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A concise and accurate solution for a polarization model in radiative transfer

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Abstract

The discrete-ordinates method is used to develop a solution to a basic polarization problem in radiative transfer. In particular, a solution for the coupled I and Q components of the Stokes vector is developed for a polarization model based on a mixture of Rayleigh and isotropic scattering. The solution is evaluated for the case of a finite plane-parallel layer that has a polarized beam incident on one surface and which has Lambertian reflection on the other surface.

1. Introduction

We consider here a polarization model of radiative transfer that is based on a combination of Rayleigh and isotropic scattering, and so we begin with the equation of transfer considered by Chandrasekhar [1] and written as

$$\mu \frac{\partial}{\partial \tau} \mathbf{I}(\tau, \mu) + \mathbf{I}(\tau, \mu) = \frac{\varpi}{2} \int_{-1}^{1} \mathbf{P}(\mu, \mu') \, \mathbf{I}(\tau, \mu') \, \mathrm{d}\mu' \tag{1}$$

where

$$\mathbf{P}(\mu, \mu') = \begin{bmatrix} 1 + \frac{c}{8}(1 - 3\mu^2)(1 - 3\mu'^2) & \frac{3c}{8}(1 - \mu'^2)(1 - 3\mu^2) \\ \frac{3c}{8}(1 - \mu^2)(1 - 3\mu'^2) & \frac{9c}{8}(1 - \mu^2)(1 - \mu'^2) \end{bmatrix}. \tag{2}$$

Here $c \in (0, 1]$ is a measure of the Rayleigh component of the scattering law: $c = 0$ would yield just isotropic scattering and $c = 1$ yields Rayleigh scattering [1]. In addition $\varpi \in [0, 1)$ is the albedo for single scattering, $\tau \in [0, \tau_0]$ is the optical variable, τ_0 is the optical thickness of the plane-parallel medium and $\mu \in [-1, 1]$ is the cosine of the polar angle (as measured from the *positive* τ axis) that describes the direction of propagation of the radiation. The scattering matrix can be factored in

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various ways; here we choose to use the factorization attributed to Rachkovsky in Ref. [2], and so we write

$$\mathbf{P}(\mu, \mu') = \mathbf{Q}(\mu)\,\mathbf{Q}^{\mathrm{T}}(\mu') \tag{3}$$

where the superscript T denotes the transpose operation, and where

$$\mathbf{Q}(\mu) = \begin{bmatrix} 1 & (\frac{c}{8})^{1/2}(1-3\mu^2) \\ 0 & 3(\frac{c}{8})^{1/2}(1-\mu^2) \end{bmatrix}. \tag{4}$$

In regard to the boundary conditions, subject to which we must solve Eq. (1), we assume that the layer is illuminated on the surface at $\tau = 0$ by a polarized beam and that there is Lambertian reflection at the surface located at $\tau = \tau_0$. We express these boundary conditions as

$$\mathbf{I}(0, \mu) = \tfrac{1}{2}\,\delta(\mu - \mu_0)\,\mathbf{F} \tag{5a}$$

and

$$\mathbf{I}(\tau_0, -\mu) = 2\lambda_0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \int_0^1 \mathbf{I}(\tau_0, \mu')\mu'\,\mathrm{d}\mu' \tag{5b}$$

for $\mu \in (0, 1]$. Here μ_0 is the direction cosine of the incident beam and the vector $\mathbf{F}$ has components F_I and F_Q. In addition, $\lambda_0 \in [0, 1]$ is the Lambertian reflection coefficient.

2. The reduced problem

Since the boundary condition given by Eq. (5a) introduces into $\mathbf{I}(\tau, \mu)$ a component that is a generalized function, we express the complete solution as

$$\mathbf{I}(\tau, \mu) = \tfrac{1}{2}\,\delta(\mu - \mu_0)\,\mathbf{F}\mathrm{e}^{-\tau/\mu} + \mathbf{Q}(\mu)\,\mathbf{G}(\tau, \mu) \tag{6}$$

where, from Eqs. (1) and (5), the vector $\mathbf{G}(\tau, \mu)$ satisfies the equation

