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A concise and accurate solution to Chandrasekhar's basic problem in radiative transfer

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Abstract

A recently developed version of the discrete-ordinates method is used along with elementary numerical linear-algebra techniques to establish an efficient and especially accurate solution to what can be called Chandrasekhar's basic problem in radiative transfer, namely the problem of computing the radiation intensity in a finite plane-parallel layer illuminated by an incident beam of radiation and in which scattering can be described by a (rather) general scattering law. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

We consider in this work what we call Chandrasekhar's basic problem in radiative transfer — the problem upon which much of Chandrasekhar's classic text *Radiative Transfer* [1] is focused, viz. the problem of computing the intensity in a finite layer illuminated by a beam of radiation incident on one surface. In Ref. [2] we reported a high-order solution based on the spherical-harmonics method for this problem, and in Ref. [3] the F_N method was used to generate accurate results for the most challenging test problem, in this general class, we have solved to date. Since many of the important works that are based on the spherical-harmonics method and the F_N method were discussed in Refs. [2, 3], additional reviewing is not done here. In this work, we use a variation of the discrete-ordinates technique and a recently developed particular solution [4] to establish a solution to Chandrasekhar's basic problem that is concise and especially accurate.

2. Basic formulation of the problem

Our formulation of the problem to be solved here and the notation we use follow directly from Ref. [2], and so we start with the equation of transfer

$$\mu \frac{\partial}{\partial \tau} I(\tau, \mu, \phi) + I(\tau, \mu, \phi) = \frac{\varpi}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\cos\Theta) I(\tau, \mu', \phi') \, d\phi' \, d\mu' \tag{1}$$

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where $\varpi \in [0, 1]$ is the albedo for single scattering, $\tau \in [0, \tau_0]$ is the optical variable, τ_0 is the optical thickness of the plane-parallel medium, Θ is the scattering angle, $\mu \in [-1, 1]$ is the cosine of the polar angle (as measured from the *positive* τ-axis) and ϕ is the azimuthal angle. Together the polar and azimuthal angles define the direction of propagation of the radiation. In addition, we consider here phase functions that can be expressed in terms of Legendre polynomials, that is

$$p(\cos\Theta) = \sum_{l=0}^{L} \beta_l P_l(\cos\Theta), \quad \beta_0 = 1, \tag{2}$$

where the β_l are the coefficients in the Lth-order expansion of the scattering law. Considering Chandrasekhar's standard problem [1], we seek to establish, for all $\mu \in [-1, 1]$, $\phi \in [0, 2\pi]$ and $\tau \in [0, \tau_0]$, a solution of Eq. (1) subject to the boundary conditions

$$I(0, \mu, \phi) = \pi\delta(\mu - \mu_0)\delta(\phi - \phi_0) \tag{3a}$$

and

$$I(\tau_0, -\mu, \phi) = 0 \tag{3b}$$

for $\mu \in (0, 1]$ and $\phi \in [0, 2\pi]$. Here μ_0 is the direction cosine of the incident beam.

3. The reduced problem

Since the boundary condition given by Eq. (3a) introduces into $I(\tau, \mu, \phi)$ a component that is a generalized function, we follow Chandrasekhar and express the complete solution to the problem defined by Eqs. (1)–(3) in the form

$$I(\tau, \mu, \phi) = I_*(\tau, \mu, \phi) + \pi\delta(\mu - \mu_0)\delta(\phi - \phi_0)\mathrm{e}^{-\tau/\mu} \tag{4}$$

where $I_*(\tau, \mu, \phi)$ is the reduced or diffuse field. Continuing, we make use of the addition theorem [5] for the Legendre polynomials and express the phase function, for scattering from $\{\mu', \phi'\}$ to $\{\mu, \phi\}$, in the form

$$p(\cos\Theta) = \sum_{m=0}^{L} (2 - \delta_{0,m}) \sum_{l=m}^{L} \beta_l P_l^m(\mu') P_l^m(\mu) \cos[m(\phi' - \phi)] \tag{5}$$

where

$$P_l^m(\mu) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} (1 - \mu^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}\mu^m} P_l(\mu) \tag{6}$$

is used to denote the *normalized* Legendre function. It follows [1–3] now that the diffuse field can be expressed as

$$I_*(\tau, \mu, \phi) = \frac{1}{2} \sum_{m=0}^{L} (2 - \delta_{0,m}) I^m(\tau, \mu) \cos[m(\phi - \phi_0)] \tag{7}$$

where the mth Fourier component satisfies the equation of transfer

$$\mu \frac{\partial}{\partial\tau} I^m(\tau, \mu) + I^m(\tau, \mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu) \int_{-1}^{1} P_l^m(\mu') I^m(\tau, \mu')\,\mathrm{d}\mu' + Q^m(\tau, \mu), \tag{8}$$

for $\mu \in [-1, 1]$ and $\tau \in (0, \tau_0)$, and the boundary conditions

$$I^m(0, \mu) = 0 \tag{9a}$$

and

$$I^m(\tau_0, -\mu) = 0 \tag{9b}$$

for $\mu \in (0, 1]$. Here the inhomogeneous source term is

$$Q^m(\tau, \mu) = \frac{\varpi}{2} e^{-\tau/\mu_0} \sum_{l=m}^{L} \beta_l P_l^m(\mu_0) P_l^m(\mu). \tag{10}$$

4. A discrete-ordinates solution

In a recent paper [6], concerning a radiative-transfer problem based on completely non-coherent scattering, a solution based on a new variation of the discrete-ordinates method was developed, evaluated and found to be very effective. And so here we wish to make use of the solution reported in Ref. [6] in order to solve efficiently and accurately the class of problems defined by Eqs. (8)–(10). For the moment, we exclude the special (conservative) case $\varpi = 1$, but in Section 6 of this work we spell-out the modifications required in our general development in order deal with that case.

As a matter of strategy, we note that we intend to use the discrete-ordinates method only to find approximate values for the integral terms in Eq. (8), and once that is done we will solve Eq. (8), with the integral terms replaced by discrete-ordinates approximations to those terms, to find the desired Fourier component $I^m(\tau, \mu)$ for all τ and μ. This second aspect of our approach is what we refer to as a "post-processing" step [1, 7].

And so, we suppress some of the explicit notation of the Fourier index m and start with our discrete-ordinates equations, relevant to the homogeneous version of Eq. (8), written as

$$\mu_i \frac{d}{d\tau} I(\tau, \mu_i) + I(\tau, \mu_i) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu_i) \sum_{k=1}^{N} w_k P_l^m(\mu_k) [I(\tau, \mu_k) + (-1)^{l-m} I(\tau, -\mu_k)] \tag{11a}$$

and

$$-\mu_i \frac{d}{d\tau} I(\tau, -\mu_i) + I(\tau, -\mu_i) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu_i) \times \sum_{k=1}^{N} w_k P_l^m(\mu_k) [(-1)^{l-m} I(\tau, \mu_k) + I(\tau, -\mu_k)] \tag{11b}$$

for $i = 1, 2, \ldots, N$. In writing Eqs. (11) as we have, we clearly are considering that the N quadrature points $\{\mu_k\}$ and the N weights $\{w_k\}$ are defined for use on the integration interval [0, 1]. We see that exponential solutions will work in Eqs. (11), and so we substitute

$$I(\tau, \pm\mu_i) = \phi(\nu, \pm\mu_i) e^{-\tau/\nu} \tag{12}$$

into Eqs. (11) to find

$$\left(1-\frac{\mu_i}{\nu}\right)\phi(\nu,\mu_i)=\frac{\varpi}{2}\sum_{l=m}^{L}\beta_l P_l^m(\mu_i)\sum_{k=1}^{N}w_k P_l^m(\mu_k)[\phi(\nu,\mu_k)+(-1)^{l-m}\phi(\nu,-\mu_k)] \tag{13a}$$

and

$$\left(1+\frac{\mu_i}{\nu}\right)\phi(\nu,-\mu_i)=\frac{\varpi}{2}\sum_{l=m}^{L}\beta_l P_l^m(\mu_i)\sum_{k=1}^{N}w_k P_l^m(\mu_k)[(-1)^{l-m}\phi(\nu,\mu_k)+\phi(\nu,-\mu_k)] \tag{13b}$$

for $i=1,2,\dots,N$. In order to write Eqs. (13) in a more convenient way, we introduce some matrix notation. So with

$$\Phi_{\pm}(\nu)=[\phi(\nu,\pm\mu_1),\phi(\nu,\pm\mu_2),\dots,\phi(\nu,\mu_N)]^{\mathrm{T}}, \tag{14}$$

$$\Pi(l,m)=[P_l^m(\mu_1),P_l^m(\mu_2),\dots,P_l^m(\mu_N)]^{\mathrm{T}}, \tag{15}$$

$$\mathbf{M}=\mathrm{diag}\{\mu_1,\mu_2,\dots,\mu_N\} \tag{16}$$

and

$$\mathbf{W}=\mathrm{diag}\{w_1,w_2,\dots,w_N\}, \tag{17}$$

we can rewrite Eqs. (13) as

$$\left(\mathbf{I}-\frac{1}{\nu}\mathbf{M}\right)\Phi_+(\nu)=\frac{\varpi}{2}\sum_{l=m}^{L}\beta_l\Pi(l,m)\Pi^{\mathrm{T}}(l,m)\mathbf{W}[\Phi_+(\nu)+(-1)^{l-m}\Phi_-(\nu)] \tag{18a}$$

and

$$\left(\mathbf{I}+\frac{1}{\nu}\mathbf{M}\right)\Phi_-(\nu)=\frac{\varpi}{2}\sum_{l=m}^{L}\beta_l\Pi(l,m)\Pi^{\mathrm{T}}(l,m)\mathbf{W}[(-1)^{l-m}\Phi_+(\nu)+\Phi_-(\nu)] \tag{18b}$$

