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A DISCRETE-ORDINATES SOLUTION FOR A POLARIZATION MODEL WITH COMPLETE FREQUENCY REDISTRIBUTION

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ABSTRACT

The discrete-ordinates method is used to develop a solution to a class of polarization problems in the theory of radiative transfer. The I and Q components of the Stokes vector are used to describe the polarized radiation field, and the model considered allows a mixture of Rayleigh and isotropic scattering with complete frequency redistribution (completely noncoherent scattering) and continuum absorption. In addition to a general formulation for a finite plane-parallel medium, specific computations for both the Doppler and the Lorentz profiles of the line-scattering coefficient are reported for a general problem with internal emission and radiation incident on one surface of the layer.

Subject headings: polarization — radiative transfer

1. INTRODUCTION

In three recent papers (Barichello & Siewert 1998a, 1998b, 1999), we used exact analysis (the H function), the F_N method, and the discrete-ordinates method to develop and evaluate solutions to some basic problems specific to a class of nongray problems in radiative transfer that are based on the equation of transfer written, following Hummer (1968), as

$$\mu \frac{\partial}{\partial \tau} I_x(\tau, \mu) + [\phi(x) + \beta] I_x(\tau, \mu) = [\phi(x) + \beta] S_x(\tau) , \tag{1}$$

where $S_x(\tau)$ is the source function,

$$[\phi(x) + \beta] S_x(\tau) = \frac{1}{2} \varpi \phi(x) \int_{-\infty}^{\infty} \phi(x') \int_{-1}^{1} I_{x'}(\tau, \mu') d\mu' \, dx' + [\rho\beta + (1 - \varpi)\phi(x)] B(\tau) , \tag{2}$$

and $B(\tau)$ is the Planck function evaluated at the center of the line. We note here that x is the normalized frequency variable measured (in dimensionless units) from the line center, $\tau \in [0, \tau_0]$ is the optical depth, τ_0 is the optical thickness of the plane-parallel medium, and $\mu \in [-1, 1]$ is the cosine of the polar angle (as measured from the positive τ-axis) that describes the direction of propagation of the radiation. In addition, $\varpi \in [0, 1)$ is the albedo for single scattering, $\beta \geq 0$ is the ratio of the continuum absorption coefficient to the average line coefficient, ρ is the ratio of the continuum source function to the Planck function, and $\phi(x)$ is the line-scattering profile.

In this work, we consider a more general model that includes some polarization effects, so we start with a generalization ($\beta \geq 0$) of the equation of transfer used by Faurobert (1987) in an early work on this subject, viz.,

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}_x(\tau, \mu) + [\phi(x) + \beta] \boldsymbol{I}_x(\tau, \mu) = \boldsymbol{S}_x(\tau, \mu) , \tag{3}$$

where

$$\boldsymbol{S}_x(\tau, \mu) = \tfrac{1}{2} \varpi \phi(x) \int_{-\infty}^{\infty} \phi(x') \int_{-1}^{1} \boldsymbol{P}(\mu, \mu') \boldsymbol{I}_{x'}(\tau, \mu') d\mu' \, dx' + [\rho\beta + (1 - \varpi)\phi(x)] \boldsymbol{B}(\tau) . \tag{4}$$

Here the vector $\boldsymbol{I}_x(\tau, \mu)$ has the two Stokes parameters $I_x(\tau, \mu)$ and $Q_x(\tau, \mu)$ as components, $\boldsymbol{P}(\mu, \mu')$ is the phase matrix, and the vector $\boldsymbol{B}(\tau)$ defines a thermal creation term. Following Faurobert-Scholl, Frisch, & Nagendra (1997) and Ivanov, Grachev, & Loskutov (1997), we make use here of the phase matrix introduced by Chandrasekhar (1950) to describe the combined effects of isotropic and Rayleigh scattering, and so we write

$$\boldsymbol{P}(\mu, \mu') = \begin{bmatrix} 1 + (c/8)(1 - 3\mu^2)(1 - 3\mu'^2) & 3(c/8)(1 - \mu'^2)(1 - 3\mu^2) \\ 3(c/8)(1 - \mu^2)(1 - 3\mu'^2) & 9(c/8)(1 - \mu^2)(1 - \mu'^2) \end{bmatrix} , \tag{5}$$

where $c \in [0, 1]$ is a measure of the Rayleigh component of the scattering law: $c = 0$ yields just isotropic scattering, while $c = 1$ yields Rayleigh scattering (Chandrasekhar 1950). The scattering matrix can be factored in various ways; here we choose to use the factorization attributed to Rachkovsky by Ivanov et al. (1997), and so we write

$$\boldsymbol{P}(\mu, \mu') = \boldsymbol{A}(\mu) \boldsymbol{A}^T(\mu') \tag{6}$$

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where the superscript T denotes the transpose operation, and where

$$\boldsymbol{A}(\mu) = \begin{bmatrix} 1 & (c/8)^{1/2}(1-3\mu^2) \\ 0 & (c/8)^{1/2}3(1-\mu^2) \end{bmatrix} . \tag{7}$$

We now consider it our job to solve equation (3) for a given $\boldsymbol{B}(\tau)$, subject to the boundary conditions

$$\boldsymbol{I}_x(0, \mu) = \boldsymbol{I}_{x,1}(\mu) \qquad \text{and} \qquad \boldsymbol{I}_x(\tau_0, -\mu) = \boldsymbol{I}_{x,2}(\mu) \tag{8}$$

for $\mu \in (0, 1]$. Here the vectors $\boldsymbol{I}_{x,1}(\mu)$ and $\boldsymbol{I}_{x,2}(\mu)$ that describe any radiation incident on the layer are assumed to be specified.

2. A REDUCTION TO A SIMPLER PROBLEM

In McCormick & Siewert (1970) some transformations were used for the case of a semi-infinite medium with no radiation incident on the surface that made it possible to construct the solution to equation (1) from the solution to what is considered to be a much simpler problem. In Barichello & Siewert (1999), transformations similar to those used by McCormick & Siewert (1970) were introduced in order to solve a general problem for a finite medium. We are now ready to extend those solutions to the case of polarization considered here. First, a change of the angular variable by $\xi = \mu\gamma_x$, with

$$\gamma_x = [\phi(x) + \beta]^{-1} , \tag{9}$$

allows us, after the changes in notation,

$$\boldsymbol{I}_x(\tau, \xi/\gamma_x) \rightarrow \gamma_x \boldsymbol{I}_x(\tau, \xi) , \qquad \boldsymbol{I}_{x,1}(\xi/\gamma_x) \rightarrow \gamma_x \boldsymbol{I}_{x,1}(\xi) , \qquad \text{and} \qquad \boldsymbol{I}_{x,2}(\xi/\gamma_x) \rightarrow \gamma_x \boldsymbol{I}_{x,2}(\xi) ,$$

to rewrite equations (3) and (8) as

$$\xi \frac{\partial}{\partial \tau} \boldsymbol{I}_x(\tau, \xi) + \boldsymbol{I}_x(\tau, \xi) = \frac{1}{2} \varpi \phi(x) \boldsymbol{A}(\xi/\gamma_x) \int_{-\gamma}^{\gamma} \int_{M_{\xi'}} \phi(x') \boldsymbol{A}^{\mathrm{T}}(\xi'/\gamma_{x'}) \boldsymbol{I}_{x'}(\tau, \xi') dx' \, d\xi' + \boldsymbol{Q}_x(\tau) \tag{10}$$

for $\xi \in (-\gamma, \gamma)$ and $\tau \in (0, \tau_0)$ and

$$\boldsymbol{I}_x(0, \xi) = \boldsymbol{I}_{x,1}(\xi) \qquad \text{and} \qquad \boldsymbol{I}_x(\tau_0, -\xi) = \boldsymbol{I}_{x,2}(\xi) \tag{11}$$

for $\xi \in (0, \gamma)$. Here

$$\boldsymbol{Q}_x(\tau) = [\rho\beta + (1 - \varpi)\phi(x)] \boldsymbol{B}(\tau) \tag{12}$$

is an inhomogeneous term (considered known), and $\gamma = \sup \gamma_x$. We consider profiles $\phi(x)$ that vanish at infinity, so $\gamma = 1/\beta$. In addition, the set M_ξ is defined such that $x \in M_\xi$ if and only if $[\phi(x) + \beta] |\xi| \leq 1$. Following Barichello & Siewert (1999), we now write

$$\boldsymbol{I}_x(\tau, \xi) = \boldsymbol{I}_x^p(\tau, \xi) + \phi(x)\boldsymbol{A}(\xi/\gamma_x)\boldsymbol{G}(\tau, \xi) - [\boldsymbol{I}_x^p(0, \xi) - \boldsymbol{I}_{x,1}(\xi) + \phi(x)\boldsymbol{A}(\xi/\gamma_x)\boldsymbol{G}(0, \xi)] e^{-\tau/\xi} \tag{13a}$$

