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A discrete-ordinates solution for Poiseuille flow in a plane channel

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Abstract. A recently established version of the discrete-ordinates method is used to develop a solution to a class of problems in the theory of rarefied-gas dynamics. In particular, an accurate solution for the flow, described by the Bhatnagar, Gross and Krook model, of a rarefied gas between two parallel plates is developed for a wide range of the Knudsen number.

Keywords. Kinetic theory, gas dynamics, discrete ordinates.

1. Introduction

In two basic papers [1,2] relevant to the field of rarefied-gas dynamics, Cercignani and Daneri used the BGK model [3] to examine theoretically and numerically the flow of a rarefied gas between two parallel plates. Following the early work of Cercignani and Daneri [1], other authors, including Boffi, De Socio, Gaffuri and Pescatore [4], Loyalka, Petrellis and Storvick [5] and Siewert, Garcia and Grandjean [6], used diverse analytical and computational methods to generate good numerical results for the mentioned Poiseuille-flow problem. Having developed in a recent work [7] a variation of the discrete-ordinates method of Chandrasekhar [8] to solve a class of non-grey radiative-transfer problems that allows scattering with complete frequency redistribution (completely non-coherent scattering), we find that our discrete-ordinates method has an immediate application here in the field of rarefied-gas dynamics. In order to introduce our discrete-ordinates method into this branch of the general area of particle-transport theory, we now use the method to solve the Poiseuille-flow problem in a plane channel.

As discussed, for example, in Ref. 6, the problem to be solved is defined, after taking some moments of the linearized BGK equation, by

$$\frac{1}{2}k\theta + \theta c_x \frac{\partial}{\partial x} Z(x, c_x) + Z(x, c_x) = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-u^2} Z(x, u) \mathrm{d}u, \tag{1}$$

for $x \in (-d/2, d/2)$ and $c_x \in (-\infty, \infty)$, and the boundary conditions

$$Z(-d/2, c_x) = (1-\alpha)Z(-d/2, -c_x) \tag{2a}$$

and

$$Z(d/2, -c_x) = (1-\alpha)Z(d/2, c_x) \tag{2b}$$

for $c_x \in (0, \infty)$. Here x is the spatial variable, d is the channel thickness, k is proportional to the pressure gradient that causes the flow, θ is the mean-free time, $\alpha \in (0, 1]$ is the accommodation coefficient and

$$Z(x, c_x) = \pi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(c_y^2+c_z^2)} c_z h(x, c_x, c_y, c_z) \mathrm{d}c_y \mathrm{d}c_z \tag{3}$$

where $h(x, c_x, c_y, c_z)$ is a perturbation from a Maxwellian distribution and c_x, c_y and c_z are the components of the molecular velocity [2].

2. Reformulation of the problem

Rather than working with Eqs. (1) and (2) we choose to follow Ref. 6 and to introduce some changes of variables, and so we let $\tau = x/\theta$, $\delta = d/\theta$ and $\mu = c_x$ and consider (after a change of notation)

$$\frac{1}{2}k\theta + \mu \frac{\partial}{\partial \tau} Z(\tau, \mu) + Z(\tau, \mu) = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-u^2} Z(\tau, u) \mathrm{d}u, \tag{4}$$

for $\tau \in (-\delta/2, \delta/2)$ and $\mu \in (-\infty, \infty)$, and the boundary conditions

$$Z(-\delta/2, \mu) = (1-\alpha)Z(-\delta/2, -\mu) \tag{5a}$$

and

$$Z(\delta/2, -\mu) = (1-\alpha)Z(\delta/2, \mu) \tag{5b}$$

for $\mu \in (0, \infty)$. Now, in order to obtain a homogeneous version of Eq. (4), we make use of a particular solution that accounts for the inhomogeneous term in that equation, and so we introduce

$$Z(\tau, \mu) = \frac{1}{2}k\theta[\tau^2 - 2\tau\mu + 2\mu^2 - a^2 - 2Y(\tau, \mu)] \tag{6}$$

into Eqs. (4) and (5) to find that $Y(\tau, \mu)$ is defined by

$$\mu \frac{\partial}{\partial \tau} Y(\tau, \mu) + Y(\tau, \mu) = \int_{-\infty}^{\infty} \Psi(u) Y(\tau, u) \mathrm{d}u, \tag{7}$$