where $\mathbf{I}$ is the $N\times N$ identity matrix. Now if we let

$$\mathbf{U}=\Phi_+(\nu)+\Phi_-(\nu) \tag{19a}$$

and

$$\mathbf{V}=\Phi_+(\nu)-\Phi_-(\nu), \tag{19b}$$

then we can take the sum and the difference of Eqs. (18) to obtain

$$\mathbf{EX}=\frac{1}{\nu}\mathbf{Y} \tag{20a}$$

and

$$\mathbf{FY}=\frac{1}{\nu}\mathbf{X} \tag{20b}$$

where

$$\mathbf{E} = \left(\mathbf{I} - \frac{\varpi}{2}\sum_{l=m}^{L}\beta_l[1 + (-1)^{l-m}]\,\mathbf{\Pi}(l,m)\mathbf{\Pi}^{\mathrm{T}}(l,m)\mathbf{W}\right)\mathbf{M}^{-1}, \tag{21a}$$

$$\mathbf{F} = \left(\mathbf{I} - \frac{\varpi}{2}\sum_{l=m}^{L}\beta_l[1 - (-1)^{l-m}]\,\mathbf{\Pi}(l,m)\mathbf{\Pi}^{\mathrm{T}}(l,m)\mathbf{W}\right)\mathbf{M}^{-1}, \tag{21b}$$

$$\mathbf{X} = \mathbf{MU} \tag{22a}$$

and

$$\mathbf{Y} = \mathbf{MV}. \tag{22b}$$

Clearly, we can eliminate between Eqs. (20) to obtain the eigenvalue problems

$$(\mathbf{FE})\mathbf{X} = \lambda\mathbf{X} \tag{23a}$$

and

$$(\mathbf{EF})\mathbf{Y} = \lambda\mathbf{Y} \tag{23b}$$

where $\lambda = 1/\nu^2$. We note that the required separation constants $\{\nu_j\}$ are readily available once we find the eigenvalues $\{\lambda_j\}$ defined by either Eq. (23a) or Eq. (23b). We choose to express our results in terms of the eigenvalues and eigenvectors defined by Eq. (23a).

Continuing, we assume that Eq. (23a) defines positive eigenvalues and a full set of eigenvectors, and so we let λ_j and $\mathbf{X}(\lambda_j)$, for $j = 1, 2, \ldots, N$, denote this collection. The separation constants we require clearly occur in plus–minus pairs, and so letting ν_j, for the $j = 1, 2, \ldots, N$, denote the reciprocal of positive root of λ_j, we can use Eqs. (19) and (20) to obtain

$$\mathbf{\Phi}_+(\nu_j) = \tfrac{1}{2}\mathbf{M}^{-1}(\mathbf{I} + \nu_j\mathbf{E})\mathbf{X}(\lambda_j) \tag{24a}$$

and

$$\mathbf{\Phi}_-(\nu_j) = \tfrac{1}{2}\mathbf{M}^{-1}(\mathbf{I} - \nu_j\mathbf{E})\mathbf{X}(\lambda_j) \tag{24b}$$

for $j = 1, 2, \ldots, N$. We note that

$$\mathbf{\Phi}_+(-\nu_j) = \mathbf{\Phi}_-(\nu_j), \tag{25}$$

and so at this point we have all we require for defining our solution to Eqs. (11). We let

$$\mathbf{I}_+(\tau) = [I(\tau,\mu_1), I(\tau,\mu_2), \ldots, I(\tau,\mu_N)]^{\mathrm{T}} \tag{26a}$$

and

$$\mathbf{I}_-(\tau) = [I(\tau,-\mu_1), I(\tau,-\mu_2), \ldots, I(\tau,-\mu_N)]^{\mathrm{T}} \tag{26b}$$

so we can express our discrete-ordinates solution to the homogeneous version of Eq. (8) as

$$\mathbf{I}^{\mathrm{h}}_+(\tau) = \sum_{j=1}^{N}[A_j\mathbf{\Phi}_+(\nu_j)\mathrm{e}^{-\tau/\nu_j} + B_j\mathbf{\Phi}_-(\nu_j)\mathrm{e}^{-(\tau_0-\tau)/\nu_j}] \tag{27a}$$

and

$$\mathbf{I}_-^{\mathrm{h}}(\tau) = \sum_{j=1}^{N} [A_j \mathbf{\Phi}_-(\nu_j) \mathrm{e}^{-\tau/\nu_j} + B_j \mathbf{\Phi}_+(\nu_j) \mathrm{e}^{-(\tau_0-\tau)/\nu_j}] \tag{27b}$$

where the constants $\{A_j\}$ and $\{B_j\}$ are at this point arbitrary. Note that in Eqs. (27) we have added the superscript h to remind us that these solutions refer to the homogeneous version of Eq. (8).

Having found our discrete-ordinates solution of the homogeneous version of Eq. (8), we are now ready to define a particular solution to account for the inhomogeneous source term $Q^m(\tau, \mu)$ that appears in Eq. (8). In a recent work [4] based on a discrete-ordinates version of a radiative-transfer problem that is sufficiently general so as to include the problem considered in this work, Barichello, Garcia and Siewert used the infinite-medium Green's function to develop a particular solution that we can use here. Taking into account some changes in notation and continuing to suppress some of the explicit notation that refers to the Fourier index m, we express the particular solution developed in Ref. [4] as

$$\mathbf{I}_+^{p}(\tau) = \sum_{j=1}^{N} [\mathscr{A}_j(\tau) \mathbf{\Phi}_+(\nu_j) + \mathscr{B}_j(\tau) \mathbf{\Phi}_-(\nu_j)] \tag{28a}$$

and

$$\mathbf{I}_-^{p}(\tau) = \sum_{j=1}^{N} [\mathscr{A}_j(\tau) \mathbf{\Phi}_-(\nu_j) + \mathscr{B}_j(\tau) \mathbf{\Phi}_+(\nu_j)]. \tag{28b}$$

Here the functions $\mathscr{A}_j(\tau)$ and $\mathscr{B}_j(\tau)$ that were developed in Ref. [4] can be expressed as

$$\mathscr{A}_j(\tau) = \frac{1}{N(\nu_j)} \int_0^{\tau} \sum_{\alpha=1}^{N} w_\alpha [Q(x, \mu_\alpha)\phi(\nu_j, \mu_\alpha) + Q(x, -\mu_\alpha)\phi(\nu_j, -\mu_\alpha)] \mathrm{e}^{-(\tau-x)/\nu_j} \mathrm{d}x \tag{29a}$$

and

$$\mathscr{B}_j(\tau) = \frac{1}{N(\nu_j)} \int_{\tau}^{\tau_0} \sum_{\alpha=1}^{N} w_\alpha [Q(x, \mu_\alpha)\phi(\nu_j, -\mu_\alpha) + Q(x, -\mu_\alpha)\phi(\nu_j, \mu_\alpha)] \mathrm{e}^{-(x-\tau)/\nu_j} \mathrm{d}x \tag{29b}$$

where

$$N(\nu_j) = \sum_{\alpha=1}^{N} w_\alpha \mu_\alpha [\phi^2(\nu_j, \mu_\alpha) - \phi^2(\nu_j, -\mu_\alpha)]. \tag{30}$$

We note that the particular solution defined by Eqs. (28)–(30) is valid for a general inhomogeneous source term $Q(\tau, \mu)$: however, for the current application where we can write

$$Q(\tau, \mu) = Q(\mu) \mathrm{e}^{-\tau/\mu_0}, \tag{31}$$

with

$$Q(\mu) = \frac{\varpi}{2} \sum_{l=m}^{L} \beta_l P_l^m(\mu_0) P_l^m(\mu), \tag{32}$$

we can follow Ref. [4] and write Eqs. (29) for this special case as

$$\mathscr{A}_j(\tau) = \frac{\mu_0 \nu_j}{N(\nu_j)} C(\tau : \nu_j, \mu_0) \sum_{\alpha=1}^{N} w_\alpha [Q(\mu_\alpha)\phi(\nu_j, \mu_\alpha) + Q(-\mu_\alpha)\phi(\nu_j, -\mu_\alpha)] \tag{33a}$$

and

$$\mathscr{B}_j(\tau) = \frac{\mu_0 \nu_j}{N(\nu_j)} \mathrm{e}^{-\tau/\mu_0} S(\tau_0 - \tau : \nu_j, \mu_0) \sum_{\alpha=1}^{N} w_\alpha [Q(\mu_\alpha)\phi(\nu_j, -\mu_\alpha) + Q(-\mu_\alpha)\phi(\nu_j, \mu_\alpha)] \tag{33b}$$

where the S and C functions are given by

$$S(\tau : x, y) = \frac{1 - \mathrm{e}^{-\tau/x}\mathrm{e}^{-\tau/y}}{x + y} \quad \text{and} \quad C(\tau : x, y) = \frac{\mathrm{e}^{-\tau/x} - \mathrm{e}^{-\tau/y}}{x - y}. \tag{34a, b}$$

Finally, if we make note of our vector notation, we can rewrite Eq. (30) as

$$N(\nu_j) = \mathbf{\Phi}_+^{\mathrm{T}}(\nu_j)\mathbf{W}\mathbf{M}\mathbf{\Phi}_+(\nu_j) - \mathbf{\Phi}_-^{\mathrm{T}}(\nu_j)\mathbf{W}\mathbf{M}\mathbf{\Phi}_-(\nu_j) \tag{35}$$

and Eqs. (33) as

$$\mathscr{A}_j(\tau) = \frac{\mu_0 \nu_j}{N(\nu_j)} C(\tau : \nu_j, \mu_0)[\mathbf{\Phi}_+^{\mathrm{T}}(\nu_j)\mathbf{W}\mathbf{Q}_+ + \mathbf{\Phi}_-^{\mathrm{T}}(\nu_j)\mathbf{W}\mathbf{Q}_-] \tag{36a}$$

and

$$\mathscr{B}_j(\tau) = \frac{\mu_0 \nu_j}{N(\nu_j)} \mathrm{e}^{-\tau/\mu_0} S(\tau_0 - \tau : \nu_j, \mu_0)[\mathbf{\Phi}_-^{\mathrm{T}}(\nu_j)\mathbf{W}\mathbf{Q}_+ + \mathbf{\Phi}_+^{\mathrm{T}}(\nu_j)\mathbf{W}\mathbf{Q}_-] \tag{36b}$$

where

$$\mathbf{Q}_\pm = [Q(\pm\mu_1), Q(\pm\mu_2), \dots, Q(\pm\mu_N)]^{\mathrm{T}}. \tag{37}$$