$$\mu\frac{\partial}{\partial\tau}\mathbf{G}(\tau, \mu) + \mathbf{G}(\tau, \mu) = \int_{-1}^{1} \boldsymbol{\Psi}(\mu')\,\mathbf{G}(\tau, \mu')\,\mathrm{d}\mu' + \mathbf{S}(\tau), \tag{7}$$

for $\tau \in (0, \tau_0)$ and $\mu \in [-1, 1]$, and the boundary conditions

$$\mathbf{G}(0, \mu) = \mathbf{0} \tag{8a}$$

and

$$\mathbf{G}(\tau_0, -\mu) = \lambda_0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left\{ \mu_0 \mathbf{F}\mathrm{e}^{-\tau_0/\mu_0} + 2\int_0^1 \mathbf{Q}(\mu')\mathbf{G}(\tau_0, \mu')\,\mu'\,\mathrm{d}\mu' \right\} \tag{8b}$$

for $\mu \in (0, 1]$. Here the known source term is

$$\mathbf{S}(\tau) = \mathbf{F}_0\mathrm{e}^{-\tau/\mu_0} \tag{9}$$

where

$$\mathbf{F}_0 = \frac{\varpi}{4}\,\mathbf{Q}^{\mathrm{T}}(\mu_0)\,\mathbf{F}, \tag{10}$$

and the "characteristic" matrix is

$$\boldsymbol{\Psi}(\mu) = \frac{\varpi}{2}\,\mathbf{Q}^{\mathrm{T}}(\mu)\,\mathbf{Q}(\mu). \tag{11}$$

In this work we intend to compute the two components $I_*(\tau, \mu)$ and $Q_*(\tau, \mu)$ of the reduced field

$$\mathbf{I}_*(\tau, \mu) = \mathbf{I}(\tau, \mu) - \tfrac{1}{2}\delta(\mu - \mu_0)\,\mathbf{F}\mathrm{e}^{-\tau/\mu} \tag{12}$$

or, after we note Eq. (6),

$$\mathbf{I}_*(\tau, \mu) = \mathbf{Q}(\mu)\,\mathbf{G}(\tau, \mu). \tag{13}$$

3. A discrete-ordinates solution

In a recent paper [3] concerning a radiative-transfer problem that was based on completely non-coherent scattering and that also included polarization effects, a discrete-ordinates solution was developed that is very similar to what we require here to solve Eq. (7) subject to the boundary conditions given by Eqs. (8). As the basic elements of the discrete-ordinates solution, we wish to develop, have already been reported in Ref. [3], we now repeat (only briefly) the part of Ref. [3] we can use in this work.

We note first of all that (what we call) the characteristic matrix $\boldsymbol{\Psi}(\mu)$ as defined by Eq. (11) is symmetric. Also $\boldsymbol{\Psi}(\mu) = \boldsymbol{\Psi}(-\mu)$, and so we write our discrete-ordinates equations, relevant to the homogeneous version of Eq. (7), as

$$\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{G}_{\mathrm{h}}(\tau, \mu_i) + \mathbf{G}_{\mathrm{h}}(\tau, \mu_i) = \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\mu_k)\,[\mathbf{G}_{\mathrm{h}}(\tau, \mu_k) + \mathbf{G}_{\mathrm{h}}(\tau, -\mu_k)] \tag{14a}$$

and

$$-\mu_i \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{G}_{\mathrm{h}}(\tau, -\mu_i) + \mathbf{G}_{\mathrm{h}}(\tau, -\mu_i) = \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\mu_k)\,[\mathbf{G}_{\mathrm{h}}(\tau, \mu_k) + \mathbf{G}_{\mathrm{h}}(\tau, -\mu_k)] \tag{14b}$$

for $i = 1, 2, \ldots, N$. In writing Eqs. (14) as we have, we clearly are considering that the N quadrature points $\{\mu_k\}$ and the N weights $\{w_k\}$ are defined for use on the integration interval [0, 1]. Now, following Ref. [3], we substitute

$$\mathbf{G}_{\mathrm{h}}(\tau, \pm\mu_i) = \boldsymbol{\Phi}(\nu, \pm\mu_i)\,\mathrm{e}^{-\tau/\nu} \tag{15}$$

