting two equations are integrated over μ from -1 to 1; finally, one is subtracted from the other to give

$$\left(\frac{1}{\eta} - \frac{1}{\eta'}\right) \int_{-1}^{1} \mu \tilde{\mathbf{\Phi}}(\eta',\mu) \mathbf{\Phi}(\eta,\mu) d\mu = 0 \,. \quad (35)$$

Here, we have used the fact that

$$c_1 \tilde{\mathbf{K}}(\mu,\mu') + c_2 \tilde{\mathbf{E}} = c_1 \mathbf{K}(\mu',\mu) + c_2 \mathbf{E} \,. \quad (36)$$

Although equation (35) proves the theorem, there remains a minor difficulty associated with the degeneracy of the continuum eigenvectors. Since $\mathbf{\Phi}_1(\eta,\mu)$ and $\mathbf{\Phi}_2(\eta,\mu)$ have the same eigenvalue spectrum, the scalar product of these two vectors does not necessarily vanish; in fact,

$$\int_{-1}^{1} \tilde{\mathbf{\Phi}}_{i}(\eta',\mu) \mathbf{\Phi}_{j}(\eta,\mu) \mu d\mu = M_{ij}(\eta) \delta(\eta - \eta') \,. \quad (37)$$

Here

$$M_{12}(\eta) = M_{21}(\eta) = -2C + (1 - \eta^2) \Lambda_1^{+}(\eta) \Lambda_1^{-}(\eta) \,, \quad (38a)$$

$$M_{11}(\eta) = (1 - \eta^2)^2\Lambda_1^+(\eta)\Lambda_1^-(\eta) + 2C(1 - \eta^2)\lambda_1(\eta) + 2C^2 , \tag{38b}$$

and

$$M_{22}(\eta) = \Lambda_1^+(\eta)\Lambda_1^-(\eta) + \Lambda_2^+(\eta)\Lambda_2^-(\eta) . \tag{38c}$$

Also, since $I_-(0,\mu)$ is not a solution to equation (4), it is not included initially in the orthogonal set; the adjoint for $I_-(0,\mu)$ is, however, easily determined.

We choose to follow the same Schmidt-type procedure used by Siewert and Zweifel (1966) to construct adjoint vectors for the continuum modes. If we define our scalar product as

$$\langle i|j\rangle \overset{\Delta}{=} \int_{-1}^{1} \tilde{\Phi}_i^\dagger(\eta,\mu)\Phi_j(\eta',\mu)d\mu , \quad i,j = +, 1, \text{ or } 2 , \tag{39}$$

we find

$$\langle i|j\rangle = 0 , \qquad i \neq j , \tag{40a}$$

$$\langle 1|1\rangle = \langle 2|2\rangle = M(\eta)\delta(\eta - \eta') , \tag{40b}$$

and

$$\langle +|+\rangle \overset{\Delta}{=} M_+ = \tfrac{4}{3} ; \tag{40c}$$

here

$$c_1^2M(\eta) = \eta(1 - \eta^2)\Lambda^+(\eta)\Lambda^-(\eta) . \tag{41}$$

In addition, the adjoint vectors are given by

$$\Phi_+^\dagger = \mu^2\Phi_+ , \tag{42a}$$

$$\Phi_1^\dagger(\eta,\mu) = \mu[M_{22}(\eta)\Phi_1(\eta,\mu) - M_{12}(\eta)\Phi_2(\eta,\mu)] , \tag{42b}$$

and

$$\Phi_2^\dagger(\eta,\mu) = \mu[M_{11}(\eta)\Phi_2(\eta,\mu) - M_{12}(\eta)\Phi_1(\eta,\mu)] . \tag{42c}$$

Finally, we note that

$$\langle -|j\rangle \overset{\Delta}{=} \int_{-1}^{1} \tilde{I}_-^\dagger(0,\mu)\Phi_j(\eta,\mu)d\mu = 0 , \qquad j = +, 1, \text{ or } 2 , \tag{43a}$$

and

$$\langle -|-\rangle \overset{\Delta}{=} \int_{-1}^{1} \tilde{I}_-^\dagger(0,\mu)I_-(0,\mu)d\mu \overset{\Delta}{=} M_- = -\tfrac{4}{3} , \tag{43b}$$

where

$$I_-^\dagger(0,\mu) = \mu\Phi_+ . \tag{44}$$

In order to illustrate the above results, we consider again the expansion given by equation (18). Using the orthogonality theorem, the adjoint vectors, and the various normalization integrals, we take full-range scalar products to obtain immediately the results for the expansion coefficients:

$$A_+M_+ = \int_{-1}^{1} \tilde{\Phi}_+^\dagger\Psi(\mu)d\mu , \tag{45a}$$

$$A_-M_- = \int_{-1}^{1} \tilde{I}_-^\dagger(0,\mu)\Psi(\mu)d\mu , \tag{45b}$$

and

$$A_i(\eta)M(\eta) = \int_{-1}^{1} \tilde{\Phi}_i^\dagger(\eta,\mu)\Psi(\mu)d\mu\ , \qquad i = 1 \text{ or } 2\ . \quad (45c)$$

These results are identical with those which follow from the completeness proof given in § III.

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APPENDIX

THE DISCRETE EIGENVALUES

The dispersion function is given by

$$\Lambda(z) = c_1\Lambda_1(z)\Lambda_2(z) + 8c_2\Lambda_0(z)\ , \quad (A1)$$

where the functions $\Lambda_a(z)$, $a = 0$, 1, and 2, are defined explicitly in equations (11) We note that $\Lambda(z)$ is an analytic function in the complex plane cut from -1 to 1 on the real axis. It thus follows from the argument principle (Churchill 1960) that the number of zeros of $\Lambda(z)$ in the finite cut plane is $1/2\pi$ times the change in the argument of the function along a contour enclosing the cut plane.

Since $\Lambda(z) \sim 1/z^2$ for large z, -4π is the change in the argument of the function as a contour which tends to infinity is traversed. We need to consider, in addition, a contour C that encloses the branch cut -1 to 1.

Noting that $\Lambda(-z) = \Lambda(z)$ and $[\Lambda^+(u)]^* = \Lambda^-(\mu)$, we conclude that the argument change of $\Lambda(z)$ on C is 4 times the change as z goes from $0 + i\epsilon$ to $1 + i\epsilon$. We denote this change by $\Delta_+(0,1)$. It follows from equation (A1) that

$$\Lambda^+(\mu) = c_1\left\{-1 + 9(1-\mu^2)^2\left[\lambda_0^2(\mu) - \frac{\pi^2}{4}\mu^2\right]\right\} + 8c_2\lambda_0(\mu) + i\pi\mu[9c_1\lambda_0(\mu)(1-\mu^2)^2 + 4c_2]\ , \quad (A2)$$

where

$$\lambda_0(\mu) = 1 - \mu \tanh^{-1}\mu\ . \quad (A3)$$

It is a simple matter to show, by utilizing equation (A2), that $\Delta_+(0,1) = \pi$. Consequently, the argument change around the branch cut is 4π.

We note finally that the total change in the argument of $\Lambda(z)$ is zero as the two enclosing contours are traversed. It follows that $\Lambda(z)$ has no zeros in the finite cut plane.

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