Having found a particular solution, we can now add it to Eqs. (27) to obtain

$$\mathbf{I}_+(\tau) = \sum_{j=1}^{N} [A_j\mathbf{\Phi}_+(\nu_j)\mathrm{e}^{-\tau/\nu_j} + B_j\mathbf{\Phi}_-(\nu_j)\mathrm{e}^{-(\tau_0-\tau)/\nu_j}] + \mathbf{I}_+^p(\tau) \tag{38a}$$

and

$$\mathbf{I}_-(\tau) = \sum_{j=1}^{N} [A_j\mathbf{\Phi}_-(\nu_j)\mathrm{e}^{-\tau/\nu_j} + B_j\mathbf{\Phi}_+(\nu_j)\mathrm{e}^{-(\tau_0-\tau)/\nu_j}] + \mathbf{I}_-^p(\tau), \tag{38b}$$

and upon substituting Eqs. (38) into the boundary conditions given by Eqs. (9) we find

$$\sum_{j=1}^{N} [A_j\mathbf{\Phi}_+(\nu_j) + B_j\mathbf{\Phi}_-(\nu_j)\mathrm{e}^{-\tau_0/\nu_j}] = -\mathbf{I}_+^p(0) \tag{39a}$$

and

$$\sum_{j=1}^{N} [B_j\mathbf{\Phi}_+(\nu_j) + A_j\mathbf{\Phi}_-(\nu_j)\mathrm{e}^{-\tau_0/\nu_j}] = -\mathbf{I}_-^p(\tau_0). \tag{39b}$$

Eqs. (39) define the system of linear algebraic equations we solve to find the constants $\{A_j\}$ and $\{B_j\}$ required to complete Eqs. (38).

As mentioned earlier in this work, we use the discrete-ordinates method only to determine approximations of the integrals in Eq. (8), and so now we substitute Eqs. (38) into our quadrature versions of those integrals and solve the resulting equation to obtain our final results. In this way we find

$$I^m(\tau, \mu) = \mu_0 C(\tau : \mu_0, \mu) Q(\mu) + \frac{\varpi}{2}[\Xi^m(\tau, \mu) + \Upsilon^m(\tau, \mu)] \tag{40a}$$

and

$$I^m(\tau, -\mu) = \mu_0 e^{-\tau/\mu_0} S(\tau_0 - \tau : \mu_0, \mu) Q(-\mu) + \frac{\varpi}{2}[\Xi^m(\tau, -\mu) + \Upsilon^m(\tau, -\mu)] \tag{40b}$$

for $\mu \in [0, 1]$ and $\tau \in [0, \tau_0]$. Here

$$\Upsilon^m(\tau, \mu) = \sum_{l=m}^{L} \beta_l P_l^m(\mu) \sum_{j=1}^{N} \nu_j [A_j C(\tau : \nu_j, \mu) + (-1)^{l-m} B_j e^{-(\tau_0 - \tau)/\nu_j} S(\tau : \nu_j, \mu)] \, G_l^m(\nu_j) \tag{41a}$$

and

$$\Upsilon^m(\tau, -\mu) = \sum_{l=m}^{L} \beta_l P_l^m(\mu) \sum_{j=1}^{N} \nu_j [(-1)^{l-m} A_j \, e^{-\tau/\nu_j} S(\tau_0 - \tau : \nu_j, \mu) + B_j C(\tau_0 - \tau : \nu_j, \mu)] \, G_l^m(\nu_j) \tag{41b}$$

where

$$G_l^m(\nu_j) = \mathbf{\Pi}^{\mathrm{T}}(l, m) \mathbf{W} [\mathbf{\Phi}_+(\nu_j) + (-1)^{l-m} \mathbf{\Phi}_-(\nu_j)]. \tag{42}$$

Also, with regard to Eqs. (40), we can write

$$\Xi^m(\tau, \mu) = \sum_{l=m}^{L} \beta_l P_l^m(\mu) \sum_{j=1}^{N} \nu_j [X_j(\tau, \mu) + (-1)^{l-m} Y_j(\tau, \mu)] \, G_l^m(\nu_j) \tag{43a}$$

and

$$\Xi^m(\tau, -\mu) = \sum_{l=m}^{L} \beta_l P_l^m(\mu) \sum_{j=1}^{N} \nu_j [(-1)^{l-m} Z_j(\tau, \mu) + W_j(\tau, \mu)] \, G_l^m(\nu_j) \tag{43b}$$

where, in general,

$$X_j(\tau, \mu) = \frac{1}{\mu \nu_j} \int_0^{\tau} \mathscr{A}_j(x) e^{-(\tau - x)/\mu} \, dx, \tag{44a}$$

$$Y_j(\tau, \mu) = \frac{1}{\mu \nu_j} \int_0^{\tau} \mathscr{B}_j(x) e^{-(\tau - x)/\mu} \, dx, \tag{44b}$$

$$Z_j(\tau, \mu) = \frac{1}{\mu \nu_j} \int_{\tau}^{\tau_0} \mathscr{A}_j(x) e^{-(x - \tau)/\mu} \, dx \tag{44c}$$

and

$$W_j(\tau,\mu)=\frac{1}{\mu\nu_j}\int_\tau^{\tau_0}\mathscr{B}_j(x)\mathrm{e}^{-(x-\tau)/\mu}\,\mathrm{d}x. \tag{44d}$$

If we now let

$$a_j=\frac{1}{N(\nu_j)}[\boldsymbol{\Phi}_+^{\mathrm{T}}(\nu_j)\mathbf{W}\mathbf{Q}_+ + \boldsymbol{\Phi}_-^{\mathrm{T}}(\nu_j)\mathbf{W}\mathbf{Q}_-] \tag{45a}$$

and

$$b_j=\frac{1}{N(\nu_j)}[\boldsymbol{\Phi}_-^{\mathrm{T}}(\nu_j)\mathbf{W}\mathbf{Q}_+ + \boldsymbol{\Phi}_+^{\mathrm{T}}(\nu_j)\mathbf{W}\mathbf{Q}_-] \tag{45b}$$

then we can enter Eqs. (36) into Eqs. (44) to obtain, for our current application,

$$X_j(\tau,\mu)=\mu_0 a_j\left[\frac{\nu_j C(\tau:\nu_j,\mu)-\mu_0 C(\tau:\mu_0,\mu)}{\nu_j-\mu_0}\right], \tag{46a}$$

$$Y_j(\tau,\mu)=\mu_0 b_j\left[\frac{\mu_0 C(\tau:\mu_0,\mu)-\nu_j\,\mathrm{e}^{-\tau_0/\mu_0}\,\mathrm{e}^{-(\tau_0-\tau)/\nu_j}S(\tau:\nu_j,\mu)}{\nu_j+\mu_0}\right], \tag{46b}$$

$$Z_j(\tau,\mu)=\mu_0 a_j\left[\frac{\nu_j\mathrm{e}^{-\tau/\nu_j}S(\tau_0-\tau:\nu_j,\mu)-\mu_0\,\mathrm{e}^{-\tau/\mu_0}S(\tau_0-\tau:\mu_0,\mu)}{\nu_j-\mu_0}\right] \tag{46c}$$

and

$$W_j(\tau,\mu)=\mu_0 b_j\left[\frac{\mu_0\mathrm{e}^{-\tau/\mu_0}S(\tau_0-\tau:\mu_0,\mu)-\nu_j\,\mathrm{e}^{-\tau_0/\mu_0}C(\tau_0-\tau:\nu_j,\mu)}{\nu_j+\mu_0}\right]. \tag{46d}$$

To this point we have been concerned with developing our solution for the radiation intensity; however, we have already all that we require to compute moments of the intensity. For example, the partial fluxes

$$q_\pm(\tau)=\int_0^1\int_0^{2\pi}\mu I(\tau,\,\pm\mu,\phi)\,\mathrm{d}\phi\,\mathrm{d}\mu \tag{47}$$

can, after we note Eqs. (4) and (7), be written as

$$q_+(\tau)=\pi\mu_0\mathrm{e}^{-\tau/\mu_0}+\pi\int_0^1\mu I(\tau,\mu)\,\mathrm{d}\mu \tag{48a}$$

and

$$q_-(\tau)=\pi\int_0^1\mu I(\tau,\,-\mu)\,\mathrm{d}\mu \tag{48b}$$

where we suppress the Fourier-component index and let $I(\tau,\mu)$ denote the $m=0$ component of $I_*(\tau,\mu,\phi)$. Keeping in mind that we are using a quadrature scheme defined for the integration interval [0,1], we now substitute Eqs. (38) into quadrature versions of Eqs. (48) to obtain

$$q_+(\tau) = \pi\mu_0 e^{-\tau/\mu_0} + \pi\sum_{j=1}^{N}[A_j e^{-\tau/\nu_j} + \mathscr{A}_j(\tau)]Q_+(\nu_j) + \pi\sum_{j=1}^{N}[B_j e^{-(\tau_0-\tau)/\nu_j} + \mathscr{B}_j(\tau)]Q_-(\nu_j) \tag{49a}$$

and

$$q_-(\tau) = \pi\sum_{j=1}^{N}[A_j e^{-\tau/\nu_j} + \mathscr{A}_j(\tau)]Q_-(\nu_j) + \pi\sum_{j=1}^{N}[B_j e^{-(\tau_0-\tau)/\nu_j} + \mathscr{B}_j(\tau)]Q_+(\nu_j) \tag{49b}$$

where, after noting Eqs. (15) and (17), we can write

$$Q_\pm(\nu_j) = \mathbf{\Pi}^{\mathrm{T}}(1,0)\mathbf{W}\mathbf{\Phi}_\pm(\nu_j). \tag{50}$$

Here, to reiterate, we note that the functions $\{\mathscr{A}_j(\tau)\}$ and $\{\mathscr{B}_j(\tau)\}$ are given by Eqs. (36) and that the constants $\{A_j\}$ and $\{B_j\}$ are the solutions to the linear system defined by Eqs. (39).