into Eqs. (14) to find

$$(\nu - \mu_i)\,\boldsymbol{\Phi}(\nu, \mu_i) = \nu \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\mu_k)\,[\boldsymbol{\Phi}(\nu, \mu_k) + \boldsymbol{\Phi}(\nu, -\mu_k)] \tag{16a}$$

and

$$(\nu + \mu_i)\,\boldsymbol{\Phi}(\nu, -\mu_i) = \nu \sum_{k=1}^{N} w_k \boldsymbol{\Psi}(\mu_k)\,[\boldsymbol{\Phi}(\nu, \mu_k) + \boldsymbol{\Phi}(\nu, -\mu_k)] \tag{16b}$$

for $i = 1, 2, \ldots, N$. If we now let $\phi_1(\nu, \pm\mu_i)$ and $\phi_2(\nu, \pm\mu_i)$ denote the two components of $\mathbf{\Phi}(\nu, \pm\mu_i)$ and if we use

$$\mathbf{\Phi}_{1\pm} = [\phi_1(\nu, \pm\mu_1), \phi_1(\nu, \pm\mu_2), \ldots, \phi_1(\nu, \pm\mu_N)]^{\mathrm{T}} \tag{17a}$$

and

$$\mathbf{\Phi}_{2\pm} = [\phi_2(\nu, \pm\mu_1), \phi_2(\nu, \pm\mu_2), \ldots, \phi_2(\nu, \pm\mu_N)]^{\mathrm{T}} \tag{17b}$$

then we can rewrite Eqs. (16) as

$$\frac{1}{\nu}\mathbf{M\Phi}_+ = (\mathbf{I} - \mathbf{W})\,\mathbf{\Phi}_+ - \mathbf{W\Phi}_- \tag{18a}$$

and

$$-\frac{1}{\nu}\mathbf{M\Phi}_- = (\mathbf{I} - \mathbf{W})\,\mathbf{\Phi}_- - \mathbf{W\Phi}_+. \tag{18b}$$

Here $\mathbf{I}$ is the $2N \times 2N$ identity matrix, the two vector elements of $\mathbf{\Phi}_\pm$ are $\mathbf{\Phi}_{1\pm}$ and $\mathbf{\Phi}_{2\pm}$, the four $N \times N$ matrix elements of $\mathbf{W}$, viz. $\mathbf{W}_{m,n}$, for $m, n = 1, 2$, are given by

$$(\mathbf{W}_{m,n})_{i,j} = w_j \Psi_{m,n}(\mu_j) \tag{19}$$

where $\Psi_{m,n}(\mu)$, for $m, n = 1, 2$, are the elements of $\mathbf{\Psi}(\mu)$ and finally

$$\mathbf{M} = \mathrm{diag}\{\mu_1, \mu_2, \ldots, \mu_N, \mu_1, \mu_2, \ldots, \mu_N\}. \tag{20}$$

Continuing to follow Ref. [3], we now let

$$\mathbf{U} = \mathbf{\Phi}_+ + \mathbf{\Phi}_- \tag{21a}$$

and

$$\mathbf{V} = \mathbf{\Phi}_+ - \mathbf{\Phi}_- \tag{21b}$$

so that we can eliminate $\mathbf{V}$ between the sum and the difference of Eqs. (18) to find

$$(\mathbf{D} - 2\mathbf{M}^{-1}\mathbf{W}\mathbf{M}^{-1})\,\mathbf{MU} = \lambda\mathbf{MU} \tag{22}$$

where $\lambda = 1/\nu^2$ and

$$\mathbf{D} = \mathrm{diag}\{\mu_1^{-2}, \mu_2^{-2}, \ldots, \mu_N^{-2}, \mu_1^{-2}, \mu_2^{-2}, \ldots, \mu_N^{-2}\}. \tag{23}$$

Considering that we have found the required separation constants $\{\pm\nu_j\}$ from the eigenvalues defined by Eq. (22), we go back to Eqs. (16) to find $\mathbf{\Phi}(\nu_j, \pm\mu_i)$, and so we write our general solution to Eqs. (14) as

$$\mathbf{G}_{\mathrm{h}}(\tau, \pm\mu_i) = \sum_{j=1}^{2N}\left[A_j \frac{\nu_j}{\nu_j \mp \mu_i}\,\mathrm{e}^{-\tau/\nu_j} + B_j \frac{\nu_j}{\nu_j \pm \mu_i}\,\mathrm{e}^{-(\tau_0 - \tau)/\nu_j}\right]\mathbf{F}(\nu_j). \tag{24}$$