and

$$\boldsymbol{I}_x(\tau, -\xi) = \boldsymbol{I}_x^p(\tau, -\xi) + \phi(x)\boldsymbol{A}(\xi/\gamma_x)\boldsymbol{G}(\tau, -\xi) - [\boldsymbol{I}_x^p(\tau_0, -\xi) - \boldsymbol{I}_{x,2}(\xi) + \phi(x)\boldsymbol{A}(\xi/\gamma_x)\boldsymbol{G}(\tau_0, -\xi)] e^{-(\tau_0 - \tau)/\xi} \tag{13b}$$

for $\xi \in (0, \gamma)$. Here $\boldsymbol{I}_x^p(\tau, \xi)$ is a particular solution of equation (10) corresponding to the specified inhomogeneous term $\boldsymbol{Q}_x(\tau)$, and the vector $\boldsymbol{G}(\tau, \xi)$ is to be defined. Substituting equations (13a) and (13b) into equation (10) and using the conditions given in equations (11), we find that equations (13a) and (13b) are correct if $\boldsymbol{G}(\tau, \xi)$ satisfies

$$\xi \frac{\partial}{\partial \tau} \boldsymbol{G}(\tau, \xi) + \boldsymbol{G}(\tau, \xi) = \int_{-\gamma}^{\gamma} \boldsymbol{\Psi}(\xi') \boldsymbol{G}(\tau, \xi') d\xi' , \tag{14}$$

for $\xi \in (-\gamma, \gamma)$, $\tau \in (0, \tau_0)$, and the boundary conditions

$$\boldsymbol{\Psi}(\xi)\boldsymbol{G}(0, \xi) = \boldsymbol{G}_1(\xi) \qquad \text{and} \qquad \boldsymbol{\Psi}(\xi)\boldsymbol{G}(\tau_0, -\xi) = \boldsymbol{G}_2(\xi) \tag{15}$$

for $\xi \in (0, \gamma)$. Here

$$\boldsymbol{\Psi}(\xi) = \frac{\varpi}{2} \int_{M_\xi} \phi^2(x) \boldsymbol{A}^{\mathrm{T}}(\xi/\gamma_x) \boldsymbol{A}(\xi/\gamma_x) dx , \tag{16}$$

$$\boldsymbol{G}_1(\xi) = \frac{\varpi}{2} \int_{M_\xi} \phi(x) \boldsymbol{A}^{\mathrm{T}}(\xi/\gamma_x) [\boldsymbol{I}_{x,1}(\xi) - \boldsymbol{I}_x^p(0, \xi)] dx , \tag{17a}$$

and

$$\boldsymbol{G}_2(\xi) = \frac{\varpi}{2} \int_{M_\xi} \phi(x) \boldsymbol{A}^{\mathrm{T}}(\xi/\gamma_x) [\boldsymbol{I}_{x,2}(\xi) - \boldsymbol{I}_x^p(\tau_0, -\xi)] dx . \tag{17b}$$

In McCormick & Siewert (1970) particular solutions of equation (1) were reported for several "simple" inhomogeneous terms, and the infinite-medium Green's function was used to develop a general procedure for constructing a particular solution. We consider for the moment that the particular solution we require for the polarization problem defined in this work is available, and so we proceed to develop our discrete-ordinates solution to the "$\boldsymbol{G}$ problem" defined by equations (14) and (15).

3. A DISCRETE-ORDINATES SOLUTION

We note first that the characteristic matrix $\boldsymbol{\Psi}(\xi)$, as defined by equation (16), is symmetric. We also note that $\boldsymbol{\Psi}(\xi) = \boldsymbol{\Psi}(-\xi)$, and so we write our discrete-ordinates equations as

$$\xi_i \frac{d}{d\tau} \boldsymbol{G}(\tau, \xi_i) + \boldsymbol{G}(\tau, \xi_i) = \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\xi_k)[\boldsymbol{G}(\tau, \xi_k) + \boldsymbol{G}(\tau, -\xi_k)] \tag{18a}$$

and

$$-\xi_i \frac{d}{d\tau} \boldsymbol{G}(\tau, -\xi_i) + \boldsymbol{G}(\tau, -\xi_i) = \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\xi_k)[\boldsymbol{G}(\tau, \xi_k) + \boldsymbol{G}(\tau, -\xi_k)] \tag{18b}$$

for $i = 1, 2, \ldots, N$. In writing equations (18a) and (18b) as we have, we are clearly considering that the N quadrature points ξ_k and the N weights w_k are defined for use on the integration interval $[0, \gamma]$. Now, seeking exponential solutions of equations (18a) and (18b) we substitute

$$\boldsymbol{G}(\tau, \pm\xi_i) = \boldsymbol{\Phi}(\nu, \pm\xi_i)e^{-\tau/\nu} \tag{19}$$

into equations (18a) and (18b) to find

$$(\nu - \xi_i)\boldsymbol{\Phi}(\nu, \xi_i) = \nu \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\xi_k)[\boldsymbol{\Phi}(\nu, \xi_k) + \boldsymbol{\Phi}(\nu, -\xi_k)] \tag{20a}$$

and

$$(\nu + \xi_i)\boldsymbol{\Phi}(\nu, -\xi_i) = \nu \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\xi_k)[\boldsymbol{\Phi}(\nu, \xi_k) + \boldsymbol{\Phi}(\nu, -\xi_k)] \tag{20b}$$

for $i = 1, 2, \ldots, N$. If we now let $\phi_1(\nu, \pm\xi_i)$ and $\phi_2(\nu, \pm\xi_i)$ denote the two components of $\boldsymbol{\Phi}(\nu, \pm\xi_i)$, and if we use

$$\boldsymbol{\Phi}_{1\pm} = [\phi_1(\nu, \pm\xi_1), \phi_1(\nu, \pm\xi_2), \ldots, \phi_1(\nu, \pm\xi_N)]^{\mathrm{T}} \tag{21a}$$

and

$$\boldsymbol{\Phi}_{2\pm} = [\phi_2(\nu, \pm\xi_1), \phi_2(\nu, \pm\xi_2), \ldots, \phi_2(\nu, \pm\xi_N)]^{\mathrm{T}} , \tag{21b}$$

then we can rewrite equations (20a) and (20b) as

$$\frac{1}{\nu} \boldsymbol{\Xi}\boldsymbol{\Phi}_+ = (\boldsymbol{I} - \boldsymbol{W})\boldsymbol{\Phi}_+ - \boldsymbol{W}\boldsymbol{\Phi}_- \quad \text{and} \quad -\frac{1}{\nu} \boldsymbol{\Xi}\boldsymbol{\Phi}_- = (\boldsymbol{I} - \boldsymbol{W})\boldsymbol{\Phi}_- - \boldsymbol{W}\boldsymbol{\Phi}_+ . \tag{22}$$

Here $\boldsymbol{I}$ is the $2N \times 2N$ identity matrix, the two vector elements of $\boldsymbol{\Phi}_\pm$ are $\boldsymbol{\Phi}_{1\pm}$ and $\boldsymbol{\Phi}_{2\pm}$, the four $N \times N$ matrix elements of $\boldsymbol{W}$, viz., $\boldsymbol{W}_{m,n}$, for $m, n = 1, 2$, are given by

$$(\boldsymbol{W}_{m,n})_{i,j} = w_j \Psi_{m,n}(\xi_j) , \tag{23}$$

where $\Psi_{m,n}(\xi)$, for $m, n = 1, 2$, are the elements of $\boldsymbol{\Psi}(\xi)$, and

$$\boldsymbol{\Xi} = \mathrm{diag}\,\{\xi_1, \xi_2, \ldots, \xi_N, \xi_1, \xi_2, \ldots, \xi_N\} . \tag{24}$$

Continuing to follow Barichello & Siewert (1999), we now let

$$\boldsymbol{U} = \boldsymbol{\Phi}_+ + \boldsymbol{\Phi}_- \tag{25a}$$

and

$$\boldsymbol{V} = \boldsymbol{\Phi}_+ - \boldsymbol{\Phi}_- , \tag{25b}$$

so that we can eliminate $\boldsymbol{V}$ between the sum and the difference of equations (22) to find

$$(\boldsymbol{D} - 2\boldsymbol{\Xi}^{-1}\boldsymbol{W}\boldsymbol{\Xi}^{-1})\boldsymbol{\Xi}\boldsymbol{U} = \lambda\boldsymbol{\Xi}\boldsymbol{U} , \tag{26}$$

where $\lambda = 1/\nu^2$ and

$$\boldsymbol{D} = \mathrm{diag}\,\{\xi_1^{-2}, \xi_2^{-2}, \ldots, \xi_N^{-2}, \xi_1^{-2}, \xi_2^{-2}, \ldots, \xi_N^{-2}\} . \tag{27}$$