5. An alternative particular solution

While the particular solution we have defined by Eqs. (28) is, for various reasons, our preferred form, we take a few lines here to list a variation of the particular solution used by Chandrasekhar. We substitute

$$I_p(\tau, \pm\mu) = F(\pm\mu)e^{-\tau/\mu_0} \tag{51}$$

into our discrete-ordinates version of Eq. (8) to find

$$\left(\mathbf{I} - \frac{1}{\mu_0}\mathbf{M}\right)\mathbf{F}_+ = \frac{\varpi}{2}\sum_{l=m}^{L}\beta_l\mathbf{\Pi}(l,m)\mathbf{\Pi}^{\mathrm{T}}(l,m)\mathbf{W}[\mathbf{F}_+ + (-1)^{l-m}\mathbf{F}_-] + \mathbf{Q}_+ \tag{52a}$$

and

$$\left(\mathbf{I} + \frac{1}{\mu_0}\mathbf{M}\right)\mathbf{F}_- = \frac{\varpi}{2}\sum_{l=m}^{L}\beta_l\mathbf{\Pi}(l,m)\mathbf{\Pi}^{\mathrm{T}}(l,m)\mathbf{W}[(-1)^{l+m}\mathbf{F}_+ + \mathbf{F}_-] + \mathbf{Q}_-. \tag{52b}$$

Here we continue to use the notation established in the previous section of this work, and in addition we note that the vectors $\mathbf{F}_\pm$ have $F(\pm\mu_i)$ as components. We can now add Eqs. (52a) and (52b) and then subtract Eq. (52b) from Eq. (52a) to find

$$[\mathbf{I} - (\mu_0)^2\mathbf{FE}]\mathbf{MP} = -\mu_0[\mathbf{D} + \mu_0\mathbf{FS}] \tag{53a}$$

and

$$[\mathbf{I} - (\mu_0)^2\mathbf{EF}]\mathbf{MH} = -\mu_0[\mathbf{S} + \mu_0\mathbf{ED}] \tag{53b}$$

where $\mathbf{E}$ and $\mathbf{F}$ are defined by Eqs. (21) and where

$$\mathbf{P} = \mathbf{F}_+ + \mathbf{F}_- \quad \text{and} \quad \mathbf{H} = \mathbf{F}_+ - \mathbf{F}_-. \tag{54a, b}$$

In addition

$$\mathbf{S} = \mathbf{Q}_+ + \mathbf{Q}_- \quad \text{and} \quad \mathbf{D} = \mathbf{Q}_+ - \mathbf{Q}_-. \tag{55a, b}$$

Clearly, we can solve Eqs. (53) to find $\mathbf{F}_+$ and $\mathbf{F}_-$. Although we can use Gaussian elimination to solve Eqs. (53), we express the results we require to complete this particular solution as

$$\mathbf{F}_+ = -\tfrac{1}{2}\mu_0\mathbf{M}^{-1}\{[\mathbf{I} - (\mu_0)^2\mathbf{FE}]^{-1}[\mathbf{D} + \mu_0\mathbf{FS}] + [\mathbf{I} - (\mu_0)^2\mathbf{EF}]^{-1}[\mathbf{S} + \mu_0\mathbf{ED}]\} \tag{56a}$$

and

$$\mathbf{F}_- = -\tfrac{1}{2}\mu_0\mathbf{M}^{-1}\{[\mathbf{I} - (\mu_0)^2\mathbf{FE}]^{-1}[\mathbf{D} + \mu_0\mathbf{FS}] - [\mathbf{I} - (\mu_0)^2\mathbf{EF}]^{-1}[\mathbf{S} + \mu_0\mathbf{ED}]\}. \tag{56b}$$

Of course, the particular solution defined by Eqs. (28) is general in the sense that it is valid for (essentially) any inhomogeneous source term $Q(\tau, \mu)$. On the other hand, Eqs. (56) are valid only for the source term given by Eq. (10). In addition, and in contrast to the particular solution defined by Eqs. (28), the particular solution given by Eqs. (56) does not exist in the (unlikely) event that μ_0 is equal to one of the separation constants $\{v_j\}$. This point is clear if we note from Eqs. (23) that μ_0 equal to one of the seperation constants would make the two factors, the inverses of which are required in Eqs. (56), singular. We note that this limitation to Eqs. (56) could be exacerbated if we think of eventually using the solution to our albedo problem as a Green's function, in which case an integration over the variable μ_0 would be encountered. Also, and again in contrast to the solution defined by Eqs. (56), the particular solution defined by Eqs. (28) is explicit and does not require the solution to systems of linear algebraic equations. We do note, however, one nice feature of the particular solution defined by Eqs. (56): it does not have to be modified for the special (conservative) case when $m = 0$ and $\varpi = 1$.

6. The conservative case

In this section we work out the modifications required to extend our discrete-ordinates solution to the conservative case ($\varpi = 1$ and $m = 0$).

The problem with the conservative case is that the largest separation constant, say v_N, becomes infinite, and so the exponential solution, introduced by Eq. (12), does not generate the two independent forms of the solution that are needed. While we can in fact modify our solution developed for $\varpi < 1$ to find a form appropriate to the $\varpi = 1$ case, it is clear that if ϖ is sufficiently close to unity, but not equal to unity, then we can anticipate some numerical difficulties (due to round off errors) in the solution developed for $\varpi < 1$. However, having considered values of $1 - \varpi$ as small, say, as 10^{-8}, we consider that essentially all $\varpi < 1$ cases of practical interest can be solved with the solution discussed in the previous section of this paper. We note that the case of very small $1 - \varpi$ has been well discussed, in the context of the spherical-harmonics method, by Karp, Greenstadt and Fillmore [8].

Continuing with the conservative case, we simply ignore the largest separation constant ν_N found from our discrete-ordinates solution and include with our solution the two (exact) solutions [2] that can be associated with an infinite separation constant. Therefore, we rewrite Eqs. (38) as

$$\mathbf{I}_+(\tau) = A[(\tau_0 - \tau)\mathbf{\Pi}(0,0) + (3/h_1)\mathbf{\Pi}(1,0)] + B[\tau\mathbf{\Pi}(0,0) - (3/h_1)\mathbf{\Pi}(1,0)] + \sum_{j=1}^{N-1} [A_j\mathbf{\Phi}_+(\nu_j)e^{-\tau/\nu_j} + B_j\mathbf{\Phi}_-(\nu_j)e^{-(\tau_0-\tau)/\nu_j}] + \mathbf{I}_+^p(\tau) \tag{57a}$$

and

$$\mathbf{I}_-(\tau) = A[(\tau_0 - \tau)\mathbf{\Pi}(0,0) - (3/h_1)\mathbf{\Pi}(1,0)] + B[\tau\mathbf{\Pi}(0,0) + (3/h_1)\mathbf{\Pi}(1,0)] + \sum_{j=1}^{N-1} [A_j\mathbf{\Phi}_-(\nu_j)e^{-\tau/\nu_j} + B_j\mathbf{\Phi}_+(\nu_j)e^{-(\tau_0-\tau)/\nu_j}] + \mathbf{I}_-^p(\tau) \tag{57b}$$

where $h_1 = 3 - \beta_1$ and the vectors $\mathbf{\Pi}(l,m)$ are given by Eq. (15). Here the constants A and B, like the $\{A_j\}$ and $\{B_j\}$, are to be determined from the boundary conditions. Of course, we must also modify the particular solution for this conservative case, and so we follow Refs. [2, 4, 9] to obtain

$$\mathbf{I}_+^p(\tau) = \mathscr{A}(\tau)[(\tau_0 - \tau)\mathbf{\Pi}(0,0) + (3/h_1)\mathbf{\Pi}(1,0)] + \mathscr{B}(\tau)[\tau\mathbf{\Pi}(0,0) - (3/h_1)\mathbf{\Pi}(1,0)] + \sum_{j=1}^{N-1} [\mathscr{A}_j(\tau)\mathbf{\Phi}_+(\nu_j) + \mathscr{B}_j(\tau)\mathbf{\Phi}_-(\nu_j)] \tag{58a}$$

and

$$\mathbf{I}_-^p(\tau) = \mathscr{A}(\tau)[(\tau_0 - \tau)\mathbf{\Pi}(0,0) - (3/h_1)\mathbf{\Pi}(1,0)] + \mathscr{B}(\tau)[\tau\mathbf{\Pi}(0,0) + (3/h_1)\mathbf{\Pi}(1,0)] + \sum_{j=1}^{N-1} [\mathscr{A}_j(\tau)\mathbf{\Phi}_-(\nu_j) + \mathscr{B}_j(\tau)\mathbf{\Phi}_+(\nu_j)] \tag{58b}$$

where $\{\mathscr{A}_j(\tau)\}$ and $\{\mathscr{B}_j(\tau)\}$ are still given by Eqs. (36) for $j = 1, 2, \ldots, N-1$, but where, in general,

$$\mathscr{A}(\tau) = \frac{1}{\tau_0}\int_0^\tau [h_1 x Q_0(x) + Q_1(x)]\,dx \tag{59a}$$

and

$$\mathscr{B}(\tau) = \frac{1}{\tau_0}\int_\tau^{\tau_0} [h_1(\tau_0 - x)Q_0(x) - Q_1(x)]\,dx \tag{59b}$$

where

$$Q_l(x) = \frac{2l+1}{2}\int_{-1}^{1} P_l(\mu)Q(x,\mu)\,d\mu. \tag{60}$$

For our current application

$$Q_0(x) = \tfrac{1}{2}e^{-x/\mu_0} \quad \text{and} \quad Q_1(x) = \tfrac{1}{2}\beta_1\mu_0 e^{-x/\mu_0} \tag{61a, b}$$

and so we find, from Eqs. (59),

$$\mathscr{A}(\tau) = \frac{\mu_0}{2\tau_0}[3\mu_0 - (3\mu_0 + h_1\tau)\mathrm{e}^{-\tau/\mu_0}] \tag{62a}$$

and

$$\mathscr{B}(\tau) = \frac{\mu_0}{2\tau_0}\{[h_1(\tau_0 - \tau) - 3\mu_0]\mathrm{e}^{-\tau/\mu_0} + 3\mu_0\mathrm{e}^{-\tau_0/\mu_0}\}. \tag{62b}$$