Here $\mathbf{F}(\nu_j)$ is a vector in the null space of

$$\mathbf{\Omega}(\nu_j) = \mathbf{I} - 2\nu_j^2 \sum_{\alpha=1}^{N} w_\alpha \mathbf{\Psi}(\mu_\alpha)\frac{1}{\nu_j^{\,2} - \mu_\alpha^{\,2}}, \tag{25}$$

I is now the 2×2 identity matrix and the arbitrary constants $\{A_j\}$ and $\{B_j\}$ are to be determined from the boundary conditions.

Since Eq. (7) has the inhomogeneous source term $\mathbf{S}(\tau)$ we must add a particular solution, for example [3]

$$\mathbf{G}_{\mathrm{p}}(\tau, \pm\mu_i) = \frac{\mu_0}{\mu_0 \mp \mu_i}\,[\boldsymbol{\Omega}(\mu_0)]^{-1}\,\mathbf{F}_0\,\mathrm{e}^{-\tau/\mu_0}, \tag{26}$$

to $\mathbf{G}_{\mathrm{h}}(\tau, \pm\mu_i)$ to obtain the desired discrete-ordinates expression for the general solution, namely

$$\mathbf{G}(\tau, \pm\mu_i) = \mathbf{G}_{\mathrm{h}}(\tau, \pm\mu_i) + \mathbf{G}_{\mathrm{p}}(\tau, \pm\mu_i). \tag{27}$$

Of course, we cannot allow $\mu_i = \nu_j$ in Eq. (24) or $\mu_i = \mu_0$ or $\mu_0 \in \{\nu_j\}$ in Eq. (26).

If we now substitute Eq. (27) into the boundary conditions, Eqs. (8), evaluated at the quadrature points we find the following system of linear algebraic equations from which we determine the constants $\{A_j\}$ and $\{B_j\}$ required to complete our solution:

$$\sum_{j=1}^{2N}\left\{\frac{\nu_j}{\nu_j-\mu_i}\mathbf{F}(\nu_j)\right\}A_j + \sum_{j=1}^{2N}\left\{\frac{\nu_j}{\nu_j+\mu_i}\mathrm{e}^{-\tau_0/\nu_j}\mathbf{F}(\nu_j)\right\}B_j = \mathbf{R}_1(\mu_i) \tag{28a}$$

and

$$\sum_{j=1}^{2N}\left\{\frac{\nu_j}{\nu_j+\mu_i}\mathrm{e}^{-\tau_0/\nu_j}\mathbf{F}(\nu_j) - \mathbf{R}_j^{\mathrm{a}}\right\}A_j + \sum_{j=1}^{2N}\left\{\frac{\nu_j}{\nu_j-\mu_i}\mathbf{F}(\nu_j) - \mathbf{R}_j^{\mathrm{b}}\right\}B_j = \mathbf{R}_2(\mu_i) \tag{28b}$$

for $i = 1, 2, \ldots, N$. Here the known right-hand sides of Eqs. (28) are given by

$$\mathbf{R}_1(\mu_i) = -\,\mathbf{G}_{\mathrm{p}}(0, \mu_i) \tag{29a}$$

and

$$\mathbf{R}_2(\mu_i) = -\,\mathbf{G}_{\mathrm{p}}(\tau_0, -\mu_i) + \lambda_0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\left\{\mu_0\mathbf{F}\mathrm{e}^{-\tau_0/\mu_0} + 2\sum_{k=1}^{N} w_k\mu_k\mathbf{Q}(\mu_k)\mathbf{G}_{\mathrm{p}}(\tau_0, \mu_k)\right\} \tag{29b}$$

and the reflection components of the matrix elements are

$$\mathbf{R}_j^{\mathrm{a}} = 2\lambda_0\nu_j\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\sum_{k=1}^{N}\left\{\mathbf{Q}(\mu_k)\,\frac{w_k\mu_k}{\nu_j-\mu_k}\right\}\mathbf{F}(\nu_j)\,\mathrm{e}^{-\tau_0/\nu_j} \tag{30a}$$

and

$$\mathbf{R}_j^{\mathrm{b}} = 2\lambda_0\nu_j\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\sum_{k=1}^{N}\left\{\mathbf{Q}(\mu_k)\,\frac{w_k\mu_k}{\nu_j+\mu_k}\right\}\mathbf{F}(\nu_j). \tag{30b}$$