Assuming that we have found the required separation constants $(\pm\nu_j)$ from the eigenvalues defined by equation (26), we go back to equations (20a) and (20b) to find $\boldsymbol{\Phi}(\nu_j, \pm\xi_i)$, and so we write our general solution to equations (18a) and (18b) as

$$\boldsymbol{G}(\tau, \pm\xi_i) = \sum_{j=1}^{2N} \left[A_j \frac{\nu_j}{\nu_j \mp \xi_i} e^{-\tau/\nu_j} + B_j \frac{\nu_j}{\nu_j \pm \xi_i} e^{-(\tau_0 - \tau)/\nu_j} \right] \boldsymbol{F}(\nu_j) . \tag{28}$$

Here $\boldsymbol{F}(\nu_j)$ is a vector in the null space of

$$\boldsymbol{\Omega}(\nu_j) = \boldsymbol{I} - 2\nu_j^2 \sum_{\alpha=1}^{N} w_\alpha \boldsymbol{\Psi}(\xi_\alpha) \frac{1}{\nu_j^2 - \xi_\alpha^2}, \tag{29}$$

where $\boldsymbol{I}$ is now the 2×2 identity matrix, and the arbitrary constants A_j and B_j are to be determined from the boundary conditions. Of course, we cannot allow $\nu_j = \xi_i$ in equation (28). To find the constants A_j and B_j, we substitute equation (28) into equations (15), evaluated at the quadrature points ξ_i, to obtain the system of linear algebraic equations

$$\boldsymbol{\Psi}(\xi_i) \sum_{j=1}^{2N} \left(A_j \frac{\nu_j}{\nu_j - \xi_i} + B_j \frac{\nu_j}{\nu_j + \xi_i} e^{-\tau_0/\nu_j} \right) \boldsymbol{F}(\nu_j) = \boldsymbol{G}_1(\xi_i) \tag{30a}$$

and

$$\boldsymbol{\Psi}(\xi_i) \sum_{j=1}^{2N} \left(B_j \frac{\nu_j}{\nu_j - \xi_i} + A_j \frac{\nu_j}{\nu_j + \xi_i} e^{-\tau_0/\nu_j} \right) \boldsymbol{F}(\nu_j) = \boldsymbol{G}_2(\xi_i) \tag{30b}$$

for $i = 1, 2, \ldots, N$. Once we have solved equations (30a) and (30b) to find the required constants A_j and B_j, we substitute equation (28) into the right-hand sides of equations (18a) and (18b). We next replace ξ_i with ξ in the resulting equations to find

$$\xi \frac{d}{d\tau} \boldsymbol{G}(\tau, \xi) + \boldsymbol{G}(\tau, \xi) = \sum_{j=1}^{2N} [A_j e^{-\tau/\nu_j} + B_j e^{-(\tau_0 - \tau)/\nu_j}] \boldsymbol{F}(\nu_j) \tag{31a}$$

and

$$-\xi \frac{d}{d\tau} \boldsymbol{G}(\tau, -\xi) + \boldsymbol{G}(\tau, -\xi) = \sum_{j=1}^{2N} [A_j e^{-\tau/\nu_j} + B_j e^{-(\tau_0 - \tau)/\nu_j}] \boldsymbol{F}(\nu_j) . \tag{31b}$$

We can now solve equations (31a) and (31b) to obtain our "postprocessed" results, viz.,

$$\boldsymbol{G}(\tau, \xi) = \boldsymbol{\Psi}^{-1}(\xi) \boldsymbol{G}_1(\xi) e^{-\tau/\xi} + \sum_{j=1}^{2N} \nu_j [A_j C(\tau : \nu_j, \xi) + B_j e^{-(\tau_0 - \tau)/\nu_j} S(\tau : \nu_j, \xi)] \boldsymbol{F}(\nu_j) \tag{32a}$$

and

$$\boldsymbol{G}(\tau, -\xi) = \boldsymbol{\Psi}^{-1}(\xi) \boldsymbol{G}_2(\xi) e^{-(\tau_0 - \tau)/\xi} + \sum_{j=1}^{2N} \nu_j [A_j e^{-\tau/\nu_j} S(\tau_0 - \tau : \nu_j, \xi) + B_j C(\tau_0 - \tau : \nu_j, \xi)] \boldsymbol{F}(\nu_j) \tag{32b}$$

for $\xi \in (0, \gamma)$. Here

$$S(\tau : x, y) = \frac{1 - e^{-\tau/x} e^{-\tau/y}}{x + y} \quad \text{and} \quad C(\tau : x, y) = \frac{e^{-\tau/x} - e^{-\tau/y}}{x - y} . \tag{33}$$

Having developed our discrete-ordinates solution to a general problem, we consider several particular problems and report on some numerical aspects of our solution.

4. SOME BASIC ELEMENTS FOR THE DOPPLER AND LORENTZ PROFILES

As can be seen from the previous section, the characteristic matrix $\boldsymbol{\Psi}(\xi)$ defined by equation (16) is the first quantity we must develop if we are to evaluate our discrete-ordinates solution for the two applications considered here, viz., the cases of the Doppler and Lorentz line-scattering profiles. So, when using equation (7) in equation (16), we can write

$$\boldsymbol{\Psi}(\xi) = a_0(\xi) \boldsymbol{\Psi}_0 + a_2(\xi) \boldsymbol{\Psi}_2 + a_4(\xi) \boldsymbol{\Psi}_4 , \tag{34}$$

where, in general,

$$a_n(\xi) = \frac{\varpi}{2} \xi^n \int_{M_\xi} \phi^2(x) [\phi(x) + \beta]^n \, dx , \tag{35}$$

and where

$$\boldsymbol{\Psi}_0 = \begin{bmatrix} 1 & (c/8)^{1/2} \\ (c/8)^{1/2} & 5c/4 \end{bmatrix}, \qquad \boldsymbol{\Psi}_2 = \begin{bmatrix} 0 & -3(c/8)^{1/2} \\ -3(c/8)^{1/2} & -3c \end{bmatrix},$$

and

$$\boldsymbol{\Psi}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 9c/4 \end{bmatrix} . \tag{36}$$

Continuing, we let

$$M_n(\xi) = \frac{1}{2} \int_{M_\xi} \phi^n(x) dx \tag{37}$$

for $n = 1, 2, \ldots$, so that we can write

$$a_0(\xi) = \varpi M_2(\xi) , \tag{38a}$$

$$a_2(\xi) = \varpi \xi^2 [M_4(\xi) + 2\beta M_3(\xi) + \beta^2 M_2(\xi)] , \tag{38b}$$

and

$$a_4(\xi) = \varpi \xi^4 [M_6(\xi) + 4\beta M_5(\xi) + 6\beta^2 M_4(\xi) + 4\beta^3 M_3(\xi) + \beta^4 M_2(\xi)] . \tag{38c}$$

At this point we must consider the two cases separately. First, for the Doppler case where

$$\phi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} , \tag{39}$$

we find that we can evaluate the integrals defined by equation (37) to obtain

$$M_n(\xi) = \begin{cases} \frac{1}{2}(n\pi^{n-1})^{-1/2} , & \xi \in [0, \gamma_0] , \\ \frac{1}{2}(n\pi^{n-1})^{-1/2} \operatorname{erfc} [\sqrt{n} m(\xi)] , & \xi \in [\gamma_0, \gamma) , \end{cases} \tag{40}$$

for $n = 1, 2, \ldots$. Here erfc (z) is the complementary error function,

$$\gamma_0 = \frac{\sqrt{\pi}}{1 + \beta\sqrt{\pi}} , \tag{41}$$

and

$$m(\xi) = \sqrt{\ln\left[\frac{\xi}{\sqrt{\pi}(1 - \beta\xi)}\right]} , \qquad \xi \in [\gamma_0, \gamma) . \tag{42}$$

In a similar way, we can evaluate the integrals given by equation (37) for the Lorentz case where

$$\phi(x) = \frac{1}{\pi(1 + x^2)} \tag{43}$$

to find for $n = 1, 2, \ldots$

$$M_n(\xi) = \begin{cases} l_n , & \xi \in [0, \gamma_0] , \\ (2/\pi) l_n \{\cot^{-1} [m(\xi)] - L_n[m(\xi)]\} , & \xi \in [\gamma_0, \gamma) , \end{cases} \tag{44}$$

where

$$\gamma_0 = \frac{\pi}{1 + \beta\pi} \tag{45}$$

and

$$m(\xi) = \sqrt{\frac{\xi(1 + \beta\pi) - \pi}{\pi(1 - \beta\xi)}} , \qquad \xi \in [\gamma_0, \gamma) . \tag{46}$$

Here

$$l_1 = \frac{1}{2} , \qquad l_2 = \frac{1}{4\pi} , \qquad l_3 = \frac{3}{16\pi^2} \qquad l_4 = \frac{5}{32\pi^3} , \qquad l_5 = \frac{35}{256\pi^4} , \qquad \text{and} \qquad l_6 = \frac{63}{512\pi^5} . \tag{47}$$