Of course, once we have solved the linear system defined by the boundary conditions so that A, B, and the $\{A_j\}$ and $\{B_j\}$ have been found, then the intensity is still given for this case by Eqs. (40), but instead of Eqs. (41) we have

$$\Upsilon(\tau,\mu) = \Upsilon_0(\tau,\mu) + \sum_{l=0}^{L}\beta_l P_l(\mu)\sum_{j=1}^{N-1}\nu_j[A_j C(\tau:\nu_j,\mu) + (-1)^l B_j\mathrm{e}^{-(\tau_0-\tau)/\nu_j}S(\tau:\nu_j,\mu)]G_l(\nu_j) \tag{63a}$$

and

$$\Upsilon(\tau,-\mu) = \Upsilon_0(\tau,-\mu) + \sum_{l=0}^{L}\beta_l P_l(\mu)\sum_{j=1}^{N-1}\nu_j[(-1)^l A_j\mathrm{e}^{-\tau/\nu_j}S(\tau_0-\tau:\nu_j,\mu) + B_j C(\tau_0-\tau:\nu_j,\mu)]G_l(\nu_j) \tag{63b}$$

where

$$\Upsilon_0(\tau,\mu) = 2A[(\tau_0 + 3\mu/h_1)(1-\mathrm{e}^{-\tau/\mu}) - \tau] + 2B[\tau - (3\mu/h_1)(1-\mathrm{e}^{-\tau/\mu})] \tag{64a}$$

and

$$\Upsilon_0(\tau,-\mu) = 2A\{\tau_0 - \tau - (3\mu/h_1)[1-\mathrm{e}^{-(\tau_0-\tau)/\mu}]\} + 2B\{\tau - \tau_0 + (\tau_0 + 3\mu/h_1)[1-\mathrm{e}^{-(\tau_0-\tau)/\mu}]\} \tag{64b}$$

for $\mu \in [0,1]$ and $\tau \in [0,\tau_0]$. And instead of Eqs. (43) we have

$$\Xi(\tau,\mu) = \Xi_0(\tau,\mu) + \sum_{l=0}^{L}\beta_l P_l(\mu)\sum_{j=1}^{N-1}\nu_j[X_j(\tau,\mu) + (-1)^l Y_j(\tau,\mu)]G_l(\nu_j) \tag{65a}$$

and

$$\Xi(\tau,-\mu) = \Xi_0(\tau,-\mu) + \sum_{l=0}^{L}\beta_l P_l(\mu)\sum_{j=1}^{N-1}\nu_j[(-1)^l Z_j(\tau,\mu) + W_j(\tau,\mu)]G_l(\nu_j) \tag{65b}$$

for $\mu \in [0,1]$ and $\tau \in [0,\tau_0]$. Here, we have defined

$$\Xi_0(\tau,\mu) = X_0(\tau,\mu) + Y_0(\tau,\mu) \tag{66a}$$

and

$$\Xi_0(\tau,-\mu) = Z_0(\tau,\mu) + W_0(\tau,\mu) \tag{66b}$$

where now

$$X_0(\tau, \mu) = \frac{2}{\mu}\int_0^{\tau} \mathscr{A}(x)\,[\tau_0 - x + (\beta_1/h_1)\mu]\mathrm{e}^{-(\tau - x)/\mu}\,\mathrm{d}x, \tag{67a}$$

$$Y_0(\tau, \mu) = \frac{2}{\mu}\int_0^{\tau} \mathscr{B}(x)\,[x - (\beta_1/h_1)\mu]\,\mathrm{e}^{-(\tau - x)/\mu}\,\mathrm{d}x, \tag{67b}$$

$$Z_0(\tau, \mu) = \frac{2}{\mu}\int_{\tau}^{\tau_0} \mathscr{A}(x)\,[\tau_0 - x - (\beta_1/h_1)\mu]\mathrm{e}^{-(x - \tau)/\mu}\,\mathrm{d}x \tag{67c}$$

and

$$W_0(\tau, \mu) = \frac{2}{\mu}\int_{\tau}^{\tau_0} \mathscr{B}(x)\,[x + (\beta_1/h_1)\mu]\,\mathrm{e}^{-(x - \tau)/\mu}\,\mathrm{d}x \tag{67d}$$

where $\mathscr{A}(\tau)$ and $\mathscr{B}(\tau)$ are given by Eqs. (62). Finally, we can use Eqs. (62) in Eqs. (67) to obtain, after we note Eqs. (66), the explicit results

$$\Xi_0(\tau, \mu) = \Gamma(\tau, \mu) - (\mu_0)^2\,(3\mu_0 + \beta_1\mu)C(\tau{:}\ \mu_0, \mu) \tag{68a}$$

and

$$\Xi_0(\tau, -\mu) = \Gamma(\tau, -\mu) - (\mu_0)^2\,(3\mu_0 - \beta_1\mu)\,\mathrm{e}^{-\tau/\mu_0}\,S(\tau_0 - \tau{:}\ \mu_0, \mu) \tag{68b}$$

for $\tau \in [0, \tau_0]$ and $\mu \in [0, 1]$. Here we continue to use the S and C functions defined by Eqs. (34), and in addition

$$\Gamma(\tau, \mu) = \frac{3(\mu_0)^2}{\tau_0}\{\tau_0(1 - \mathrm{e}^{-\tau/\mu}) - \tau(1 - \mathrm{e}^{-\tau_0/\mu_0}) + (3\mu/h_1)\,(1 - \mathrm{e}^{-\tau/\mu})(1 - \mathrm{e}^{-\tau_0/\mu_0})\} \tag{69a}$$

and

$$\Gamma(\tau, -\mu) = \frac{3(\mu_0)^2}{\tau_0}\{\tau_0\,[1 - \mathrm{e}^{-(\tau_0 - \tau)/\mu}] + (\tau_0 - \tau)\,(1 - \mathrm{e}^{-\tau_0/\mu_0})$$
$$- (\tau_0 + 3\mu/h_1)\,[1 - \mathrm{e}^{-(\tau_0 - \tau)/\mu}]\,(1 - \mathrm{e}^{-\tau_0/\mu_0})\}\,. \tag{69b}$$

Now, to conclude our formulation for the conservative case, we note that for the partial fluxes we have modified Eqs. (49) to obtain

$$q_+(\tau) = \pi\mu_0\mathrm{e}^{-\tau/\mu_0} + \pi\,[A + \mathscr{A}(\tau)]\,[(\tau_0 - \tau)/2 + 1/h_1] + \pi[B + \mathscr{B}(\tau)]\,(\tau/2 - 1/h_1)$$
$$+ \pi\sum_{j=1}^{N-1}\,[A_j\,\mathrm{e}^{-\tau/\nu_j} + \mathscr{A}_j(\tau)]Q_+(\nu_j) + \pi\sum_{j=1}^{N-1}\,[B_j\mathrm{e}^{-(\tau_0 - \tau)/\nu_j} + \mathscr{B}_j(\tau)]Q_-(\nu_j) \tag{70a}$$

and

$$q_-(\tau) = \pi\,[A + \mathscr{A}(\tau)]\,[(\tau_0 - \tau)/2 - 1/h_1] + \pi[B + \mathscr{B}(\tau)]\,(\tau/2 + 1/h_1)$$
$$+ \pi\sum_{j=1}^{N-1}\,[A_j\mathrm{e}^{-\tau/\nu_j} + \mathscr{A}_j(\tau)]Q_-(\nu_j) + \pi\sum_{j=1}^{N-1}\,[B_j\mathrm{e}^{-(\tau_0 - \tau)/\nu_j} + \mathscr{B}_j(\tau)]Q_+(\nu_j). \tag{70b}$$

7. Computational details and numerical results

Of course, the first thing we must do in order to evaluate our discrete-ordinates solution numerically is to define a quadrature scheme, and so at this point we can emphasize that our discrete-ordinates solution is essentially independent of the quadrature scheme to be used. The