Clearly, once we have solved Eqs. (28) to find the constants $\{A_j\}$ and $\{B_j\}$ we have a first version of the desired solution available from Eqs. (24), (26) and (27). However, to have a better and more general result we follow Ref. [3] and substitute Eq. (27) into the right-hand side of

$$\mu\frac{\partial}{\partial\tau}\mathbf{G}(\tau,\mu) + \mathbf{G}(\tau,\mu) = \sum_{k=1}^{N} w_k\boldsymbol{\Psi}(\mu_k)\,[\mathbf{G}(\tau,\mu_k) + \mathbf{G}(\tau,-\mu_k)] + \mathbf{S}(\tau) \tag{31}$$

which we can then solve to find

$$\mathbf{G}(\tau, \mu) = \boldsymbol{\Xi}(\tau, \mu) + \boldsymbol{\Upsilon}(\tau, \mu) \tag{32a}$$

and

$$\mathbf{G}(\tau, -\mu) = \mathbf{R}\mathrm{e}^{-(\tau_0-\tau)/\mu} + \boldsymbol{\Xi}(\tau, -\mu) + \boldsymbol{\Upsilon}(\tau, -\mu) \tag{32b}$$

for $\mu \in [0, 1]$ and $\tau \in [0, \tau_0]$. Here

$$\boldsymbol{\Xi}(\tau, \mu) = \sum_{j=1}^{2N} \nu_j [A_j C(\tau\colon \nu_j, \mu) + B_j \mathrm{e}^{-(\tau_0-\tau)/\nu_j} S(\tau\colon \nu_j, \mu)]\, \mathbf{F}(\nu_j), \tag{33a}$$

$$\boldsymbol{\Xi}(\tau, -\mu) = \sum_{j=1}^{2N} \nu_j [A_j \mathrm{e}^{-\tau/\nu_j} S(\tau_0 - \tau\colon \nu_j, \mu) + B_j C(\tau_0 - \tau\colon \nu_j, \mu)]\, \mathbf{F}(\nu_j), \tag{33b}$$

$$\boldsymbol{\Upsilon}(\tau, \mu) = \mu_0 C(\tau\colon \mu_0, \mu)\, [\boldsymbol{\Omega}(\mu_0)]^{-1} \mathbf{F}_0 \tag{34a}$$

and

$$\boldsymbol{\Upsilon}(\tau, -\mu) = \mu_0 \mathrm{e}^{-\tau/\mu_0} S(\tau_0 - \tau\colon \mu_0, \mu)\, [\boldsymbol{\Omega}(\mu_0)]^{-1} \mathbf{F}_0 \tag{34b}$$

where the S and C functions are given by

$$S(\tau\colon x, y) = \frac{1 - \mathrm{e}^{-\tau/x}\mathrm{e}^{-\tau/y}}{x + y} \tag{35a}$$

and

$$C(\tau\colon x, y) = \frac{\mathrm{e}^{-\tau/x} - \mathrm{e}^{-\tau/y}}{x - y}. \tag{35b}$$

To complete Eqs. (32) we find that the constant reflection vector $\mathbf{R}$ can be computed by using Eq. (27) in

$$\mathbf{R} = \lambda_0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left\{ \mu_0 \mathbf{F} \mathrm{e}^{-\tau_0/\mu_0} + 2 \sum_{k=1}^{N} w_k \mu_k \mathbf{Q}(\mu_k)\, \mathbf{G}(\tau_0, \mu_k) \right\}. \tag{36}$$

Having completed our discrete-ordinates solution, we are ready to discuss some of the computational aspects of the solution and to report some typical numerical results.

4. Computational details and numerical results

Of course, the first thing we must do in order to evaluate our discrete-ordinates solution numerically is to define a quadrature scheme. We simply use the usual Gauss–Legendre scheme (of order N) defined for the interval $[-1, 1]$ mapped onto the interval $[0, 1]$. Having defined our quadrature scheme, we obtain the required separation constants $\{\nu_j\}$ by using the driver program RG from the EISPACK collection [4] to find the eigenvalues defined by Eq. (22), and so we consider our solution complete.