To complete equation (44), we list

$$L_1(x) = 0 , \qquad L_2(x) = \frac{x}{1 + x^2} , \qquad L_3(x) = \frac{x(3x^2 + 5)}{3(1 + x^2)^2} , \qquad L_4(x) = \frac{x(15x^4 + 40x^2 + 33)}{15(1 + x^2)^3} ,$$

$$L_5(x) = \frac{x(105x^6 + 385x^4 + 511x^2 + 279)}{105(1 + x^2)^4} , \qquad \text{and} \qquad L_6(x) = \frac{x(315x^8 + 1470x^6 + 2688x^4 + 2370x^2 + 965)}{315(1 + x^2)^5} . \tag{48}$$

Having completed our explicit evaluation of the characteristic matrix $\mathbf{\Psi}(\xi)$, we list here some additional results that we will use in the next two sections, where we develop and evaluate our discrete-ordinates solution. First, we write the dispersion matrix for the considered polarization problem as

$$\boldsymbol{\Lambda}(\xi) = \boldsymbol{I} + \xi \int_{-\gamma}^{\gamma} \boldsymbol{\Psi}(z) \frac{dz}{z - \xi} , \tag{49}$$

where $\boldsymbol{I}$ is the 2×2 identity matrix, and we find, for $\xi \to \infty$,

$$\boldsymbol{\Lambda}(\infty) = \boldsymbol{I} - \int_{-\gamma}^{\gamma} \boldsymbol{\Psi}(z)dz \tag{50a}$$

or

$$\boldsymbol{\Lambda}(\infty) = \boldsymbol{I} - \varpi \int_{-\infty}^{\infty} \phi^2(x)\gamma_x\, dx \int_0^1 \boldsymbol{A}^{\mathrm{T}}(\mu)\boldsymbol{A}(\mu)d\mu . \tag{50b}$$

We also list here our results for the quantities

$$\boldsymbol{K}_1(\beta) = \int_{-\infty}^{\infty} \phi(x)\gamma_x\, dx \int_0^1 \boldsymbol{A}^{\mathrm{T}}(\mu)d\mu \tag{51a}$$

and

$$\boldsymbol{K}_2(\beta) = \int_{-\infty}^{\infty} \phi^2(x)\gamma_x\, dx \int_0^1 \boldsymbol{A}^{\mathrm{T}}(\mu)d\mu \tag{51b}$$

that will be used in the expression for a particular solution we develop in the next section of this work. We define

$$k_n(\beta) = \int_{-\infty}^{\infty} \phi^n(x)\gamma_x\, dx \tag{52}$$

so that we can express equations (50b), (51a), and (51b) as

$$\boldsymbol{\Lambda}(\infty) = \boldsymbol{I} - \varpi k_2(\beta)\boldsymbol{M} , \tag{53}$$

$$\boldsymbol{K}_1(\beta) = k_1(\beta)\boldsymbol{N} , \qquad \text{and} \qquad \boldsymbol{K}_2(\beta) = k_2(\beta)\boldsymbol{N} , \tag{54}$$

where

$$\boldsymbol{M} = \int_0^1 \boldsymbol{A}^{\mathrm{T}}(\mu)\boldsymbol{A}(\mu)d\mu \qquad \text{and} \qquad \boldsymbol{N} = \int_0^1 \boldsymbol{A}^{T}(\mu)d\mu . \tag{55}$$

We note that

$$\boldsymbol{M} = \begin{bmatrix} 1 & 0 \\ 0 & 7c/10 \end{bmatrix} \qquad \text{and} \qquad \boldsymbol{N} = \begin{bmatrix} 1 & 0 \\ 0 & (c/2)^{1/2} \end{bmatrix} . \tag{56}$$

To complete this part of our work, we can evaluate the integrals defined by equation (52) for $n = 1, 2$. For the case of the Doppler line-scattering coefficient, we find

$$\beta k_1(\beta) = I_1(\beta\sqrt{\pi}) \qquad \text{and} \qquad k_2(\beta) = I_2(\beta\sqrt{\pi}) \tag{57}$$

where

$$I_1(z) = \frac{2z}{\sqrt{\pi}} \int_0^{\infty} \frac{\exp(-x^2)}{\exp(-x^2) + z}\, dx \tag{58a}$$

and

$$I_2(z) = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{\exp(-2x^2)}{\exp(-x^2) + z}\, dx . \tag{58b}$$

We note that $I_1(z) + I_2(z) = 1$. For the Lorentz case, we can write

$$\beta k_1(\beta) = f \qquad \text{and} \qquad k_2(\beta) = 1 - f \tag{59}$$

where

$$f = \sqrt{\frac{\beta\pi}{1 + \beta\pi}} . \tag{60}$$

5. SOME APPLICATIONS OF THE DEVELOPED SOLUTION

Having discussed our solution to the given class of radiative-transfer problems, we are ready to solve some specific problems and report some numerical results. Because we want the problems solved here to include the problems discussed in

Faurobert-Scholl et al. (1997) and Ivanov et al. (1997), we consider a finite layer with a monoenergetic beam incident on one surface, and we take the Planck function $\boldsymbol{B}(\tau)$ to be the constant $\boldsymbol{B}$, with components B_I and B_Q. We therefore start with the equation of transfer

$$\mu \frac{\partial}{\partial \tau} \boldsymbol{I}_x(\tau, \mu) + [\phi(x) + \beta]\boldsymbol{I}_x(\tau, \mu) = \frac{1}{2} \varpi\phi(x) \int_{-\infty}^{\infty} \phi(x') \int_{-1}^{1} \boldsymbol{P}(\mu, \mu')\boldsymbol{I}_{x'}(\tau, \mu')d\mu' \, dx' + \boldsymbol{Q}_x \tag{61}$$

for $\mu \in [-1, 1]$ and $\tau \in (0, \tau_0)$. Here

$$\boldsymbol{Q}_x = [\rho\beta + (1 - \varpi)\phi(x)]\boldsymbol{B} , \tag{62}$$

and the boundary conditions are

$$\boldsymbol{I}_x(0, \mu) = \delta(x - x_0)[1 - \Delta + \Delta\delta(\mu - \mu_0)]\boldsymbol{F} \tag{63a}$$

and

$$\boldsymbol{I}_x(\tau_0, -\mu) = \boldsymbol{0} \tag{63b}$$

for $\mu \in (0, 1]$. We note that (x_0, μ_0) defines the energy and direction of an incident beam and that the constant $\boldsymbol{F}$ has components F_I and F_Q. In addition, we use $\Delta \in [0, 1]$ in equation (63a) in order to be able to model a mixture of isotropic incident radiation and a monodirectional beam.

As discussed in § 2, we can use the ξ variable to rewrite the problem defined by equations (61), (62), (63a), and (63b) as the transfer equation

$$\xi \frac{\partial}{\partial \tau} \boldsymbol{I}_x(\tau, \xi) + \boldsymbol{I}_x(\tau, \xi) = \frac{1}{2} \varpi\phi(x)\boldsymbol{A}(\xi/\gamma_x) \int_{-\gamma}^{\gamma} \int_{M_{\xi'}} \phi(x')\boldsymbol{A}^{\mathrm{T}}(\xi'/\gamma_{x'})\boldsymbol{I}_{x'}(\tau, \xi')dx' \, d\xi' + \boldsymbol{Q}_x , \tag{64}$$

for $\xi \in (-\gamma, \gamma)$ and $\tau \in (0, \tau_0)$, and the boundary conditions

$$\boldsymbol{I}_x(0, \xi) = \delta(x - x_0)\{(1 - \Delta)[\phi(x_0) + \beta] + \Delta\delta(\xi - \xi_0)\}\boldsymbol{F} \tag{65a}$$

and

$$\boldsymbol{I}_x(\tau_0, -\xi) = \boldsymbol{0} \tag{65b}$$

for $\xi \in (0, \gamma)$. Here $\xi_0 = \mu_0 \gamma_{x0}$.

Considering now the inhomogeneous term given in equation (64), we find that we can use the particular solution

$$\boldsymbol{I}_x^p(\tau, \xi) = \{\rho\beta\boldsymbol{I} + [\boldsymbol{A}(\xi/\gamma_x)\boldsymbol{C} + (1 - \varpi)\boldsymbol{\Gamma}]\phi(x)\}\boldsymbol{B} , \tag{66}$$

where

$$\boldsymbol{C} = \varpi\boldsymbol{\Lambda}^{-1}(\infty)[\rho\beta\boldsymbol{K}_1(\beta) + (1 - \varpi)\boldsymbol{K}_2(\beta)] , \tag{67}$$

and where $\boldsymbol{\Lambda}(\infty)$, $\boldsymbol{K}_1(\beta)$, and $\boldsymbol{K}_2(\beta)$ are conveniently expressed by equations (53) and (54).