Table 1
The Legendre coefficients for the cloud C_1 phase function

l	β_l	β_{l+35}	β_{l+70}	β_{l+105}	β_{l+140}	β_{l+175}	β_{l+210}	β_{l+245}	β_{l+280}
0	1.000	19.884	16.144	6.990	2.025	0.440	0.079	0.012	0.002
1	2.544	20.024	15.883	6.785	1.940	0.422	0.074	0.011	0.002
2	3.883	20.145	15.606	6.573	1.869	0.401	0.071	0.011	0.001
3	4.568	20.251	15.338	6.377	1.790	0.384	0.067	0.010	0.001
4	5.235	20.330	15.058	6.173	1.723	0.364	0.064	0.009	0.001
5	5.887	20.401	14.784	5.986	1.649	0.349	0.060	0.009	0.001
6	6.457	20.444	14.501	5.790	1.588	0.331	0.057	0.008	0.001
7	7.177	20.477	14.225	5.612	1.518	0.317	0.054	0.008	0.001
8	7.859	20.489	13.941	5.424	1.461	0.301	0.052	0.008	0.001
9	8.494	20.483	13.662	5.255	1.397	0.288	0.049	0.007	0.001
10	9.286	20.467	13.378	5.075	1.344	0.273	0.047	0.007	0.001
11	9.856	20.427	13.098	4.915	1.284	0.262	0.044	0.006	0.001
12	10.615	20.382	12.816	4.744	1.235	0.248	0.042	0.006	0.001
13	11.229	20.310	12.536	4.592	1.179	0.238	0.039	0.006	0.001
14	11.851	20.236	12.257	4.429	1.134	0.225	0.038	0.005	0.001
15	12.503	20.136	11.978	4.285	1.082	0.215	0.035	0.005	0.001
16	13.058	20.036	11.703	4.130	1.040	0.204	0.034	0.005	0.001
17	13.626	19.909	11.427	3.994	0.992	0.195	0.032	0.005	0.001
18	14.209	19.785	11.156	3.847	0.954	0.185	0.030	0.004	0.001
19	14.660	19.632	10.884	3.719	0.909	0.177	0.029	0.004	0.001
20	15.231	19.486	10.618	3.580	0.873	0.167	0.027	0.004	
21	15.641	19.311	10.350	3.459	0.832	0.160	0.026	0.004	
22	16.126	19.145	10.090	3.327	0.799	0.151	0.024	0.003	
23	16.539	18.949	9.827	3.214	0.762	0.145	0.023	0.003	
24	16.934	18.764	9.574	3.090	0.731	0.137	0.022	0.003	
25	17.325	18.551	9.318	2.983	0.696	0.131	0.021	0.003	
26	17.673	18.348	9.072	2.866	0.668	0.124	0.020	0.003	
27	17.999	18.119	8.822	2.766	0.636	0.118	0.018	0.003	
28	18.329	17.901	8.584	2.656	0.610	0.112	0.018	0.002	
29	18.588	17.659	8.340	2.562	0.581	0.107	0.017	0.002	
30	18.885	17.428	8.110	2.459	0.557	0.101	0.016	0.002	
31	19.103	17.174	7.874	2.372	0.530	0.097	0.015	0.002	
32	19.345	16.931	7.652	2.274	0.508	0.091	0.014	0.002	
33	19.537	16.668	7.424	2.193	0.483	0.087	0.013	0.002	
34	19.721	16.415	7.211	2.102	0.463	0.082	0.013	0.002	

Table 2
The diffuse component of the intensity $I_*(\eta\tau_0, \mu, \phi)$ for the cloud C_1 phase function with $\tau_0 = 64.0$, $\varpi = 0.9$, $\mu_0 = 0.2$ and $\phi - \phi_0 = 0$

μ	$\eta = 0.00$	$\eta = 0.05$	$\eta = 0.10$	$\eta = 0.20$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
–1.0	1.56935(–2)	4.31093(–3)	1.91728(–3)	4.17708(–4)	4.69286(–6)	1.11762(–7)	
–0.9	3.14608(–2)	4.48311(–3)	1.95662(–3)	4.32182(–4)	4.89715(–6)	1.16675(–7)	
–0.8	5.58201(–2)	5.13016(–3)	2.11072(–3)	4.57172(–4)	5.16878(–6)	1.23166(–7)	
–0.7	9.44525(–2)	6.06885(–3)	2.33466(–3)	4.91078(–4)	5.51932(–6)	1.31526(–7)	
–0.6	1.54447(–1)	7.34407(–3)	2.63384(–3)	5.35020(–4)	5.96299(–6)	1.42095(–7)	
–0.5	2.48011(–1)	9.03821(–3)	3.02086(–3)	5.90620(–4)	6.51661(–6)	1.55273(–7)	
–0.4	3.95633(–1)	1.12707(–2)	3.51388(–3)	6.59959(–4)	7.20019(–6)	1.71538(–7)	
–0.3	6.33521(–1)	1.42106(–2)	4.13716(–3)	7.45642(–4)	8.03773(–6)	1.91459(–7)	
–0.2	1.03022	1.80991(–2)	4.92258(–3)	8.50905(–4)	9.05809(–6)	2.15722(–7)	
–0.1	1.73200	2.32935(–2)	5.91202(–3)	9.79764(–4)	1.02962(–5)	2.45157(–7)	
–0.0	2.37252	3.03778(–2)	7.16112(–3)	1.13722(–3)	1.17948(–5)	2.80774(–7)	
0.0		3.03778(–2)	7.16112(–3)	1.13722(–3)	1.17948(–5)	2.80774(–7)	3.31330(–9)
0.1		4.07951(–2)	8.74536(–3)	1.32950(–3)	1.36058(–5)	3.23808(–7)	5.91271(–9)
0.2		6.67342(–2)	1.07739(–2)	1.56440(–3)	1.57933(–5)	3.75777(–7)	7.55766(–9)
0.3		9.39509(–2)	1.34341(–2)	1.85168(–3)	1.84366(–5)	4.38560(–7)	9.32979(–9)
0.4		1.11784(–1)	1.69798(–2)	2.20358(–3)	2.16343(–5)	5.14492(–7)	1.13416(–8)
0.5		1.23053(–1)	2.13665(–2)	2.63511(–3)	2.55098(–5)	6.06501(–7)	1.36868(–8)
0.6		1.20147(–1)	2.57860(–2)	3.16194(–3)	3.02185(–5)	7.18273(–7)	1.64673(–8)
0.7		1.05252(–1)	2.89528(–2)	3.78867(–3)	3.59569(–5)	8.54498(–7)	1.98045(–8)
0.8		8.33081(–2)	2.97753(–2)	4.48327(–3)	4.29740(–5)	1.02118(–6)	2.38490(–8)
0.9		5.81233(–2)	2.75115(–2)	5.14014(–3)	5.15794(–5)	1.22608(–6)	2.87904(–8)
1.0		2.45441(–2)	1.83657(–2)	5.22684(–3)	6.20352(–5)	1.47920(–6)	3.48741(–8)

only restriction we have imposed is that the N quadrature points $\{\mu_k\}$ and the N weights $\{w_k\}$ must be defined for use on the integration interval [0, 1]. In a recent work [10] concerning the equivalence between the spherical-harmonics method and the classical discrete-ordinates method that uses a quadrature scheme defined for use on the integration interval $[-1, 1]$, we confirmed that the weights and nodes defined by the zeros of the associated Legendre functions $P^m_{m+2N}(\mu)$ were a natural choice for a "full-range" quadrature scheme. We therefore can suggest that a "half-range" quadrature scheme defined in terms of the "weight function" $(1-\mu^2)^m$ on the integration interval [0, 1] seems the natural choice [11, 12] to use in this work. As reported by Chalhoub and Garcia in Refs. [11, 12], this quadrature scheme has been used to good effect in radiative-transfer calculations. On the other hand, we have seen [13] a case where the inclusion in the boundary data of a "step function" was well solved by subdividing the integration interval [0, 1] so as to have a "break point" that coincided with the rise in the step-function boundary data. And so we consider there to be some merit in using a simple integration scheme that can naturally be mapped onto the integration interval [0, 1] or various subintervals of that basic interval. While we intend to investigate (in future work) the effectiveness of other integration schemes, in this work we follow a simple approach: we start with the usual Gauss-Legendre scheme (of order N)

Table 3
The diffuse component of the intensity $I_*(\eta\tau_0, \mu, \phi)$ for the cloud C_1 phase function with $\tau_0 = 64.0$, $\varpi = 0.9$, $\mu_0 = 0.2$ and $\phi - \phi_0 = \pi/2$

μ	$\eta = 0.00$	$\eta = 0.05$	$\eta = 0.10$	$\eta = 0.20$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
−1.0	1.56935(−2)	4.31093(−3)	1.91728(−3)	4.17708(−4)	4.69286(−6)	1.11762(−7)	
−0.9	1.81328(−2)	4.64405(−3)	2.02015(−3)	4.36492(−4)	4.89825(−6)	1.16676(−7)	
−0.8	2.10149(−2)	5.05580(−3)	2.15480(−3)	4.61409(−4)	5.16997(−6)	1.23167(−7)	
−0.7	2.44400(−2)	5.56443(−3)	2.32628(−3)	4.93502(−4)	5.52018(−6)	1.31527(−7)	
−0.6	2.83993(−2)	6.18988(−3)	2.54077(−3)	5.34027(−4)	5.96314(−6)	1.42095(−7)	
−0.5	3.30050(−2)	6.95443(−3)	2.80590(−3)	5.84491(−4)	6.51563(−6)	1.55273(−7)	
−0.4	3.81247(−2)	7.88409(−3)	3.13102(−3)	6.46703(−4)	7.19765(−6)	1.71536(−7)	
−0.3	4.37074(−2)	9.00936(−3)	3.52735(−3)	7.22843(−4)	8.03307(−6)	1.91455(−7)	
−0.2	4.91547(−2)	1.03654(−2)	4.00840(−3)	8.15544(−4)	9.05067(−6)	2.15715(−7)	
−0.1	5.28131(−2)	1.19911(−2)	4.59024(−3)	9.27992(−4)	1.02853(−5)	2.45146(−7)	
−0.0	3.66127(−2)	1.39262(−2)	5.29178(−3)	1.06405(−3)	1.17792(−5)	2.80758(−7)	
0.0		1.39262(−2)	5.29178(−3)	1.06405(−3)	1.17792(−5)	2.80758(−7)	3.31329(−9)
0.1		1.62038(−2)	6.13482(−3)	1.22844(−3)	1.35842(−5)	3.23786(−7)	5.91269(−9)
0.2		1.88152(−2)	7.14313(−3)	1.42687(−3)	1.57641(−5)	3.75748(−7)	7.55763(−9)
0.3		2.15527(−2)	8.33897(−3)	1.66631(−3)	1.83976(−5)	4.38520(−7)	9.32975(−9)
0.4		2.39306(−2)	9.73007(−3)	1.95513(−3)	2.15826(−5)	5.14440(−7)	1.13416(−8)
0.5		2.55761(−2)	1.12835(−2)	2.30318(−3)	2.54419(−5)	6.06432(−7)	1.36868(−8)
0.6		2.64190(−2)	1.29162(−2)	2.72117(−3)	3.01300(−5)	7.18184(−7)	1.64672(−8)
0.7		2.65711(−2)	1.45215(−2)	3.21908(−3)	3.58435(−5)	8.54384(−7)	1.98044(−8)
0.8		2.62034(−2)	1.60037(−2)	3.80338(−3)	4.28336(−5)	1.02104(−6)	2.38489(−8)
0.9		2.54826(−2)	1.72967(−2)	4.47469(−3)	5.14244(−5)	1.22593(−6)	2.87903(−8)
1.0		2.45441(−2)	1.83657(−2)	5.22684(−3)	6.20352(−5)	1.47920(−6)	3.48741(−8)

defined by the zeros of the Legendre polynomial $P_N(\mu)$ for use on the integration interval $[-1, 1]$, and then we map (linearly) this scheme into a scheme defined for use on the interval $[0, 1]$.