for $\tau \in (-a, a)$ and $\mu \in (-\infty, \infty)$, and the boundary conditions

$$Y(-a, \mu) = (1 - \alpha)Y(-a, -\mu) + \alpha\mu^2 + a\mu(2 - \alpha) \tag{8a}$$

and

$$Y(a, -\mu) = (1 - \alpha)Y(a, \mu) + \alpha\mu^2 + a\mu(2 - \alpha) \tag{8b}$$

for $\mu \in (0, \infty)$. Here

$$\Psi(u) = \pi^{-1/2} e^{-u^2} \tag{9}$$

and $2a = \delta$. We proceed now to report our discrete-ordinates solution to the problem defined by Eqs. (7) and (8).

3. A discrete-ordinates solution

We note first of all that $\Psi(u)$ as defined by Eq. (9) is an even function, and so we follow Ref. 7 and write our discrete-ordinates equations as

$$\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau} Y(\tau, \mu_i) + Y(\tau, \mu_i) = \sum_{k=1}^{N} w_k \Psi(\mu_k)[Y(\tau, \mu_k) + Y(\tau, -\mu_k)] \tag{10a}$$

and

$$-\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau} Y(\tau, -\mu_i) + Y(\tau, -\mu_i) = \sum_{k=1}^{N} w_k \Psi(\mu_k)[Y(\tau, \mu_k) + Y(\tau, -\mu_k)] \tag{10b}$$

for $i = 1, 2, \ldots, N$. In writing Eqs. (10) as we have, we are considering that the N quadrature points $\{\mu_k\}$ and the N weights $\{w_k\}$ are defined for use on the integration interval $[0, \infty)$. We note that it is to this feature of using a "half-range" quadrature scheme that we partially attribute the especially good accuracy we have obtained from the solution reported here.

Seeking exponential solutions of Eqs. (10), we substitute

$$Y(\tau, \pm\mu_i) = \phi(\nu, \pm\mu_i) e^{-\tau/\nu} \tag{11}$$

into Eqs. (10) to find

$$\frac{1}{\nu} \mathbf{M} \mathbf{\Phi}_+ = (\mathbf{I} - \mathbf{W}) \mathbf{\Phi}_+ - \mathbf{W} \mathbf{\Phi}_- \tag{12a}$$

and

$$-\frac{1}{\nu} \mathbf{M} \mathbf{\Phi}_- = (\mathbf{I} - \mathbf{W}) \mathbf{\Phi}_- - \mathbf{W} \mathbf{\Phi}_+ \tag{12b}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix,

$$\mathbf{\Phi}_{\pm} = [\phi(\nu, \pm\mu_1), \phi(\nu, \pm\mu_2), \dots, \phi(\nu, \pm\mu_N)]^{\mathrm{T}}, \tag{13}$$

the superscript T denotes the transpose operation, the elements of the matrix $\mathbf{W}$ are

$$w_{i,j} = w_j \Psi(\mu_j) \tag{14}$$

and

$$\mathbf{M} = \mathrm{diag}\{\mu_1, \mu_2, \dots, \mu_N\}. \tag{15}$$

If we now let

$$\mathbf{U} = \mathbf{\Phi}_{+} + \mathbf{\Phi}_{-} \tag{16}$$

then we can eliminate between the sum and the difference of Eqs. (12) to find

$$(\mathbf{D} - 2\mathbf{M}^{-1}\mathbf{W}\mathbf{M}^{-1})\mathbf{M}\mathbf{U} = \frac{1}{\nu^2}\mathbf{M}\mathbf{U} \tag{17}$$

where

$$\mathbf{D} = \mathrm{diag}\{\mu_1^{-2}, \mu_2^{-2}, \dots, \mu_N^{-2}\}. \tag{18}$$

Multiplying Eq. (17) by a diagonal matrix $\mathbf{T}$, we find

$$(\mathbf{D} - 2\mathbf{V})\mathbf{X} = \frac{1}{\nu^2}\mathbf{X} \tag{19}$$

where

$$\mathbf{V} = \mathbf{M}^{-\mathbf{1}}\mathbf{T}\mathbf{W}\mathbf{T}^{-\mathbf{1}}\mathbf{M}^{-\mathbf{1}} \tag{20}$$