Looking back to equations (13a) and (13b), we see that in order to have the desired final result, we must also solve the $\boldsymbol{G}$ problem defined by equations (14) and (15), which we rewrite here as

$$\xi \frac{\partial}{\partial \tau} \boldsymbol{G}(\tau, \xi) + \boldsymbol{G}(\tau, \xi) = \int_{-\gamma}^{\gamma} \boldsymbol{\Psi}(\xi')\boldsymbol{G}(\tau, \xi')d\xi' \tag{68}$$

for $\xi \in (-\gamma, \gamma)$ and $\tau \in (0, \tau_0)$ with, for the considered application,

$$\boldsymbol{\Psi}(\xi)\boldsymbol{G}(0, \xi) = \frac{\varpi}{2} \phi(x_0)\Theta(x_0, \xi)\{(1 - \Delta)[\phi(x_0) + \beta] + \Delta\delta(\xi - \xi_0)\}\boldsymbol{A}^T(\xi/\gamma_{x0})\boldsymbol{F} - \boldsymbol{T}(\xi)\boldsymbol{B} \tag{69a}$$

and

$$\boldsymbol{\Psi}(\xi)\boldsymbol{G}(\tau_0, -\xi) = -\boldsymbol{T}(\xi)\boldsymbol{B} \tag{69b}$$

for $\xi \in (0, \gamma)$. Here we have introduced

$$\Theta(x_0, \xi) = \begin{cases} 0 , & \xi > \gamma_{x0} , \\ 1 , & \text{otherwise} , \end{cases} \tag{70}$$

where, to be specific, we write

$$\gamma_{x0} = \frac{\sqrt{\pi}}{e^{-x_0^2} + \beta\sqrt{\pi}} \tag{71a}$$

for the Doppler case and

$$\gamma_{x0} = \frac{\pi(1 + x_0^2)}{1 + \beta\pi(1 + x_0^2)} \tag{71b}$$

for the Lorentz case. In writing equations (69a) and (69b), we have also introduced

$$\boldsymbol{T}(\xi) = \frac{\varpi}{2}\int_{M_\xi} \phi(x)\boldsymbol{A}^{\mathrm{T}}(\xi/\gamma_x)\{\rho\beta\boldsymbol{I} + [\boldsymbol{A}(\xi/\gamma_x)\boldsymbol{C} + (1-\varpi)\boldsymbol{I}]\phi(x)\}dx\,, \tag{72}$$

which can be rewritten in terms of quantities defined by equations (40) and (44), viz.,

$$\boldsymbol{T}(\xi) = \boldsymbol{\Psi}(\xi)\boldsymbol{C} + \varpi T_0(\xi)\boldsymbol{A}_0^{\mathrm{T}} + \varpi\xi^2 T_2(\xi)\boldsymbol{A}_2^{\mathrm{T}}\,, \tag{73}$$

where

$$T_0(\xi) = \rho\beta M_1(\xi) + (1-\varpi)M_2(\xi) \tag{74a}$$

and

$$T_2(\xi) = \rho\beta[M_3(\xi) + 2\beta M_2(\xi) + \beta^2 M_1(\xi)] + (1-\varpi)[M_4(\xi) + 2\beta M_3(\xi) + \beta^2 M_2(\xi)]\,, \tag{74b}$$

and where

$$\boldsymbol{A}_0^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ (c/8)^{1/2} & 3(c/8)^{1/2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}_2^{\mathrm{T}} = -3(c/8)^{1/2}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}. \tag{75}$$

Here the elements of the diagonal $\boldsymbol{C}$ matrix are

$$C_{11} = \frac{\varpi}{\Lambda_{11}(\infty)}\,[\rho\beta k_1(\beta) + (1-\varpi)k_2(\beta)] \tag{76a}$$

and

$$C_{22} = \frac{\varpi}{\Lambda_{22}(\infty)}\,(c/2)^{1/2}\,[\rho\beta k_1(\beta) + (1-\varpi)k_2(\beta)]\,, \tag{76b}$$

where $k_1(\beta)$ and $k_2(\beta)$ are defined by equations (57)–(60), and where

$$\Lambda_{11}(\infty) = 1 - \varpi k_2(\beta) \tag{77a}$$

and

$$\Lambda_{22}(\infty) = 1 - \frac{7}{10}\,\varpi c k_2(\beta)\,. \tag{77b}$$

It is clear that because of the presence of the generalized function $\delta(\xi - \xi_0)$ in equation (69a), we cannot use our discrete-ordinates solution to solve the $\boldsymbol{G}$ problem as defined by equations (68), (69a), and (69b), so we substitute

$$\boldsymbol{G}(\tau, \xi) = \boldsymbol{G}_*(\tau, \xi) + \frac{\varpi}{2}\,\phi(x_0)\Delta\delta(\xi - \xi_0)\boldsymbol{\Psi}^{-1}(\xi_0)\boldsymbol{A}^{\mathrm{T}}(\mu_0)\boldsymbol{F}e^{-\tau/\xi_0} \tag{78}$$

into equations (68), (69a), and (69b) to obtain a problem that is free of generalized functions, viz.,

$$\xi\,\frac{\partial}{\partial\tau}\,\boldsymbol{G}_*(\tau, \xi) + \boldsymbol{G}_*(\tau, \xi) = \int_{-\gamma}^{\gamma} \boldsymbol{\Psi}(\xi')\boldsymbol{G}_*(\tau, \xi')d\xi' + \boldsymbol{S}(\tau) \tag{79}$$

for $\xi \in (-\gamma, \gamma)$ and $\tau \in (0, \tau_0)$, with the boundary conditions

$$\boldsymbol{\Psi}(\xi)\boldsymbol{G}_*(0, \xi) = \frac{\varpi}{2}\,\phi(x_0)\Theta(x_0, \xi)(1-\Delta)[\phi(x_0) + \beta]\boldsymbol{A}^{\mathrm{T}}(\xi/\gamma_{x0})\boldsymbol{F} - \boldsymbol{T}(\xi)\boldsymbol{B} \tag{80a}$$

and

$$\boldsymbol{\Psi}(\xi)\boldsymbol{G}_*(\tau_0, -\xi) = -\boldsymbol{T}(\xi)\boldsymbol{B} \tag{80b}$$

for $\xi \in (0, \gamma)$. Here the inhomogeneous term in equation (79) is given by

$$\boldsymbol{S}(\tau) = \boldsymbol{F}_0\,e^{-\tau/\xi_0} \qquad \text{where} \qquad \boldsymbol{F}_0 = \frac{\varpi}{2}\,\phi(x_0)\Delta\boldsymbol{A}^{\mathrm{T}}(\mu_0)\boldsymbol{F}\,. \tag{81}$$

In § 3 of this work, we developed our discrete-ordinates solution to a homogeneous version of equation (79) that we can, of course, use. However, we also require here a particular solution to account for the inhomogeneous term $\boldsymbol{S}(\tau)$ in equation (79). We follow a procedure similar to the one used in § 3 to find

$$\boldsymbol{G}_*^p(\tau, \pm\xi_i) = \frac{\xi_0}{\xi_0 \mp \xi_i}\,\boldsymbol{\Omega}^{-1}(\xi_0)\boldsymbol{F}_0\,e^{-\tau/\xi_0}\,, \tag{82}$$

where

$$\boldsymbol{\Omega}(z) = \boldsymbol{I} - 2z^2 \sum_{\alpha=1}^{N} w_\alpha \boldsymbol{\Psi}(\xi_\alpha) \frac{1}{z^2 - \xi_\alpha^2} . \tag{83}$$

While the particular solution given by equation (82) can be used in many calculations, it cannot be considered a general result, since in that solution we cannot allow ξ_0 either to be one of the quadrature points ξ_i or, since $\boldsymbol{\Omega}(\xi_0)$ would be singular, to be one of the eigenvalues v_j. We note that a particular solution that does not suffer from either of these restrictions can be found by extending to the vector model considered here the results of the recent work of Barichello, Garcia, & Siewert (1999) on particular solutions for the discrete-ordinates method. For the moment, at least, we choose to use the simple form given in equation (82), especially since if one or both of these restrictions were violated for a given value of N, they would not be violated for other, equally valid, values of N.