To report some numerical results we consider the problem of a layer of thickness $\tau_0 = 2.0$ illuminated by a normally incident beam ($\mu_0 = 1.0$) defined by $F_I = 1.0$ and $F_Q = 0.8$. For the scattering law, we take $\varpi = 0.99$ and $c = 0.5$. In Tables 1 and 2 we list our results for the two components $I_*(\tau, \mu)$ and $Q_*(\tau, \mu)$ of the reduced field. The results listed in Tables 1 and 2 were obtained (with $N = 20$) by first solving Eqs. (28) and then using Eqs. (32) in Eqs. (13). We have compared our numerical results with a generalized spherical-harmonics solution [5] of the same problem, and so, after some additional studies where we varied the order of the quadrature scheme, we believe we have reasons to think that the results given here are correct to within one unit in the last place given. In addition, since a FORTRAN implementation of our discrete-ordinates solution (with $N = 20$) runs in less than a second on a 166 MHz Pentium-based PC, we believe our solution to be very efficient as well as very accurate.

To conclude this work, we mention two special cases for which the solution developed here does not apply. The first is the $c = 0$ case where the problem breaks down into two simple scalar problems, and the second is the conservative case ($\varpi = 1$) where special attention is required to take into account a repeated eigenvalue at infinity. Needless to say, if the solution does not apply for either of these special cases, then we can expect also to encounter numerical difficulties if we have data sufficiently close to one of these cases. However, for essentially all practical cases, say c as small as 10^{-3} or $1 - \varpi$ as small as 10^{-7}, we have found that the solution yields excellent results. Of

Table 1
The component $I_*(\eta\tau_0, \mu)$ for the case of $\varpi = 0.99$, $c = 0.5$ and $\tau_0 = 2.0$ with $\mu_0 = 1.0$, $F_I = 1.0$ and $F_Q = 0.8$

μ	$\eta = 0.00$	$\eta = 0.10$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
− 1.0	5.1625(− 1)	4.9254(− 1)	3.2822(− 1)	2.1140(− 1)	1.0578(− 1)
− 0.9	5.2315(− 1)	5.0263(− 1)	3.4016(− 1)	2.1891(− 1)	1.0578(− 1)
− 0.8	5.3052(− 1)	5.1370(− 1)	3.5398(− 1)	2.2798(− 1)	1.0578(− 1)
− 0.7	5.3802(− 1)	5.2559(− 1)	3.7000(− 1)	2.3906(− 1)	1.0578(− 1)
− 0.6	5.4512(− 1)	5.3794(− 1)	3.8852(− 1)	2.5278(− 1)	1.0578(− 1)
− 0.5	5.5096(− 1)	5.5014(− 1)	4.0976(− 1)	2.7000(− 1)	1.0578(− 1)
− 0.4	5.5429(− 1)	5.6120(− 1)	4.3366(− 1)	2.9193(− 1)	1.0578(− 1)
− 0.3	5.5335(− 1)	5.6972(− 1)	4.5938(− 1)	3.1999(− 1)	1.0578(− 1)
− 0.2	5.4580(− 1)	5.7414(− 1)	4.8467(− 1)	3.5479(− 1)	1.0578(− 1)
− 0.1	5.2752(− 1)	5.7274(− 1)	5.0664(− 1)	3.9110(− 1)	1.0578(− 1)
− 0.0	4.7883(− 1)	5.6188(− 1)	5.2542(− 1)	4.1834(− 1)	1.0578(− 1)
0.0		5.6188(− 1)	5.2542(− 1)	4.1834(− 1)	2.6389(− 1)
0.1		4.7038(− 1)	5.4088(− 1)	4.4219(− 1)	3.0219(− 1)
0.2		3.4231(− 1)	5.4775(− 1)	4.6301(− 1)	3.3215(− 1)
0.3		2.6479(− 1)	5.3846(− 1)	4.7726(− 1)	3.5792(− 1)
0.4		2.1625(− 1)	5.1795(− 1)	4.8256(− 1)	3.7817(− 1)
0.5		1.8372(− 1)	4.9321(− 1)	4.8042(− 1)	3.9225(− 1)
0.6		1.6073(− 1)	4.6830(− 1)	4.7349(− 1)	4.0088(− 1)
0.7		1.4382(− 1)	4.4503(− 1)	4.6398(− 1)	4.0529(− 1)
0.8		1.3101(− 1)	4.2408(− 1)	4.5338(− 1)	4.0673(− 1)
0.9		1.2111(− 1)	4.0559(− 1)	4.4263(− 1)	4.0618(− 1)
1.0		1.1332(− 1)	3.8945(− 1)	4.3228(− 1)	4.0441(− 1)