and

$$\mathbf{X} = \mathbf{T}\mathbf{M}\mathbf{U}. \tag{21}$$

We can define the elements $T_1, T_2, \dots, T_N$ of $\mathbf{T}$ so as to make $\mathbf{V}$ symmetric; and therefore, since $\mathbf{V}$ is a symmetric, rank one matrix, we can write our eigenvalue problem in the form

$$(\mathbf{D} - 2\mathbf{z}\mathbf{z}^T)\mathbf{X} = \lambda\mathbf{X} \tag{22}$$

where $\lambda = 1/\nu^2$ and

$$\mathbf{z} = \left[\frac{\sqrt{w_1\Psi(\mu_1)}}{\mu_1}, \frac{\sqrt{w_2\Psi(\mu_2)}}{\mu_2}, \dots, \frac{\sqrt{w_N\Psi(\mu_N)}}{\mu_N}\right]^{\mathrm{T}}. \tag{23}$$

We note that the eigenvalue problem defined by Eq. (22) is of a form that is encountered when the so-called "divide and conquer" method [9] is used to find the eigenvalues of tridiagonal matrices.

Considering that we have found the required eigenvalues from Eq. (22), we impose the normalization condition

$$\sum_{k=1}^{N} w_k \Psi(\mu_k)[\phi(\nu,\mu_k)+\phi(\nu,-\mu_k)] = 1 \tag{24}$$

so that we can write our discrete-ordinates solution as

$$Y(\tau,\pm\mu_i) = \sum_{j=1}^{N}\left[A_j \frac{\nu_j}{\nu_j \mp \mu_i} e^{-(a+\tau)/\nu_j} + B_j \frac{\nu_j}{\nu_j \pm \mu_i} e^{-(a-\tau)/\nu_j}\right] \tag{25}$$

where the arbitrary constants $\{A_j\}$ and $\{B_j\}$ are to be determined from the boundary conditions and the separation constants $\{\nu_j\}$ are the reciprocals of the positive square roots of the eigenvalues defined by Eq. (22).

At this point we find it convenient to modify slightly the discrete-ordinates solution we reported in Ref. 7. We note that problems based on Eq. (7) are "conservative" since

$$\int_{-\infty}^{\infty} \Psi(\mu)\mathrm{d}\mu = 1, \tag{26}$$

and so we expect that one of the eigenvalues defined by Eq. (22) should tend to zero as N tends to infinity. We choose to take this fact into account by explicitly neglecting ν_N, the largest of the computed separation constants $\{\nu_j\}$ and, subsequently, by writing Eq. (25) as

$$Y(\tau,\pm\mu_i) = A + B(\tau\mp\mu_i) + \sum_{j=1}^{N-1}\left[A_j \frac{\nu_j}{\nu_j \mp \mu_i} e^{-(a+\tau)/\nu_j} + B_j \frac{\nu_j}{\nu_j \pm \mu_i} e^{-(a-\tau)/\nu_j}\right]. \tag{27}$$

To define the constants A, B, $\{A_j\}$ and $\{B_j\}$ we substitute Eq. (27) into Eqs. (8) evaluated at the quadrature points $\{\mu_i\}$ to find the system of linear algebraic equations

$$\sum_{j=1}^{N-1}\left\{M_{i,j}A_j + N_{i,j}B_j e^{-2a/\nu_j}\right\} + \alpha A - B[\alpha a + \mu_i(2-\alpha)] = \alpha\mu_i^2 + a\mu_i(2-\alpha) \tag{28a}$$

and

$$\sum_{j=1}^{N-1}\left\{M_{i,j}B_j + N_{i,j}A_j e^{-2a/\nu_j}\right\} + \alpha A + B[\alpha a + \mu_i(2-\alpha)] = \alpha\mu_i^2 + a\mu_i(2-\alpha) \tag{28b}$$

for $i = 1, 2, \ldots, N$. Here the matrix elements are defined by

$$M_{i,j} = \nu_j\left[\frac{\alpha\nu_j + \mu_i(2-\alpha)}{\nu_j^2 - \mu_i^2}\right] \tag{29a}$$

and

$$N_{i,j} = \nu_j \left[\frac{\alpha \nu_j - \mu_i (2-\alpha)}{\nu_j^2 - \mu_i^2} \right]. \tag{29b}$$

Considering that we have solved Eqs. (28) to find the required constants, we can use our discrete-ordinates solution, as given by Eq. (27), along with the normalization condition given by Eq. (24) to express

$$Y_0(\tau) = \int_{-\infty}^{\infty} \Psi(\mu) Y(\tau,\mu) \mathrm{d}\mu \tag{30}$$

as

$$Y_0(\tau) = A + B\tau + \sum_{j=1}^{N-1} [A_j e^{-(a+\tau)/\nu_j} + B_j e^{-(a-\tau)/\nu_j}]. \tag{31}$$

Having developed our discrete-ordinates solution, we are ready to discuss some numerical aspects of the solution and to report some numerical results.