Having found a particular solution, we can now express the desired solution to the $\boldsymbol{G}_*$ problem as

$$\boldsymbol{G}_*(\tau, \pm\xi_i) = \sum_{j=1}^{2N} \left[A_j \frac{v_j}{v_j \mp \xi_i} e^{-\tau/v_j} + B_j \frac{v_j}{v_j \pm \xi_i} e^{-(\tau_0 - \tau)/v_j} \right] \boldsymbol{F}(v_j) + \boldsymbol{G}_*^p(\tau, \pm\xi_i) , \tag{84}$$

where the constants A_j and B_j are to be found from the system of linear algebraic equations obtained when equation (84) is substituted into equations (80a) and (80b) evaluated at the quadrature points, viz.,

$$\boldsymbol{\Psi}(\xi_i) \sum_{j=1}^{2N} \left(A_j \frac{v_j}{v_j - \xi_i} + B_j \frac{v_j}{v_j + \xi_i} e^{-\tau_0/v_j} \right) \boldsymbol{F}(v_j) = \boldsymbol{F}_1(\xi_i) \tag{85a}$$

and

$$\boldsymbol{\Psi}(\xi_i) \sum_{j=1}^{2N} \left(B_j \frac{v_j}{v_j - \xi_i} + A_j \frac{v_j}{v_j + \xi_i} e^{-\tau_0/v_j} \right) \boldsymbol{F}(v_j) = \boldsymbol{F}_2(\xi_i) \tag{85b}$$

for $i = 1, 2, \ldots, N$. Here

$$\boldsymbol{F}_1(\xi_i) = \frac{\varpi}{2} \phi(x_0)\Theta(x_0, \xi_i)(1 - \Delta)[\phi(x_0) + \beta]\boldsymbol{A}^{\mathrm{T}}(\xi_i/\gamma_{x_0})\boldsymbol{F} - \boldsymbol{T}(\xi_i)\boldsymbol{B} - \boldsymbol{\Psi}(\xi_i)\boldsymbol{G}_*^p(0, \xi_i) \tag{86a}$$

and

$$\boldsymbol{F}_2(\xi_i) = -\boldsymbol{T}(\xi_i)\boldsymbol{B} - \boldsymbol{\Psi}(\xi_i)\boldsymbol{G}_*^p(\tau_0, -\xi_i) . \tag{86b}$$

Once we have solved equations (85a) and (85b) to find the constants A_j and B_j, we can construct our postprocessed result. The development of the postprocessed result here is very similar to the procedure used in § 3 to obtain equations (32a) and (32b). However, since equation (79) has the inhomogeneous term $\boldsymbol{S}(\tau)$, the computation that yielded equations (32a) and (32b) requires a minor modification to account for the addition of the particular solution we have used. We omit some details here and simply list our final result for the $\boldsymbol{G}_*$ problem as

$$\boldsymbol{G}_*(\tau, \xi) = \boldsymbol{G}_*(0, \xi)e^{-\tau/\xi} + \boldsymbol{\Xi}(\tau, \xi) + \boldsymbol{\Upsilon}(\tau, \xi) \tag{87a}$$

and

$$\boldsymbol{G}_*(\tau, -\xi) = \boldsymbol{G}_*(\tau_0, -\xi)e^{-(\tau_0 - \tau)/\xi} + \boldsymbol{\Xi}(\tau, -\xi) + \boldsymbol{\Upsilon}(\tau, -\xi) \tag{87b}$$

for $\xi \in (0, \gamma)$. Here $\boldsymbol{G}_*(0, \xi)$ and $\boldsymbol{G}_*(\tau_0, -\xi)$ are defined by equations (80a) and (80b),

$$\boldsymbol{\Xi}(\tau, \xi) = \sum_{j=1}^{2N} v_j[A_j C(\tau : v_j, \xi) + B_j e^{-(\tau_0 - \tau)/v_j} S(\tau : v_j, \xi)]\boldsymbol{F}(v_j) , \tag{88a}$$

$$\boldsymbol{\Xi}(\tau, -\xi) = \sum_{j=1}^{2N} v_j[A_j e^{-\tau/v_j} S(\tau_0 - \tau : v_j, \xi) + B_j C(\tau_0 - \tau : v_j, \xi)]\boldsymbol{F}(v_j) , \tag{88b}$$

$$\boldsymbol{\Upsilon}(\tau, \xi) = \xi_0 C(\tau : \xi_0, \xi)\boldsymbol{\Omega}^{-1}(\xi_0)\boldsymbol{F}_0 \tag{89a}$$

and

$$\boldsymbol{\Upsilon}(\tau, -\xi) = \xi_0 e^{-\tau/\xi_0} S(\tau_0 - \tau : \xi_0, \xi)\boldsymbol{\Omega}^{-1}(\xi_0)\boldsymbol{F}_0 , \tag{89b}$$

where the S and C functions are given by equations (33).

It is clear that the solution we have developed here includes the solutions of two special cases of the general problem defined by equations (61), (62), (63a), and (63b). That is, if we consider $\boldsymbol{F} = \boldsymbol{0}$ and $\boldsymbol{B} \neq \boldsymbol{0}$, then we have the case of a radiation field generated only by the internal source, viz., the Planck function. On the other hand, if we consider $\boldsymbol{B} = \boldsymbol{0}$ and $\boldsymbol{F} \neq \boldsymbol{0}$, then we have an arbitrary mixture of the classical albedo problem and the case of isotropic incident radiation.

Clearly, our job here is finished once we find the constants A_j and B_j; however, before proceeding to some numerical work, we state our final solution to the problem defined by equations (61), (62), (63a), and (63b). We use equations (65a), (65b), (66), (78), (80a), (80b), (87a), and (87b) in equations (13a) and (13b) to find

$$\boldsymbol{I}_x(\tau, \mu) = \delta(x - x_0)[1 - \Delta + \Delta\delta(\mu - \mu_0)]\boldsymbol{F}e^{-\tau[\phi(x) + \beta]/\mu} + \gamma_x \boldsymbol{B}_x(\mu)\{1 - e^{-\tau[\phi(x) + \beta]/\mu}\} + \gamma_x \phi(x)\boldsymbol{A}(\mu)\boldsymbol{\Gamma}(\tau, \mu\gamma_x) \tag{90a}$$

and

$$\boldsymbol{I}_x(\tau, -\mu) = \gamma_x \boldsymbol{B}_x(\mu)\{1 - e^{-(\tau_0 - \tau)[\phi(x) + \beta]/\mu}\} + \gamma_x \phi(x)\boldsymbol{A}(\mu)\boldsymbol{\Gamma}(\tau, -\mu\gamma_x) \tag{90b}$$

for $\tau \in [0, \tau_0]$, $\mu \in (0, 1]$, and for all x. In writing equations (90a) and (90b), we have used

$$\boldsymbol{B}_x(\mu) = \{\rho\beta \boldsymbol{I} + [\boldsymbol{A}(\mu)\boldsymbol{C} + (1 - \varpi)\boldsymbol{I}]\phi(x)\}\boldsymbol{B} , \tag{91}$$

along with

$$\boldsymbol{\Gamma}(\tau, \mu\gamma_x) = \boldsymbol{\Xi}(\tau, \mu\gamma_x) + \boldsymbol{\Upsilon}(\tau, \mu\gamma_x) , \tag{92}$$

where $\boldsymbol{\Xi}$ and $\boldsymbol{Y}$ are given by equations (88a) and (88b) and (89a) and (89b).

Now we are ready to use our discrete-ordinates solution to compute the only quantities that are not already known explicitly, viz., $\boldsymbol{\Gamma}(\tau, \xi)$ and $\boldsymbol{\Gamma}(\tau, -\xi)$ for $\xi \in (0, \gamma)$.

6. SOME NUMERICAL ASPECTS OF OUR DISCRETE-ORDINATES SOLUTION

In regard to our numerical work, we note that it closely follows our work in Barichello & Siewert (1999), but for completeness we repeat here some of the discussion given there. First, we must define the quadrature scheme to be used in our discrete-ordinates solution.

Noting that the characteristic matrix $\boldsymbol{\Psi}(\xi)$ is defined differently in each of two subintervals $[0, \gamma_0]$ and $[\gamma_0, \gamma]$, we use a Gauss-Legendre scheme on the interval [0, 1], after using a linear transformation to map the first part of the integration interval, viz., $[0, \gamma_0]$, onto [0, 1]. For the second part of the integration interval, we used either the transformation (B. Rutily 1997, private communication)

$$u(z) = \frac{1}{1 + m(z)} , \tag{93}$$

with $u(\gamma) = 0$, for the Lorentz case, or the transformation

$$u(z) = e^{-am(z)} , \tag{94}$$

with $u(\gamma) = 0$, for the Doppler case, to map the interval $[\gamma_0, \gamma]$ onto [0, 1]; we then again used a Gauss-Legendre scheme on the interval [0, 1]. In equation (94), we generally used $a = 1$. If we put N_1 Gauss points in the interval $[0, \gamma_0]$ and N_2 points in the interval $[\gamma_0, \gamma]$, then we clearly have $N = N_1 + N_2$ quadrature points in the interval $[0, \gamma]$.

While we found that the quadrature scheme defined in the preceding discussion works well as long as $\Delta = 1$, we also concluded that a modification is desirable in order to take into account the step function $\Theta(x_0, \xi)$ that is present in equation (69a). To this end, we subdivided the interval $u \in [0, 1]$ into $u \in [0, u_0]$ and $u \in [u_0, 1]$, where $u_0 = u(\gamma_{x0})$, and we used a separate Gauss-Legendre quadrature scheme on each of the subintervals. We note that this interesting case, in which a discontinuous function appears in the boundary conditions of a radiative-transfer problem, is the first of this class we have seen. We therefore see, as a result of this step function $\Theta(x_0, \xi)$ appearing in the boundary conditions, that the choice of a good quadrature scheme in a discrete-ordinates solution in the field of radiative transfer can depend on both the equation of transfer and the boundary conditions.