Having defined our quadrature scheme, we obtain the required separation constants $\{\nu_j\}$ and the associated eigenvectors by using the driver program RG from the EISPACK collection [14] to solve the eigenvalue problem defined by Eq. (23a). We have also used a Gaussian elimination package from the LINPACK collection [15] to solve the system of linear algebraic equations defined by Eqs. (39). At this point our solution is complete, and so we are ready to look at some numerical results.

For our example calculations we consider the cloud C_1 problem that was posed as a basic test problem by the Radiation Commission of the International Association of Meteorology and Atmospheric Physics [16]. This cloud problem was also used to define test cases in Refs. [2, 3]. For this model the 300-term phase function is defined by the Legendre coefficients that were accurately computed by de Haan [17] and Karp [18]. For the sake of completeness we reproduce in our Table 1 the defining Legendre coefficients that were first tabulated in Ref. [2]. Our first problem is

Table 4
The diffuse component of the intensity $I_*(\eta\tau_0, \mu, \phi)$ for the cloud C_1 phase function with $\tau_0 = 64.0$, $\varpi = 0.9$, $\mu_0 = 0.2$ and $\phi - \phi_0 = \pi$

μ	$\eta = 0.00$	$\eta = 0.05$	$\eta = 0.10$	$\eta = 0.20$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
−1.0	1.56935(−2)	4.31093(−3)	1.91728(−3)	4.17708(−4)	4.69286(−6)	1.11762(−7)	
−0.9	2.17431(−2)	5.11215(−3)	2.10612(−3)	4.41062(−4)	4.89936(−6)	1.16677(−7)	
−0.8	3.78221(−2)	5.62344(−3)	2.25453(−3)	4.66487(−4)	5.17116(−6)	1.23168(−7)	
−0.7	4.61092(−2)	6.13263(−3)	2.41651(−3)	4.97557(−4)	5.52106(−6)	1.31528(−7)	
−0.6	4.80969(−2)	6.68373(−3)	2.59985(−3)	5.35633(−4)	5.96332(−6)	1.42095(−7)	
−0.5	5.52736(−2)	7.29004(−3)	2.81068(−3)	5.82115(−4)	6.51471(−6)	1.55272(−7)	
−0.4	6.60110(−2)	7.96452(−3)	3.05524(−3)	6.38599(−4)	7.19517(−6)	1.71533(−7)	
−0.3	9.22168(−2)	8.72469(−3)	3.34008(−3)	7.06954(−4)	8.02850(−6)	1.91450(−7)	
−0.2	1.62931(−1)	9.59029(−3)	3.67231(−3)	7.89394(−4)	9.04337(−6)	2.15707(−7)	
−0.1	1.30445(−1)	1.05863(−2)	4.05996(−3)	8.88548(−4)	1.02744(−5)	2.45135(−7)	
−0.0	8.61035(−2)	1.17417(−2)	4.51252(−3)	1.00754(−3)	1.17638(−5)	2.80743(−7)	
0.0		1.17417(−2)	4.51252(−3)	1.00754(−3)	1.17638(−5)	2.80743(−7)	3.31328(−9)
0.1		1.30933(−2)	5.04114(−3)	1.15012(−3)	1.35629(−5)	3.23764(−7)	5.91266(−9)
0.2		1.47042(−2)	5.65889(−3)	1.32074(−3)	1.57353(−5)	3.75718(−7)	7.55760(−9)
0.3		1.65402(−2)	6.38083(−3)	1.52485(−3)	1.83590(−5)	4.38481(−7)	9.32971(−9)
0.4		1.82756(−2)	7.22013(−3)	1.76904(−3)	2.15316(−5)	5.14388(−7)	1.13415(−8)
0.5		1.89516(−2)	8.16162(−3)	2.06151(−3)	2.53749(−5)	6.06363(−7)	1.36867(−8)
0.6		1.81268(−2)	9.13873(−3)	2.41248(−3)	3.00430(−5)	7.18095(−7)	1.64671(−8)
0.7		1.69801(−2)	1.01214(−2)	2.83519(−3)	3.57322(−5)	8.54270(−7)	1.98043(−8)
0.8		1.59154(−2)	1.11720(−2)	3.34996(−3)	4.26963(−5)	1.02090(−6)	2.38487(−8)
0.9		1.56553(−2)	1.26104(−2)	4.00401(−3)	5.12731(−5)	1.22577(−6)	2.87901(−8)
1.0		2.45441(−2)	1.83657(−2)	5.22684(−3)	6.20352(−5)	1.47920(−6)	3.48741(−8)

for a layer of optical thickness $\tau_0 = 64$ with $\varpi = 0.9$, and the incident beam is defined by the direction cosine $\mu_0 = 0.2$. In Ref. [2] the spherical-harmonics method was used to solve this problem, with essentially five figures of accuracy, for the case of normal incidence ($\mu_0 = 1$) which requires only the $m = 0$ Fourier component to define the complete solution. In a more recent work [3], the F_N method was used to solve, again with essentially five figures of accuracy, a case of non-normal incidence ($\mu_0 = 0.2$). This problem is considered a severe test of a computational method since all 300 Fourier components $I^m(\tau, \mu)$ are required for the complete solution. And so with our test problem defined, we have used a FORTRAN implementation of our solution to obtain the results listed for three values of the azimuthal angle in Tables 2–4. These results given with what we believe to be six figures of accuracy were obtained using various orders N of the quadrature scheme for each of the Fourier component problems. For example, the results listed in our tables were obtained with a maximum value of $N = 350$ for $m \in [0, 6]$ and a minimum value of $N = 100$ for $m \in [201, 299]$. While we did not attempt to find minimum values of N that would yield the six-figure results that are listed in Tables 2–4, we did double the mentioned values of N to see that our algorithm was in fact very stable and that the results listed in Tables 2–4 did not change.

Table 5
The diffuse component of the intensity $I_*(\eta\tau_0, \mu, \phi)$ for the cloud C_1 phase function with $\tau_0 = 64.0$, $\varpi = 1.0$, $\mu_0 = 0.2$ and $\phi - \phi_0 = 0$

μ	$\eta = 0.00$	$\eta = 0.05$	$\eta = 0.10$	$\eta = 0.20$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
−1.0	1.11696(−1)	9.48001(−2)	8.87177(−2)	7.82799(−2)	4.77964(−2)	2.24161(−2)	
−0.9	1.38292(−1)	9.63736(−2)	8.97374(−2)	7.93064(−2)	4.88402(−2)	2.34552(−2)	
−0.8	1.76327(−1)	9.90274(−2)	9.10961(−2)	8.03709(−2)	4.98841(−2)	2.44968(−2)	
−0.7	2.33788(−1)	1.02336(−1)	9.26214(−2)	8.14527(−2)	5.09280(−2)	2.55397(−2)	
−0.6	3.19725(−1)	1.06383(−1)	9.43133(−2)	8.25509(−2)	5.19719(−2)	2.65833(−2)	
−0.5	4.49274(−1)	1.11325(−1)	9.61919(−2)	8.36665(−2)	5.30159(−2)	2.76272(−2)	
−0.4	6.47359(−1)	1.17390(−1)	9.82892(−2)	8.48016(−2)	5.40599(−2)	2.86712(−2)	
−0.3	9.57154(−1)	1.24890(−1)	1.00649(−1)	8.59591(−2)	5.51039(−2)	2.97153(−2)	
−0.2	1.45849	1.34260(−1)	1.03326(−1)	8.71430(−2)	5.61480(−2)	3.07593(−2)	
−0.1	2.31370	1.46129(−1)	1.06391(−1)	8.83579(−2)	5.71921(−2)	3.18034(−2)	
−0.0	2.86066	1.61526(−1)	1.09932(−1)	8.96094(−2)	5.82363(−2)	3.28474(−2)	
0.0		1.61526(−1)	1.09932(−1)	8.96094(−2)	5.82363(−2)	3.28474(−2)	3.57779(−3)
0.1		1.83158(−1)	1.14062(−1)	9.09038(−2)	5.92806(−2)	3.38914(−2)	6.56432(−3)
0.2		2.31498(−1)	1.18935(−1)	9.22479(−2)	6.03249(−2)	3.49354(−2)	8.10518(−3)
0.3		2.78928(−1)	1.24813(−1)	9.36480(−2)	6.13693(−2)	3.59794(−2)	9.48945(−3)
0.4		2.99396(−1)	1.31852(−1)	9.51078(−2)	6.24139(−2)	3.70233(−2)	1.07873(−2)
0.5		3.02958(−1)	1.39199(−1)	9.66228(−2)	6.34585(−2)	3.80673(−2)	1.20272(−2)
0.6		2.81238(−1)	1.44522(−1)	9.81612(−2)	6.45032(−2)	3.91112(−2)	1.32253(−2)
0.7		2.40623(−1)	1.45108(−1)	9.96161(−2)	6.55479(−2)	4.01551(−2)	1.43934(−2)
0.8		1.90312(−1)	1.39120(−1)	1.00730(−1)	6.65924(−2)	4.11991(−2)	1.55402(−2)
0.9		1.36594(−1)	1.25482(−1)	1.00986(−1)	6.76359(−2)	4.22430(−2)	1.66711(−2)
1.0		6.89992(−2)	9.47993(−2)	9.79271(−2)	6.86730(−2)	4.32868(−2)	1.77898(−2)

Finally, to test our solution for the conservative case we consider a second problem defined by the same data as our first problem except that now we take $\varpi = 1$. As mentioned in Section 6, the conservative case must be treated differently (for the $m = 0$ Fourier component), and so this case is considered an important test of our general solution. In Tables 5–7 we list our results which (again) we believe to be correct to all of the six figures given.