4. Numerical results

The first thing we must do is to define the quadrature scheme to be used in our discrete-ordinates solution. First of all we have used either the transformation

$$u(\mu) = \frac{1}{1+\mu} \tag{32}$$

or the transformation

$$u(\mu) = e^{-\mu} \tag{33}$$

to map the interval $[0,\infty)$ onto $[0,1]$, and we then used a Gauss–Legendre scheme mapped onto the interval $[0,1]$. Of course other quadrature schemes could be used. In fact we note that recent works by Garcia [10] and Gander and Karp [11] have reported special quadrature schemes for use in the general area of particle transport theory. Such an approach clearly could be used here. In fact the choice of a quadrature scheme based on the integration interval $[0,\infty)$ with a weight function as defined by Eq. (9) seems a natural choice for this work. However, we have found the use of mapping defined by either Eq. (32) or Eq. (33) followed by the use of the Gauss–Legendre integration formulas to be so effective that we have not tried other integration techniques.

Having defined our quadrature scheme, we found the required separation constants $\{\nu_j\}$ by solving the eigenvalue problem defined by Eq. (22). To date, in computing these eigenvalues we have not used any specialized software that takes into account the special form of Eq. (22). We have simply used the driver program RG from the EISPACK collection [12] to find the eigenvalues; however we expect to try to improve this aspect of our calculation.

Finally, but importantly, we note that since the function $\Psi(\mu)$ defined by Eq. (9) can be zero, from a computational point-of-view, we can have some, say a total of N_0, of the separation constants $\{\nu_j\}$ equal to some of the quadrature points $\{\mu_i\}$. Of course this is not allowed in Eq. (25), and so, since the quadrature points where $\Psi(\mu)$ is effectively zero make no contribution to the right-hand side of Eqs. (10), we have simply omitted from our calculation these quadrature points. Omitting these N_0 quadrature points changes N to $N - N_0$ in our final solution.

In order to illustrate the effectiveness of our discrete-ordinates solution we report our results for the macroscopic velocity profile

$$q(\tau) = \frac{1}{k\theta}\int_{-\infty}^{\infty} \Psi(\mu) Z(\tau,\mu)\mathrm{d}\mu \tag{34}$$

and the flow rate [5]

$$Q = -\frac{1}{2a^2}\int_{-a}^{a} q(\tau)\mathrm{d}\tau. \tag{35}$$

Making use of Eq. (6), we find

$$q(\tau) = \frac{1}{2}(1 - a^2 + \tau^2) - Y_0(\tau) \tag{36}$$

and

$$Q = \frac{1}{2a^2}\int_{-a}^{a} Y_0(\tau)\mathrm{d}\tau - \frac{1}{2a}(1 - \frac{2}{3}a^2) \tag{37}$$

where $Y_0(\tau)$ is defined by Eq. (30) and expressed in terms of our discrete-ordinates solution by Eq. (31). Upon using Eq. (31) in Eq. (37), we find

$$Q = \frac{1}{2a^2}\left[2aA + \sum_{j=1}^{N-1} \nu_j (A_j + B_j)(1 - e^{-2a/\nu_j})\right] - \frac{1}{2a}(1 - \frac{2}{3}a^2). \tag{38}$$

In regard to our numerical results we note, first of all, that Loyalka, Petrellis and Storvick [5] have reported, for various values of the accommodation coefficient α, the flow rate Q for values of the inverse Knudsen number $2a \in [0.05, 10]$ and the velocity profile $q(\tau)$ for the case $2a = 2.0$. We have used our discrete-ordinates solution to confirm to within plus or minus one unit in the last digit given all of the results for Poiseuille flow given, with five figures of accuracy, in Ref. 5. In Ref. 6 the flow rate Q was reported, also with five figures of accuracy, for $2a \in [0.001, 100]$ for the case $\alpha = 1$. Again, we have used our discrete-ordinates solution to confirm, to within plus or minus one unit in the last digit given, all of the Poiseuille-flow results reported in Ref. 6. Having used the found agreement with the results of Refs. 5 and 6 to establish some confidence in the solution developed in this work, we now list in Tables 1 and 2 some results typical of what we found here using our discrete-ordinates solution. The results for the macroscopic velocity listed in Table

1 and the results for the flow rate listed in Table 2 are given with six figures of accuracy and are thought to be correct to within one unit in the last digit given. While, not surprisingly, we have found that the order N of our approximation required to generate "six-figure" results depends on the parameters of a given problem, we note that we have used a maximum value of $N = 100$ to generate the results listed in Tables 1 and 2.