Having defined our quadrature scheme, we found the required separation constants ν_j by using the driver program RG from the EISPACK collection (Smith et al. 1976) to find the eigenvalues of the problem defined by equation (26). In addition, we used a Gaussian elimination package from the LINPACK collection (Dongarra et al. 1979) to solve the system of linear algebraic equations that defines the constants A_j and B_j required in the solution.

At this point, we consider it important to note that since the characteristic matrix $\boldsymbol{\Psi}(\xi)$ used in this work can be zero, from a computational point of view, we can have some of the separation constants ν_j equal to some of the quadrature points ξ_j. Of course, this is not allowed in equation (28), and so, since the quadrature points where $\boldsymbol{\Psi}(\xi)$ is effectively zero make no contribution to the right-hand side of equations (18a) and (18b), we have resolved this issue by omitting these quadrature points from our calculation.

In order to use fully the FORTRAN program written to implement the discrete-ordinates solution we have developed here, we consider two basic problems, one for the Doppler line-scattering coefficient and the other for the Lorentz case, that have none of the defining physical parameters set equal to zero. While it may be that the test problems we define are more general than might be required, we wish to use and evaluate all aspects of our solution.

In Table 1 we list the values of the physical data used to define the problems for which we choose to report our numerical results. In Tables 2 and 3 we list the first and second components of the vector $\boldsymbol{\Gamma}(\tau, \xi)$, the fundamental quantity required to define the solution given by equations (90a) and (90b), for the case of the Doppler line-scattering coefficient, and in Tables 4

TABLE 1

BASIC DATA

τ_0	ϖ	c	β	ρ	x_0	μ_0	Δ	B_I	B_Q	F_I	F_Q
2.0	0.999999	0.5	0.0001	0.3	0.2	0.5	0.4	1.0	0.9	1.0	0.8

TABLE 2

THE FIRST COMPONENT OF $\Gamma(\eta\tau_0, \pi^{1/2}z)$ FOR THE DOPPLER CASE

z	$\eta = 0.00$	$\eta = 0.10$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
$-\gamma/\pi^{1/2}$	−1.5115(−5)	−1.6301(−5)	−1.4274(−5)	−8.5646(−6)	
−5.0(3)	−1.7055(−5)	−1.8393(−5)	−1.6106(−5)	−9.6641(−6)	
−2.0(3)	−4.2626(−5)	−4.5973(−5)	−4.0262(−5)	−2.4159(−5)	
−1.0(3)	−8.5213(−5)	−9.1912(−5)	−8.0511(−5)	−4.8315(−5)	
−5.0(2)	−1.7027(−4)	−1.8369(−4)	−1.6097(−4)	−9.6615(−5)	
−2.0(2)	−4.2453(−4)	−4.5823(−4)	−4.0204(−4)	−2.4143(−4)	
−1.0(2)	−8.4525(−4)	−9.1313(−4)	−8.0280(−4)	−4.8250(−4)	
−5.0(1)	−1.6753(−3)	−1.8131(−3)	−1.6005(−3)	−9.6357(−4)	
−2.0(1)	−4.0766(−3)	−4.4351(−3)	−3.9631(−3)	−2.3982(−3)	
−1.0(1)	−7.7936(−3)	−8.5551(−3)	−7.8011(−3)	−4.7610(−3)	
−5.0	−1.4240(−2)	−1.5920(−2)	−1.5117(−2)	−9.3822(−3)	
−2.0	−2.7098(−2)	−3.2122(−2)	−3.4425(−2)	−2.2447(−2)	
−1.0	−3.4050(−2)	−4.5241(−2)	−5.9234(−2)	−4.1777(−2)	
−5.0(−1)	−2.4461(−2)	−4.5844(−2)	−8.9400(−2)	−7.2717(−2)	
−2.0(−1)	1.7853(−2)	−1.4920(−2)	−1.1160(−1)	−1.2463(−1)	
−1.0(−1)	4.5741(−2)	9.9758(−3)	−1.0454(−1)	−1.5002(−1)	
−0.0	7.9069(−2)	4.0227(−2)	−8.3576(−2)	−1.4333(−1)	
0.0		4.0227(−2)	−8.3576(−2)	−1.4333(−1)	−2.0165(−1)
1.0(−1)		3.7688(−2)	−5.9175(−2)	−1.2256(−1)	−1.7963(−1)
2.0(−1)		2.4778(−2)	−3.9060(−2)	−1.0098(−1)	−1.5904(−1)
5.0(−1)		1.1828(−2)	−1.5692(−2)	−5.9367(−2)	−1.0878(−1)
1.0		6.2861(−3)	−6.8809(−3)	−3.3604(−2)	−6.7776(−2)
2.0		3.2416(−3)	−3.0022(−3)	−1.7723(−2)	−3.8013(−2)
5.0		1.3211(−3)	−1.0611(−3)	−7.2829(−3)	−1.6295(−2)
1.0(1)		6.6468(−4)	−5.0384(−4)	−3.6702(−3)	−8.3365(−3)
2.0(1)		3.3338(−4)	−2.4490(−4)	−1.8419(−3)	−4.2162(−3)
5.0(1)		1.3360(−4)	−9.6231(−5)	−7.3834(−4)	−1.6981(−3)
1.0(2)		6.6843(−5)	−4.7823(−5)	−3.6943(−4)	−8.5096(−4)
2.0(2)		3.3432(−5)	−2.3838(−5)	−1.8478(−4)	−4.2597(−4)
5.0(2)		1.3375(−5)	−9.5176(−6)	−7.3926(−5)	−1.7050(−4)
1.0(3)		6.6880(−6)	−4.7559(−6)	−3.6966(−5)	−8.5271(−5)
2.0(3)		3.3441(−6)	−2.3772(−6)	−1.8483(−5)	−4.2640(−5)
5.0(3)		1.3377(−6)	−9.5070(−7)	−7.3935(−6)	−1.7057(−5)
$\gamma/\pi^{1/2}$		1.1855(−6)	−8.4253(−7)	−6.5524(−6)	−1.5117(−5)

TABLE 3

THE SECOND COMPONENT OF $\Gamma(\eta\tau_0, \pi^{1/2}z)$ FOR THE DOPPLER CASE

z	$\eta = 0.00$	$\eta = 0.10$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
$-\gamma/\pi^{1/2}$	1.1281(−5)	8.8329(−6)	2.9276(−6)	1.0478(−6)	
−5.0(3)	1.2729(−5)	9.9668(−6)	3.3035(−6)	1.1823(−6)	
−2.0(3)	3.1819(−5)	2.4914(−5)	8.2581(−6)	2.9556(−6)	
−1.0(3)	6.3626(−5)	4.9820(−5)	1.6514(−5)	5.9109(−6)	
−5.0(2)	1.2720(−4)	9.9604(−5)	3.3021(−5)	1.1820(−5)	
−2.0(2)	3.1765(−4)	2.4874(−4)	8.2496(−5)	2.9540(−5)	
−1.0(2)	6.3409(−4)	4.9660(−4)	1.6480(−4)	5.9043(−5)	
−5.0(1)	1.2634(−3)	9.8969(−4)	3.2886(−4)	1.1794(−4)	
−2.0(1)	3.1231(−3)	2.4481(−3)	8.1656(−4)	2.9375(−4)	
−1.0(1)	6.1310(−3)	4.8111(−3)	1.6147(−3)	5.8387(−4)	
−5.0	1.1821(−2)	9.2955(−3)	3.1577(−3)	1.1534(−3)	
−2.0	2.6605(−2)	2.1036(−2)	7.3904(−3)	2.7797(−3)	
−1.0	4.5337(−2)	3.6097(−2)	1.3304(−2)	5.2362(−3)	
−5.0(−1)	6.9070(−2)	5.5421(−2)	2.1916(−2)	9.3336(−3)	
−2.0(−1)	9.8444(−2)	7.9193(−2)	3.4296(−2)	1.7113(−2)	
−1.0(−1)	1.1430(−1)	9.1434(−2)	4.0613(−2)	2.2610(−2)	
−0.0	1.4013(−1)	1.0849(−1)	4.8081(−2)	2.8458(−2)	
0.0		1.0849(−1)	4.8081(−2)	2.8458(−2)	1.3164(−2)
1.0(−1)		8.0931(−2)	5.7908(−2)	3.5043(−2)	1.8890(−2)
2.0(−1)		5.2171(−2)	6.2635(−2)	4.2044(−2)	2.4831(−2)
5.0(−1)		2.4615(−2)	5.1292(−2)	4.4996(−2)	3.4333(−2)
1.0		1.3031(−2)	3.4312(−2)	3.5007(−2)	3.1468(−2)
2.0		6.7072(−3)	2.0042(−2)	2.2321(−2)	2.2119(−2)
5.0		2.7303(−3)	8.8282(−3)	1.0401(−2)	1.0977(−2)
1.0(1)		1.3732(−3)	4.5606(−3)	5.4783(−3)	5.9079(−3)
2.0(1)		6.8860(−4)	2.3181(−3)	2.8119(−3)	3.0658(−3)
5.0(1)		2.7593(−4)	9.3643(−4)	1.1426(−3)	1.2541(−3)
1.0(2)		1.3804(−4)	4.6976(−4)	5.7434(−4)	6.3175(−4)
2.0(2)		6.9042(−5)	2.3527(−4)	2.8793(−4)	3.1706(−4)
5.0(2)		2.7622(−5)	9.4201(−5)	1.1535(−4)	1.2711(−4)
1.0(3)		1.3812(−5)	4.7116(−5)	5.7708(−5)	6.3602(−5)
2.0(3)		6.9061(−6)	2.3562(−5)	2.8861(−5)	3.1813(−5)
5.0(3)		2.7625(−6)	9.4257(−6)	1.1546(−5)	1.2728(−5)
$\gamma/\pi^{1/2}$		2.4482(−6)	8.3534(−6)	1.0233(−5)	1.1280(−5)