Table 2
The component $Q_*(\eta\tau_0, \mu)$ for the case of $\varpi = 0.99$, $c = 0.5$ and $\tau_0 = 2.0$ with $\mu_0 = 1.0$, $F_I = 1.0$ and $F_Q = 0.8$

μ	$\eta = 0.00$	$\eta = 0.10$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
− 1.0	0.0	0.0	0.0	0.0	
− 0.9	− 9.8423(− 3)	− 7.9047(− 3)	− 3.2507(− 3)	− 1.5848(− 3)	
− 0.8	− 1.9763(− 2)	− 1.5884(− 2)	− 6.6045(− 3)	− 3.2786(− 3)	
− 0.7	− 2.9761(− 2)	− 2.3935(− 2)	− 1.0073(− 2)	− 5.1109(− 3)	
− 0.6	− 3.9834(− 2)	− 3.2051(− 2)	− 1.3665(− 2)	− 7.1213(− 3)	
− 0.5	− 4.9978(− 2)	− 4.0216(− 2)	− 1.7376(− 2)	− 9.3618(− 3)	
− 0.4	− 6.0195(− 2)	− 4.8416(− 2)	− 2.1177(− 2)	− 1.1895(− 2)	
− 0.3	− 7.0516(− 2)	− 5.6646(− 2)	− 2.4984(− 2)	− 1.4770(− 2)	
− 0.2	− 8.1030(− 2)	− 6.4950(− 2)	− 2.8644(− 2)	− 1.7903(− 2)	
− 0.1	− 9.1932(− 2)	− 7.3437(− 2)	− 3.2088(− 2)	− 2.0663(− 2)	
− 0.0	− 1.0413(− 1)	− 8.2268(− 2)	− 3.5552(− 2)	− 2.2484(− 2)	
0.0		− 8.2268(− 2)	− 3.5552(− 2)	− 2.2484(− 2)	− 1.8374(− 2)
0.1		− 7.6246(− 2)	− 3.9143(− 2)	− 2.4355(− 2)	− 1.8028(− 2)
0.2		− 5.5003(− 2)	− 4.2112(− 2)	− 2.6286(− 2)	− 1.8499(− 2)
0.3		− 4.0386(− 2)	− 4.2155(− 2)	− 2.7603(− 2)	− 1.9064(− 2)
0.4		− 3.0241(− 2)	− 3.9183(− 2)	− 2.7411(− 2)	− 1.9177(− 2)
0.5		− 2.2667(− 2)	− 3.4238(− 2)	− 2.5516(− 2)	− 1.8384(− 2)
0.6		− 1.6653(− 2)	− 2.8158(− 2)	− 2.2161(− 2)	− 1.6519(− 2)
0.7		− 1.1645(− 2)	− 2.1458(− 2)	− 1.7676(− 2)	− 1.3620(− 2)
0.8		− 7.3213(− 3)	− 1.4432(− 2)	− 1.2349(− 2)	− 9.8073(− 3)
0.9		− 3.4827(− 3)	− 7.2476(− 3)	− 6.4027(− 3)	− 5.2233(− 3)
1.0		0.0	0.0	0.0	0.0

course, the solution developed in this work can easily be modified to solve both the special cases. For the conservative case we can ignore the infinite eigenvalue and use the exact "discrete" solutions, as was done in a recent work in the area of rarefied-gas dynamics [6]. The special case $c = 0$ is not consider interesting since the Q component can be solved exactly in one line, and since the resulting equation for the I component is just the usual monochromatic equation with isotropic scattering (a problem easily solved by any number of methods, including the one discussed in this work).

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