8. Concluding remarks

Needless to say, this is not the first work concerning the use of the discrete-ordinates method for solving radiative-transfer problems in plane geometry. However, we consider that we have defined (and tested numerically for a quite difficult problem) a very efficient and concise version of the method. Having said that, we must note that Stamnes, Tsay, Wiscombe and Jayaweera [19] have reported a discrete-ordinates algorithm that has many features common to our work here. However, we note something we find strange about the development reported in

Table 6
The diffuse component of the intensity $I_*(\eta\tau_0, \mu, \phi)$ for the cloud C_1 phase function with $\tau_0 = 64.0$, $\varpi = 1.0$, $\mu_0 = 0.2$ and $\phi - \phi_0 = \pi/2$

μ	$\eta = 0.00$	$\eta = 0.05$	$\eta = 0.10$	$\eta = 0.20$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
–1.0	1.11696(–1)	9.48001(–2)	8.87177(–2)	7.82799(–2)	4.77964(–2)	2.24161(–2)	
–0.9	1.16231(–1)	9.64067(–2)	8.98900(–2)	7.93320(–2)	4.88402(–2)	2.34552(–2)	
–0.8	1.21145(–1)	9.80605(–2)	9.10772(–2)	8.03856(–2)	4.98841(–2)	2.44968(–2)	
–0.7	1.26441(–1)	9.97763(–2)	9.22823(–2)	8.14406(–2)	5.09280(–2)	2.55397(–2)	
–0.6	1.31940(–1)	1.01567(–1)	9.35073(–2)	8.24972(–2)	5.19718(–2)	2.65833(–2)	
–0.5	1.37480(–1)	1.03442(–1)	9.47543(–2)	8.35554(–2)	5.30157(–2)	2.76272(–2)	
–0.4	1.42496(–1)	1.05402(–1)	9.60251(–2)	8.46153(–2)	5.40595(–2)	2.86712(–2)	
–0.3	1.46276(–1)	1.07441(–1)	9.73204(–2)	8.56773(–2)	5.51034(–2)	2.97153(–2)	
–0.2	1.47001(–1)	1.09539(–1)	9.86396(–2)	8.67414(–2)	5.61473(–2)	3.07593(–2)	
–0.1	1.40326(–1)	1.11654(–1)	9.99786(–2)	8.78076(–2)	5.71911(–2)	3.18033(–2)	
–0.0	8.45445(–2)	1.13697(–1)	1.01328(–1)	8.88759(–2)	5.82350(–2)	3.28474(–2)	
0.0		1.13697(–1)	1.01328(–1)	8.88759(–2)	5.82350(–2)	3.28474(–2)	3.57779(–3)
0.1		1.15499(–1)	1.02667(–1)	8.99458(–2)	5.92789(–2)	3.38914(–2)	6.56432(–3)
0.2		1.16680(–1)	1.03957(–1)	9.10159(–2)	6.03227(–2)	3.49354(–2)	8.10518(–3)
0.3		1.16363(–1)	1.05121(–1)	9.20835(–2)	6.13666(–2)	3.59793(–2)	9.48945(–3)
0.4		1.13579(–1)	1.06010(–1)	9.31433(–2)	6.24105(–2)	3.70233(–2)	1.07873(–2)
0.5		1.08299(–1)	1.06376(–1)	9.41850(–2)	6.34543(–2)	3.80672(–2)	1.20272(–2)
0.6		1.01224(–1)	1.05950(–1)	9.51890(–2)	6.44982(–2)	3.91112(–2)	1.32253(–2)
0.7		9.31800(–2)	1.04556(–1)	9.61214(–2)	6.55420(–2)	4.01551(–2)	1.43934(–2)
0.8		8.48414(–2)	1.02162(–1)	9.69316(–2)	6.65858(–2)	4.11990(–2)	1.55402(–2)
0.9		7.66825(–2)	9.88565(–2)	9.75559(–2)	6.76295(–2)	4.22429(–2)	1.66711(–2)
1.0		6.89992(–2)	9.47993(–2)	9.79271(–2)	6.86730(–2)	4.32868(–2)	1.77898(–2)

Ref. [19]: the order of the Legendre expansion of the phase function (our L) and the order of the half-range quadrature scheme (our N) seem to be related by the (impossible) condition $2N - 1 = L$.

While we have also used (for the incident-beam problem) a particular solution (see Section 5) of the form given by Eq. (9a) of Ref. [19], we are of the opinion that the particular solution as given by Eqs. (28) and (33) has two advantages over the form used in Ref. [19]: (i) our particular solution, as defined by Eqs. (28) and (33), is given explicitly and so does not depend on solutions to systems of linear algebraic equations, and (ii) our particular solution is not singular in the event that μ_0 happens to be one of the separation constants $\{v_j\}$. We are happy to report, however, that we have obtained the results given in our tables from formulations based on the two different forms of the particular solution. Finally, we note that our particular solution as given by Eqs. (28) and (29) is general in the sense that it is valid for a general form of the inhomogeneous source term.

Table 7
The diffuse component of the intensity $I_*(\eta\tau_0, \mu, \phi)$ for the cloud C_1 phase function with $\tau_0 = 64.0$, $\varpi = 1.0$, $\mu_0 = 0.2$ and $\phi - \phi_0 = \pi$

μ	$\eta = 0.00$	$\eta = 0.05$	$\eta = 0.10$	$\eta = 0.20$	$\eta = 0.50$	$\eta = 0.75$	$\eta = 1.00$
–1.0	1.11696(–1)	9.48001(–2)	8.87177(–2)	7.82799(–2)	4.77964(–2)	2.24161(–2)	
–0.9	1.20383(–1)	9.74329(–2)	9.01621(–2)	7.93608(–2)	4.88403(–2)	2.34552(–2)	
–0.8	1.41593(–1)	9.91923(–2)	9.13338(–2)	8.04086(–2)	4.98841(–2)	2.44968(–2)	
–0.7	1.53157(–1)	1.00701(–1)	9.24149(–2)	8.14434(–2)	5.09279(–2)	2.55397(–2)	
–0.6	1.56512(–1)	1.02044(–1)	9.34165(–2)	8.24662(–2)	5.19717(–2)	2.65833(–2)	
–0.5	1.65210(–1)	1.03220(–1)	9.43364(–2)	8.34765(–2)	5.30155(–2)	2.76272(–2)	
–0.4	1.77131(–1)	1.04211(–1)	9.51671(–2)	8.44730(–2)	5.40592(–2)	2.86712(–2)	
–0.3	2.06090(–1)	1.04994(–1)	9.58966(–2)	8.54538(–2)	5.51029(–2)	2.97152(–2)	
–0.2	2.83360(–1)	1.05535(–1)	9.65079(–2)	8.64166(–2)	5.61466(–2)	3.07593(–2)	
–0.1	2.36858(–1)	1.05799(–1)	9.69792(–2)	8.73582(–2)	5.71902(–2)	3.18033(–2)	
–0.0	1.44304(–1)	1.05733(–1)	9.72834(–2)	8.82751(–2)	5.82337(–2)	3.28474(–2)	
0.0		1.05733(–1)	9.72834(–2)	8.82751(–2)	5.82337(–2)	3.28474(–2)	3.57779(–3)
0.1		1.05259(–1)	9.73883(–2)	8.91626(–2)	5.92772(–2)	3.38914(–2)	6.56432(–3)
0.2		1.04246(–1)	9.72546(–2)	9.00153(–2)	6.03206(–2)	3.49354(–2)	8.10518(–3)
0.3		1.02179(–1)	9.68326(–2)	9.08269(–2)	6.13639(–2)	3.59793(–2)	9.48945(–3)
0.4		9.82614(–2)	9.60459(–2)	9.15906(–2)	6.24071(–2)	3.70233(–2)	1.07873(–2)
0.5		9.14511(–2)	9.47548(–2)	9.23000(–2)	6.34502(–2)	3.80672(–2)	1.20272(–2)
0.6		8.19737(–2)	9.27974(–2)	9.29515(–2)	6.44932(–2)	3.91111(–2)	1.32253(–2)
0.7		7.23240(–2)	9.02283(–2)	9.35523(–2)	6.55362(–2)	4.01551(–2)	1.43934(–2)
0.8		6.35982(–2)	8.74289(–2)	9.41472(–2)	6.65793(–2)	4.11990(–2)	1.55402(–2)
0.9		5.73107(–2)	8.54715(–2)	9.49212(–2)	6.76232(–2)	4.22429(–2)	1.66711(–2)
1.0		6.89992(–2)	9.47993(–2)	9.79271(–2)	6.86730(–2)	4.32868(–2)	1.77898(–2)

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