TABLE 4

THE FIRST COMPONENT OF $\Gamma(\eta\tau_0, \pi z)$ FOR THE LORENTZ CASE

z	$\eta = 0.00$	$\eta = 0.10$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
$-\gamma/\pi$	−3.6097(−5)	−3.3627(−5)	−2.0846(−5)	−1.1013(−5)	
−2.0(3)	−5.7446(−5)	−5.3515(−5)	−3.3177(−5)	−1.7527(−5)	
−1.0(3)	−1.1487(−4)	−1.0701(−4)	−6.6348(−5)	−3.5052(−5)	
−5.0(2)	−2.2966(−4)	−2.1396(−4)	−1.3267(−4)	−7.0099(−5)	
−2.0(2)	−5.7355(−4)	−5.3440(−4)	−3.3152(−4)	−1.7520(−4)	
−1.0(2)	−1.1451(−3)	−1.0671(−3)	−6.6249(−4)	−3.5027(−4)	
−5.0(1)	−2.2822(−3)	−2.1277(−3)	−1.3228(−3)	−6.9997(−4)	
−2.0(1)	−5.6457(−3)	−5.2699(−3)	−3.2907(−3)	−1.7457(−3)	
−1.0(1)	−1.1096(−2)	−1.0378(−2)	−6.5274(−3)	−3.4773(−3)	
−5.0	−2.1436(−2)	−2.0128(−2)	−1.2843(−2)	−6.8987(−3)	
−2.0	−4.8391(−2)	−4.5963(−2)	−3.0581(−2)	−1.6837(−2)	
−1.0	−8.2180(−2)	−7.9468(−2)	−5.6492(−2)	−3.2363(−2)	
−5.0(−1)	−1.2144(−1)	−1.2119(−1)	−9.6998(−2)	−5.9881(−2)	
−2.0(−1)	−1.4726(−1)	−1.5633(−1)	−1.6121(−1)	−1.2004(−1)	
−1.0(−1)	−1.3859(−1)	−1.5285(−1)	−1.8935(−1)	−1.7308(−1)	
−0.0	−1.1365(−1)	−1.3222(−1)	−1.8395(−1)	−2.0880(−1)	
0.0		−1.3222(−1)	−1.8395(−1)	−2.0880(−1)	−2.3228(−1)
1.0(−1)		−5.8625(−2)	−1.6082(−1)	−1.9189(−1)	−2.1682(−1)
2.0(−1)		−3.3810(−2)	−1.2831(−1)	−1.6673(−1)	−1.9693(−1)
5.0(−1)		−1.4792(−2)	−7.3519(−2)	−1.0728(−1)	−1.3824(−1)
1.0		−7.6253(−3)	−4.2077(−2)	−6.4874(−2)	−8.7781(−2)
2.0		−3.8718(−3)	−2.2576(−2)	−3.5907(−2)	−5.0021(−2)
5.0		−1.5632(−3)	−9.4304(−3)	−1.5299(−2)	−2.1722(−2)
1.0(1)		−7.8401(−4)	−4.7845(−3)	−7.8151(−3)	−1.1170(−2)
2.0(1)		−3.9261(−4)	−2.4099(−3)	−3.9499(−3)	−5.6645(−3)
5.0(1)		−1.5719(−4)	−9.6821(−4)	−1.5902(−3)	−2.2852(−3)
1.0(2)		−7.8620(−5)	−4.8482(−4)	−7.9684(−4)	−1.1458(−3)
2.0(2)		−3.9316(−5)	−2.4259(−4)	−3.9885(−4)	−5.7374(−4)
5.0(2)		−1.5728(−5)	−9.7078(−5)	−1.5964(−4)	−2.2969(−4)
1.0(3)		−7.8642(−6)	−4.8546(−5)	−7.9839(−5)	−1.1488(−4)
2.0(3)		−3.9322(−6)	−2.4275(−5)	−3.9924(−5)	−5.7448(−5)
γ/π		−2.4707(−6)	−1.5253(−5)	−2.5086(−5)	−3.6097(−5)

TABLE 5

THE SECOND COMPONENT OF $\Gamma(\eta\tau_0, \pi z)$ FOR THE LORENTZ CASE

z	$\eta = 0.00$	$\eta = 0.10$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
$-\gamma/\pi$	7.0812(−6)	5.7835(−6)	2.2476(−6)	8.9448(−7)	
−2.0(3)	1.1270(−5)	9.2043(−6)	3.5770(−6)	1.4236(−6)	
−1.0(3)	2.2537(−5)	1.8407(−5)	7.1535(−6)	2.8471(−6)	
−5.0(2)	4.5062(−5)	3.6805(−5)	1.4305(−5)	5.6937(−6)	
−2.0(2)	1.1257(−4)	9.1951(−5)	3.5748(−5)	1.4231(−5)	
−1.0(2)	2.2488(−4)	1.8370(−4)	7.1447(−5)	2.8452(−5)	
−5.0(1)	4.4869(−4)	3.6658(−4)	1.4270(−4)	5.6862(−5)	
−2.0(1)	1.1137(−3)	9.1037(−4)	3.5528(−4)	1.4184(−4)	
−1.0(1)	2.2012(−3)	1.8008(−3)	7.0572(−4)	2.8265(−4)	
−5.0	4.3002(−3)	3.5236(−3)	1.3923(−3)	5.6119(−4)	
−2.0	1.0037(−2)	8.2612(−3)	3.3431(−3)	1.3727(−3)	
−1.0	1.8003(−2)	1.4916(−2)	6.2610(−3)	2.6487(−3)	
−5.0(−1)	2.9540(−2)	2.4722(−2)	1.1044(−2)	4.9381(−3)	
−2.0(−1)	4.6404(−2)	3.9348(−2)	1.9789(−2)	1.0121(−2)	
−1.0(−1)	5.6023(−2)	4.7534(−2)	2.5676(−2)	1.5103(−2)	
−0.0	7.2250(−2)	5.9139(−2)	3.2446(−2)	2.2164(−2)	
0.0		5.9139(−2)	3.2446(−2)	2.2164(−2)	1.3545(−2)
1.0(−1)		3.0250(−2)	3.8593(−2)	2.8379(−2)	1.9170(−2)
2.0(−1)		1.7601(−2)	3.4834(−2)	3.0202(−2)	2.3142(−2)
5.0(−1)		7.7423(−3)	2.1853(−2)	2.3200(−2)	2.1691(−2)
1.0		3.9984(−3)	1.2912(−2)	1.4974(−2)	1.5387(−2)
2.0		2.0321(−3)	7.0404(−3)	8.5669(−3)	9.2757(−3)
5.0		8.2086(−4)	2.9694(−3)	3.7234(−3)	4.1666(−3)
1.0(1)		4.1178(−4)	1.5114(−3)	1.9147(−3)	2.1667(−3)
2.0(1)		2.0623(−4)	7.6250(−4)	9.7091(−4)	1.1050(−3)
5.0(1)		8.2572(−5)	3.0665(−4)	3.9167(−4)	4.4727(−4)
1.0(2)		4.1300(−5)	1.5360(−4)	1.9639(−4)	2.2453(−4)
2.0(2)		2.0653(−5)	7.6868(−5)	9.8334(−5)	1.1249(−4)
5.0(2)		8.2621(−6)	3.0764(−5)	3.9367(−5)	4.5048(−5)
1.0(3)		4.1312(−6)	1.5385(−5)	1.9689(−5)	2.2533(−5)
2.0(3)		2.0656(−6)	7.6930(−6)	9.8459(−6)	1.1269(−5)
γ/π		1.2979(−6)	4.8338(−6)	6.1867(−6)	7.0809(−6)

and 5 we report our results for the Lorentz case. Concerning the accuracy of the results listed in these tables, we note that we have been unable to find any comparison results applicable to the general case we consider; however, by noting the stability in our results as we varied the order of the quadrature scheme about the final quadrature schemed used, we have developed some confidence that our results are correct to within 1 unit in the last digit given. At the same time, we cannot be certain that the FORTRAN implementation of our solution is free of errors; additional computational work is planned that will (we hope) support the numerical results reported here.

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