Table 1. The velocity profile $q(\tau)$ for a channel of half width $a = 1.0$.

τ	$\alpha = 0.50$	$\alpha = 0.80$	$\alpha = 0.88$	$\alpha = 0.96$	$\alpha = 1.00$
0.0	−3.65222	−2.31962	−2.11741	−1.94880	−1.87458
0.1	−3.64484	−2.31215	−2.10992	−1.94129	−1.86706
0.2	−3.62258	−2.28964	−2.08735	−1.91866	−1.84440
0.3	−3.58512	−2.25176	−2.04937	−1.88058	−1.80627
0.4	−3.53185	−2.19790	−1.99537	−1.82644	−1.75206
0.5	−3.46179	−2.12707	−1.92435	−1.75524	−1.68078
0.6	−3.37332	−2.03767	−1.83472	−1.66539	−1.59082
0.7	−3.26373	−1.92699	−1.72378	−1.55421	−1.47952
0.8	−3.12792	−1.79004	−1.58657	−1.41674	−1.34193
0.9	−2.95402	−1.61528	−1.41163	−1.24164	−1.16676
1.0	−2.67641	−1.34037	−1.13753	−9.68381(−1)	−8.93925(−1)

Table 2. The flow rate Q.

$2a$	$\alpha = 0.50$	$\alpha = 0.80$	$\alpha = 0.88$	$\alpha = 0.96$	$\alpha = 1.00$
0.05	5.22330	3.08971	2.73834	2.43735	2.30226
0.10	4.55641	2.70774	2.40605	2.14824	2.03271
0.30	3.77847	2.24477	2.00107	1.79451	1.70247
0.50	3.54437	2.10227	1.87662	1.68634	1.60187
0.70	3.43767	2.03877	1.82201	1.63985	1.55919
0.90	3.38389	2.00924	1.79764	1.62022	1.54180
1.00	3.36822	2.00187	1.79206	1.61631	1.53868
2.00	3.37657	2.04139	1.83856	1.66937	1.59486
5.00	3.77440	2.43823	2.23506	2.06548	1.99077
7.00	4.08811	2.74611	2.54144	2.37038	2.29493
9.00	4.41019	3.06346	2.85756	2.68530	2.60925

The problem of Poiseuille flow in a channel that we have considered here is typical of a class of problems that has, over the years, been solved in one way or another by many techniques (some more rigorous than others, naturally). Even so, we believe the solution reported here is especially easy to follow, would be easy to generalize and has yielded numerical results which are, in general, one digit better than those previously reported. Finally, since a Fortran implementation of our discrete-ordinates solution (with $N = 100$) runs in less than three seconds on a 166 MHz Pentium-based PC, we believe our solution to be very efficient as well as very accurate.

5. Concluding remarks

While we have focused our attention here on the Poiseuille-flow problem, it is clear that our discrete-ordinates method is well suited to solve other basic problems (such as the Couette-flow problem, the thermal-creep problem and the viscous-slip problem discussed, for example, in Refs. 5 and 13) based on minor variations of Eqs. (1) and (2). We note also that a discrete-ordinates solution recently developed [14] for a radiative-transfer problem based on a non-coherent scattering model that includes some polarization effects is expected to be used soon to solve several of the two-vector problems in rarefied-gas dynamics where the coupling effects of temperature and density are considered [15]. In addition, since the implementation of our discrete-ordinates solution has proved to be algebraically and computationally straightforward, we expect to be able to extend the use of the method in order to solve problems in rarefied-gas dynamics that are based on models more general than the linearized BGK model considered here. It is also clear that our implementation of the discrete-ordinates method will be computationally even faster once we incorporate into the calculation an especially efficient way of computing the eigenvalues defined by Eq. (22).

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